# A DIRICHLET SERIES RELATED TO EIGENVALUES OF THE LAPLACIAN FOR CONGRUENCE SUBGROUPS

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Abstract. For any congruence subgroup $\Gamma_0(N)$, an explicit Dirichlet series is given which represents an analytic function of $s$ in the half-plane $\operatorname{Re} s > 1/2$ except for having simple poles at $s = 1, 1/2 + \sqrt{1/4 - \lambda_j}$, $j = 1, 2, \cdots, S$, where $\lambda_j$, $j = 1, 2, \cdots, S$, are the exceptional eigenvalues of the non-Euclidean Laplacian for the congruence subgroup.

## 1. Introduction

Let $N$ be a positive integer greater than one. Denote by $\Gamma_0(N)$ the Hecke congruence subgroup of level $N$. The Laplacian $\Delta$ on the upper half-plane $\mathcal{H}$ is given by

$$\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

Let $D$ be the fundamental domain of $\Gamma_0(N)$. Eigenfunctions of the discrete spectrum of $\Delta$ are nonzero real-analytic solutions of the equation

$$\Delta\psi = \lambda\psi$$

such that $\psi(\gamma z) = \psi(z)$ for all $\gamma$ in $\Gamma_0(N)$ and such that

$$\int_D |\psi(z)|^2 dz < \infty$$

where $dz$ represents the Poincaré measure of the upper half-plane.

Let $\mathfrak{a}$ be a cusp of $\Gamma_0(N)$. Its stabilizer is denoted by $\Gamma_{\mathfrak{a}}$. An element $\sigma_{\mathfrak{a}} \in PSL(2, \mathbb{R})$ exists such that $\sigma_{\mathfrak{a}}\infty = \mathfrak{a}$ and $\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \Gamma_\infty$. Let $f$

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be a $\Gamma_0(N)$-invariant function. If $\mathfrak{a}$ is a cusp of $\Gamma_0(N)$, then $f(\sigma_\mathfrak{a} z)$ is $\Gamma_\infty$-invariant, and hence it admits a formal Fourier expansion

$$f(\sigma_\mathfrak{a} z) = \sum_{n\in\mathbb{Z}} c_{\mathfrak{a}n}(y)e^{2n\pi ix}.$$

A $\Gamma_0(N)$-invariant function is said to be a Maass cusp form if it is square-integrable and is an eigenvector of $\Delta$, such that the Fourier coefficient $c_{\mathfrak{a}0}(y) = 0$ for every cusp $\mathfrak{a}$ of $\Gamma_0(N)$. If $\psi$ is a cusp form associated with a positive discrete eigenvalue $\lambda$, then it has the Fourier expansion [8]

$$\psi(\sigma_\mathfrak{a} z) = \sqrt{y}\sum_{m\neq 0} \rho_\mathfrak{a}(m)K_{i\kappa}(2\pi|m|y)e^{2m\pi ix},$$

where $\kappa = \sqrt{\lambda - 1/4}$ and $K_\nu(y)$ is given by the formula §6.32, [23]

$$K_\nu(y) = \frac{2^\nu\Gamma(\nu+\frac{1}{2})}{y^\nu\sqrt{\pi}}\int_0^\infty \frac{\cos(yt)}{(1+t^2)^{\nu+\frac{1}{2}}}dt. \tag{1.1}$$

The complex numbers $\rho_\mathfrak{a}(m)$, $m(\neq 0) \in \mathbb{Z}$, are called the Fourier coefficients of $\psi$ around the cusp $\mathfrak{a}$.

Let $h_d$ be the class number of indefinite rational quadratic forms with discriminant d. Define

$$\epsilon_d = \frac{v_0 + u_0\sqrt{d}}{2}$$

where the pair $(v_0, u_0)$ is the fundamental solution [13] of Pell's equation $v^2 - du^2 = 4$. Denote by $\Omega$ the set of all the positive integers $d$ such that $d \equiv 0$ or $1 \pmod 4$ and such that $d$ is not a square of an integer. The Möbius function $\mu(n)$ is defined to be one if $n = 1$, to be $(-1)^k$ if $n$ is the product of $k$ distinct primes, and to be zero otherwise.

In this paper we obtain the following theorem, which is a generalization of the main theorem of Conrey and Li [1] in the case when $n = 1$. The result is related to that of Hejhal [6] in spirit, and some of our computation is implicit in Hejhal [5]. Let $\lambda_j$, $j = 1, 2, \cdots$, be an enumeration in nondecreasing order of all positive discrete eigenvalues of the Laplacian for $\Gamma_0(N)$ with an eigenvalue of multiplicity $m$ appearing $m$ times, and let $\kappa_j = \sqrt{\lambda_j - 1/4}$.

**Theorem 1.** *Let*

$$L_N(s) = \sum_{\substack{m|N \\ m\text{ square-free}}} \sum_{k|N} k^{1-2s}\frac{\mu((m,k))}{(m,k)}\sum_{d\in\Omega}\sum_{u}\left(\frac{d}{m}\right)\frac{h_d\ln\epsilon_d}{(du^2)^s}\prod_{p^{2l}|(d,\frac{N}{k})} p^l,$$

*where the summation on $u$ is taken over all the positive integers $u$ such that $\sqrt{4+dk^2u^2} \in \mathbb{Z}$ and $p^{2l}$ is the greatest even prime $p$-power dividing $(d, N/k)$. Then $L_N(s)$ is analytic for $Re\, s > 1$ and can be extended by analytic continuation to the half-plane $Re\, s > 0$ except for having a possible pole at $s = 1/2$ and for having simple poles at $s = 1, \frac{1}{2} \pm i\kappa_j$, $j = 1, 2, \cdots$.*

Discrete eigenvalues $\lambda$ of the Laplacian for congruence subgroups are said to be exceptional if $0 < \lambda < 1/4$. In particular, by using Theorem 1 we obtain the following result for the case when $N$ is a prime number $p$.

**Corollary 2.** *Let $\lambda_j$, $j = 1, 2, \cdots, S$, be the exceptional eigenvalues of the Laplacian for $\Gamma_0(p)$. If*

$$L_p(s) = \sum_{u=1}^{\infty} \frac{1}{u^{2s}} \{ \sum_{\substack{d \in \Omega \\ v^2 - du^2 = 4}} \left( \frac{v^2-4}{p} \right) d^{\frac{1}{2}-s} \sum_{k=1}^{d} \frac{1}{k} \left( \frac{d}{k} \right)$$

$$+ p^{1-2s} \sum_{\substack{d \in \Omega \\ v^2 - dp^2u^2 = 4}} d^{\frac{1}{2}-s} \sum_{k=1}^{d} \frac{1}{k} \left( \frac{d}{k} \right) \},$$

*then $L_p(s)$ represents an analytic function of $s$ in the half-plane $Re\, s > 1/2$ except for having simple poles at $s = \frac{1}{2} + \sqrt{1/4 - \lambda_j}$, $j = 1, 2, \cdots, S$.*

The paper is organized as follows. In section 2, we recall Selberg's trace formula (cf. Murty [12] and Selberg [15]). Elements of $\Gamma_0(N)$ can be divided into four types, the identity, hyperbolic, elliptic and parabolic elements. By using the Maass-Selberg relation (cf. Theorem 2.3.1 of [9]) and some computations of Kubota [9], we compute contributions of the identity, elliptic, hyperbolic and parabolic elements to the trace formula in section 3. From contributions of the identity, elliptic, and parabolic elements, we obtain analyticity information about a series formed by contributions of hyperbolic elements, and a precise statement is given in Theorem 3.4. In section 4, we compute explicitly the total contribution of hyperbolic elements by using results of Sarnak [13]. The result is stated in Theorem 4.4. Then Theorem 1 follows from Theorem 3.4 and Theorem 4.4. Finally we prove Corollary 2 in section 5 by using Dirichlet's class number formula [7] and the Pólya-Vinogradov inequality [2].

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## 2. Trace formulas

Let $s$ be a complex number with $\operatorname{Re} s > 1$. Define

$$k(t) = (1 + \frac{t}{4})^{-s}$$

and

$$k(z, z') = k\left(\frac{|z - z'|^2}{yy'}\right),$$

for $z = x + iy$ and $z' = x' + iy'$ in the upper half-plane. Then $k(mz, mz') = k(z, z')$ for every $2 \times 2$ matrix $m$ of determinant one with real entries. The kernel $k(z, z')$ is of (a)-(b) type in the sense of Selberg [15], p.60. Let

$$g(u) = \int_w^\infty k(t) \frac{dt}{\sqrt{t - w}}$$

with $w = e^u + e^{-u} - 2$. Write

$$h(r) = \int_{-\infty}^\infty g(u) e^{iru} du.$$

Then

$$g(u) = \sqrt{w} \int_0^1 (t + \frac{w}{4})^{-s} t^{s - \frac{3}{2}} \frac{dt}{\sqrt{1 - t}} = c(1 + \frac{w}{4})^{\frac{1}{2} - s} \tag{2.1}$$

where $c = 2\sqrt{\pi}\Gamma(s - \frac{1}{2})\Gamma^{-1}(s)$. Since

$$\begin{aligned} h(r) &= c4^{s - \frac{1}{2}} \int_0^\infty (u + \frac{1}{u} + 2)^{\frac{1}{2} - s} u^{ir - 1} du \\ &= c4^{s - \frac{1}{2}} \int_1^\infty (u + \frac{1}{u} + 2)^{\frac{1}{2} - s} \left(u^{ir} + u^{-ir}\right) \frac{du}{u} \\ &= c \frac{4^s (s - \frac{1}{2})}{(s - \frac{1}{2})^2 + r^2} + A(r, s), \end{aligned}$$

where $A(r, s)$ is finite for $|\operatorname{Im} r| \leqslant 1/2$ and for $\operatorname{Re} s > 0$, we obtain that

$$\lim_{s \to 1/2 + i\kappa} (s - \frac{1}{2} - i\kappa) h(r) = \begin{cases} 4^{1/2 + i\kappa} \sqrt{\pi} \frac{\Gamma(i\kappa)}{\Gamma(1/2 + i\kappa)}, & \text{for } r = \pm\kappa; \\ 0, & \text{for } r \neq \pm\kappa. \end{cases} \tag{2.2}$$

Let $\nu$ be the number of inequivalent cusps of $\Gamma_0(N)$, and let $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_\nu$ be a complete set of inequivalent cusps of $\Gamma_0(N)$. We choose an element $\sigma_{\mathfrak{a}_i} \in PSL(2,\mathbb{R})$ such that $\sigma_{\mathfrak{a}_i}\infty = \mathfrak{a}_i$ and $\sigma_{\mathfrak{a}_i}^{-1}\Gamma_{\mathfrak{a}_i}\sigma_{\mathfrak{a}_i} = \Gamma_\infty$ for $i = 1, 2, \cdots, \nu$. The Eisenstein series $E_i(z,s)$ for the cusp $\mathfrak{a}_i$ is defined by

$$E_i(z,s) = \sum_{\gamma \in \Gamma_{\mathfrak{a}_i} \backslash \Gamma_0(N)} \left(\mathrm{Im}(\sigma_{\mathfrak{a}_i}^{-1}\gamma z)\right)^s$$

for $\mathrm{Re}\, s > 1$ when $z$ is in the upper half-plane. Define

$$K(z,z') = \sum_{T \in \Gamma_0(N)} k(z, Tz')$$

and

$$H(z,z') = \sum_{i=1}^{\nu} \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) E_i(z, \frac{1}{2} + ir) E_i(z', \frac{1}{2} - ir) dr.$$

It follows from (2.14) of [15], Theorem 5.3.3 of [9], and the spectral decomposition formula (5.3.12) of [9] that

$$h(-\frac{i}{2}) + \sum_{j=1}^{\infty} h(\kappa_j) = \int_D \{K(z,z) - H(z,z)\} dz \tag{2.3}$$

for $\mathrm{Re}\, s > 1$.

## 3. Evaluation of components of the trace

For every element $T$ of $\Gamma_0(N)$, we denote by $\Gamma_T$ the set of all the elements of $\Gamma_0(N)$ commuting with $T$. Put $D_T = \Gamma_T \backslash \mathcal{H}$. Elements of $\Gamma_0(N)$ can be divided into four types, of which the first consists of the identity element, while the others are respectively the hyperbolic, the elliptic and the parabolic elements. If $T$ is not a parabolic element, put

$$c(T) = \int_{D_T} k(z, Tz) dz.$$

Every cusp of $\Gamma_0(N)$ is equivalent to one of the following inequivalent cusps

$$\frac{u}{w} \text{ with } u, w > 0,\ (u,w) = 1,\ w | N. \tag{3.1}$$

Two such cusps $u/w$ and $u_1/w_1$ are $\Gamma_0(N)$-equivalent if and only if $w = w_1$ and $u \equiv u_1$ modulo $(w, N/w)$. Let $\mathfrak{a} = u/w$ be given as in (3.1). By (2.2) and (2.3) of [3], we have

$$\Gamma_{\mathfrak{a}} = \left\{ \begin{pmatrix} 1 + cu/w & -cu^2/w^2 \\ c & 1 - cu/w \end{pmatrix} : \ c \equiv 0 \ (\mathrm{mod}\, [w^2, N]) \right\}$$

and

$$\sigma_{\mathfrak{a}}\infty = \mathfrak{a} \ \text{ and } \ \sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \Gamma_\infty$$

where

$$\sigma_{\mathfrak{a}} = \begin{pmatrix} \mathfrak{a}\sqrt{[w^2, N]} & 0 \\ \sqrt{[w^2, N]} & 1/\mathfrak{a}\sqrt{[w^2, N]} \end{pmatrix}. \tag{3.2}$$

**3.1. The identity component.**
We have

$$c(I) = \int_{\Gamma_0(N)\backslash\mathcal{H}} dz.$$

**3.2. Elliptic components.**
There are only a finite number of elliptic conjugacy classes.

**Lemma 3.1.** *Let $R$ be an elliptic element of $\Gamma_0(N)$. Then*

$$c(R) = \frac{\pi}{2m\sin\theta}\int_0^\infty \frac{k(t)}{\sqrt{t + 4\sin^2\theta}}dt,$$

*where $m$ is the order of a primitive element of $\Gamma_R$ and where $\theta$ is defined by the formula trace$(R) = 2\cos\theta$.*

*Proof.* Since $R$ is an elliptic element of $\Gamma_0(N)$, an element $\sigma \in PSL(2, \mathbb{R})$ exists such that

$$\sigma R \sigma^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \widetilde{R}$$

for some real number $0 < \theta < \pi$. Denote by $(\sigma\Gamma_0(N)\sigma^{-1})_{\widetilde{R}}$ the set of all the elements of $\sigma\Gamma_0(N)\sigma^{-1}$ which commute with $\widetilde{R}$. We have

$$c(R) = \int_{D_{\widetilde{R}}} k(z, \widetilde{R}z)dz$$

where $D_{\widetilde{R}} = (\sigma\Gamma_0(N)\sigma^{-1})_{\widetilde{R}}\backslash\mathcal{H}$.

Let $\gamma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ be an element of $\Gamma_0(N)$ which has the same fixed points as $R = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then $(\alpha - \delta)c = \gamma(a-d)$ and $\beta c = \gamma b$. It follows that $\gamma$ commutes with $R$. By Proposition 1.16 of [17], a primitive elliptic element $\gamma_0$ of $\Gamma_0(N)$ exists such that $(\eta\Gamma\eta^{-1})_{\widetilde{R}}$ is generated by $\eta\gamma_0\eta^{-1}$. Since $\eta\gamma_0\eta^{-1}$ commutes with $\widetilde{R}$, it is of the form

$$\begin{pmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{pmatrix}$$

for some real number $\theta_0$. By Proposition 1.16 of [17], $\theta_0 = \pi/m$ for some positive integer $m$. It follows from the argument of [9], p.99 that

$$c(R) = \frac{1}{m} \int_0^\infty \int_{-\infty}^\infty k\left( \frac{|z^2+1|^2}{y^2} \sin^2\theta \right) dz.$$

By the argument of [9], p.100 we have

$$c(R) = \frac{\pi}{2m\sin\theta} \int_0^\infty \frac{k(t)}{\sqrt{t + 4\sin^2\theta}} dt. \quad \square$$

**3.3. Hyperbolic components.**

Let $P$ be a hyperbolic element of $\Gamma_0(N)$. Then an element $\rho \in SL_2(\mathbb{R})$ exists such that

$$\rho P \rho^{-1} = \begin{pmatrix} \lambda_P & 0 \\ 0 & \lambda_P^{-1} \end{pmatrix} = \widetilde{P}$$

with $\lambda_P > 1$. The number $\lambda_P^2$ is called the norm of $P$, and is denoted by $NP$. It follows that

$$c(P) = \int_{D_{\widetilde{P}}} k(z, NPz) dz$$

where $D_{\widetilde{P}} = (\rho\Gamma_0(N)\rho^{-1})_{\widetilde{P}} \backslash \mathcal{H}$. Let $P_0$ be a primitive hyperbolic element of $SL_2(\mathbb{Z})$, which generates the group of all elements of $SL_2(\mathbb{Z})$ commuting with $P$. Then there exists a hyperbolic element $P_1 \in \Gamma_0(N)$, which generates $\Gamma_P$, such that $P_1$ is the smallest positive integer power of $P_0$ among all the generators of $\Gamma_P$ in $\Gamma_0(N)$. Detail discussions about the "primitive" hyperbolic element $P_1$ are given in the proof of Lemma 4.1.

**Theorem 3.2.** *Let $P$ be a hyperbolic element of $\Gamma_0(N)$. If $P_1$ is a "primitive" hyperbolic element of $\Gamma_0(N)$ which generates $\Gamma_P$, then*

$$c(P) = \frac{\ln NP_1}{(NP)^{1/2} - (NP)^{-1/2}} g(\ln NP).$$

*Proof.* An argument similar to that made for elliptic elements shows that every element of $\Gamma_0(N)$, which has the same fixed points as $P$, commutes with $P$. Because $\rho P_1 \rho^{-1}$ commutes with $\widetilde{P}$, it is of the form

$$\begin{pmatrix} \lambda_{P_1} & 0 \\ 0 & \lambda_{P_1}^{-1} \end{pmatrix}$$

for some real number $\lambda_{P_1} > 1$. Then

$$c(P) = \int_1^{NP_1} \frac{dy}{y^2} \int_{-\infty}^{\infty} k\left( \frac{(NP-1)^2}{NP} \frac{|z|^2}{y^2} \right) dx.$$

The stated identity follows. $\square$

**3.4. Parabolic components.**

Let $S$ be a parabolic element of $\Gamma_0(N)$. An argument similar to that made for the elliptic elements shows that every element of $\Gamma_0(N)$, which has the same fixed point as $S$, commutes with $S$. If $\mathfrak{a} = u/w$ is the fixed point of $S$, then $\Gamma_S = \Gamma_{\mathfrak{a}}$, and hence we have $\sigma_{\mathfrak{a}}^{-1} \Gamma_S \sigma_{\mathfrak{a}} = \Gamma_\infty$ where $\sigma_{\mathfrak{a}}$ is given as in (3.2). It follows that

$$\sigma_{\mathfrak{a}}^{-1} S \sigma_{\mathfrak{a}} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

for some integer $b$. Furthermore, elements of the form

$$S = \sigma_{\mathfrak{a}} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \sigma_{\mathfrak{a}}^{-1}, \quad 0 \neq b \in \mathbb{Z}$$

constitute a complete set of representatives for the conjugacy classes of parabolic elements of $\Gamma_0(N)$ having $\mathfrak{a}$ as its fixed point. For a large positive number $Y$, let

$$D_Y = \{z \in D : \mathrm{Im} \sigma_{\mathfrak{a}_i}^{-1} z < Y, \ i = 1, 2, \cdots, h\}.$$

Then we have

$$\sum_{\{S\}} \int_{D_Y} k(z, Sz) dz = \int_0^Y \int_0^1 \sum_{0 \neq b \in \mathbb{Z}} k(z, z+b) dz + o(1),$$

where the summation on $\{S\}$ is taken over all parabolic classes represented by parabolic elements whose fixed point is $\mathfrak{a} = u/w$ and $o(1)$ tends to zero as $Y \to \infty$.

**Theorem 3.3.** *Let*

$$c(\infty)_Y = \nu \int_0^Y \int_0^1 \sum_{0\neq b\in\mathbb{Z}} k(z, z+b)dz - \int_{D_Y} H(z,z)dz.$$

*Then*

$$\begin{aligned} c(\infty)_Y &= -\nu g(0)\ln 2 + \sum_{i,j=1}^{\nu} \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r) Re\, \varphi'_{ij}(\frac{1}{2}+ir)\varphi_{ij}(\frac{1}{2}-ir)dr \\ &\quad - \frac{\nu}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}(1+ir)dr + \frac{1}{4}h(0)\{\nu - \sum_{i=1}^{\nu}\varphi_{ii}(\frac{1}{2})\} + o(1) \end{aligned}$$

*where $o(1)$ tends to zero as $Y \to \infty$.*

*Proof.* By the argument of [9], pp.102–106 we have

$$\begin{aligned} &\int_0^Y \int_0^1 \sum_{0\neq b\in\mathbb{Z}} k(z, z+b)dz \\ &= g(0)\ln Y - \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}(1+ir)dr - g(0)\ln 2 + \frac{1}{4}h(0) + o(1). \end{aligned}$$

Let

$$\varphi_{ij,m}(s) = \sum_c \frac{1}{|c|^{2s}}\left(\sum_d e(md/c)\right)$$

where summations are taken over $c > 0, d$ modulo $c$ with $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$ belonging to $\sigma_{\mathfrak{a}_i}^{-1}\Gamma_0(N)\sigma_{\mathfrak{a}_j}$. Then we have

$$\begin{aligned} E_i(\sigma_{\mathfrak{a}_j}z, s) &= \delta_{ij}y^s + \varphi_{ij}(s)y^{1-s} \\ &\quad + \frac{2\pi^s\sqrt{y}}{\Gamma(s)}\sum_{m\neq 0}|m|^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2|m|\pi y)\varphi_{ij,m}(s)e(mx) \end{aligned}$$

where

$$\varphi_{ij}(s) = \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)}\varphi_{ij,0}(s).$$

By (1.1) and the Maass-Selberg relation (cf. Theorem 2.3.1 of [9]), we obtain that

$$\begin{aligned} \int_{D_Y} E_i(z,s)E_i(z,\bar{s})dz &= \frac{Y^{s+\bar{s}-1} - \sum_{j=1}^{\nu}\varphi_{ij}(s)\varphi_{ij}(\bar{s})Y^{1-s-\bar{s}}}{s+\bar{s}-1} \\ &\quad + \frac{\varphi_{ii}(\bar{s})Y^{s-\bar{s}} - \varphi_{ii}(s)Y^{\bar{s}-s}}{s-\bar{s}} + o(1) \end{aligned} \tag{3.3}$$

for nonreal $s$ with $\operatorname{Re} s > 1/2$, where $o(1)$ tends to zero as $Y \to \infty$. By partial integration, we obtain

$$h(r) = \frac{1}{r^4} \int_0^\infty g^{(4)}(\ln u) u^{ir-1} du \tag{3.4}$$

for nonzero $r$. It follows that

$$\begin{aligned} &\lim_{S \to \frac{1}{2}^+} \int_{-\infty}^{\infty} h(r) \{ \int_{D_Y} E_i(z, S+ir) E_i(z, S-ir) dz \} dr \\ &= 4\pi g(0) \ln Y - \sum_{j=1}^{\nu} \int_{-\infty}^{\infty} h(r) \operatorname{Re} \varphi_{ij}'(\frac{1}{2}+ir)\varphi_{ij}(\frac{1}{2}-ir) dr \\ &+ \int_{-\infty}^{\infty} h(r) \frac{\varphi_{ii}(\frac{1}{2}-ir) Y^{2ir}}{ir} dr + o(1). \end{aligned} \tag{3.5}$$

By the Riemann-Lebesgue theorem (cf. §1.8 of [21]), we have

$$\begin{aligned} &\lim_{Y\to\infty} \int_{-\infty}^{\infty} h(r) \frac{\varphi_{ii}(\frac{1}{2}-ir) Y^{2ir}}{ir} dr \\ &= \lim_{Y\to\infty} \int_{-\infty}^{\infty} h(r) Re \left( \varphi_{ii}(\frac{1}{2}-ir) \right) \frac{\sin(2r\ln Y)}{r} dr = \pi h(0) \varphi_{ii}(\frac{1}{2}). \end{aligned}$$

Then it follows from (3.5) that

$$\begin{aligned} \int_{D_Y} H(z,z) dz = &\sum_{i=1}^{\nu} \{ g(0) \ln Y + \frac{1}{4} h(0) \varphi_{ii}(\frac{1}{2}) \\ &- \sum_{j=1}^{\nu} \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \operatorname{Re} \varphi_{ij}'(\frac{1}{2}+ir) \varphi_{ij}(\frac{1}{2}-ir) dr \} + o(1). \end{aligned}$$

The stated identity then follows. □

It follows Theorem 3.3 that

(3.6)

$$\begin{aligned} &\lim_{Y\to\infty} c(\infty)_Y = -\nu g(0) \ln 2 + \frac{1}{4} h(0) \left( \nu - \sum_{i=1}^{\nu} \varphi_{ii}(\frac{1}{2}) \right) \\ &- \frac{\nu}{2\pi} \int_{-\infty}^{\infty} h(r) \frac{\Gamma'}{\Gamma}(1+ir) dr + \sum_{i,j=1}^{\nu} \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \varphi_{ij}'(\frac{1}{2}+ir)\varphi_{ij}(\frac{1}{2}-ir) dr. \end{aligned}$$

Denote by $c(\infty)$ the right side of the identity (3.6). We conclude that the trace formula (2.3) can be written as

$$h(-\frac{i}{2}) + \sum_{j=1}^{\infty} h(\kappa_j) = c(I) + \sum_{\{R\}} c(R) + \sum_{\{P\}} c(P) + c(\infty) \tag{3.7}$$

for $\operatorname{Re} s > 1$, where the summations on the right side of the identity are taken over conjugacy classes.

**Theorem 3.4.** *Let $c(P) = \int_{\Gamma_P \backslash \mathcal{H}} k(z, Pz)dz$ for hyperbolic elements $P \in \Gamma_0(N)$. Then the series*

$$\sum_{\{P\}} c(P)$$

*represents an analytic function in the half-plane $Res > 0$ except for having a possible pole at $s = 1/2$ and for having simple poles at $s = 1, \frac{1}{2} \pm i\kappa_j$, $j = 1, 2, \cdots$.*

*Proof.* We have

$$g^{(4)}(\log u) = A(s)u^{\frac{1}{2}-s} + O_s(u^{-\frac{1}{2}}),$$

where $A(s)$ is an analytic function of $s$ for $\operatorname{Re} s > 0$ and where $O_s(u^{-\frac{1}{2}})$ means that, for every complex number $s$ with $\operatorname{Re} s > 0$, there exists a finite constant $B(s)$ depending only on $s$ such that

$$|O_s(u^{-\frac{1}{2}})| \leqslant B(s)u^{-\frac{1}{2}}.$$

Moreover, for every fixed value of $u$, the term $O_s(u^{-\frac{1}{2}})$ also represents an analytic function of $s$ for $\operatorname{Re} s > 0$. Since

$$h(r) = \frac{1}{r^4} \int_0^{\infty} g^{(4)}(\ln u)u^{ir-1}du$$

for nonzero $r$, we have

$$\begin{aligned} h(r) &= \frac{1}{r^4} \int_1^{\infty} g^{(4)}(\ln u)(u^{ir} + u^{-ir})\frac{du}{u} \\ &= \frac{A(s)}{r^4} \int_1^{\infty} u^{-\frac{1}{2}-s}(u^{ir} + u^{-ir})du + O_s\left(\frac{1}{r^4}\int_1^{\infty} u^{-1-\epsilon}du\right) \\ &= \frac{A(s)}{r^4}\left(\frac{1}{s-\frac{1}{2}-ir} + \frac{1}{s-\frac{1}{2}+ir}\right) + O_s(r^{-4}) \end{aligned}$$

for $\mathrm{Re}\, s > 1$ and for nonzero $r$ with $|\mathrm{Im}\, r| < \frac{1}{2} - \epsilon$. By analytic continuation, we obtain that

$$h(r) = \frac{2A(s)(s-\frac{1}{2})}{r^4[(s-\frac{1}{2})^2 + r^2]} + O_s(r^{-4}) \tag{3.8}$$

for $\mathrm{Re}\, s > 0$ and for nonzero $r$ with $|\mathrm{Im} r| < \frac{1}{2} - \epsilon$. It follows from results of [22] and (2.2) that the left side of (3.7) is an analytic function of $s$ for $\mathrm{Re}\, s > 0$ except for having simple poles at $s = 1, \frac{1}{2} \pm i\kappa_j,\ j = 1, 2, \cdots$. Then the right side of (3.7) can be interpreted as an analytic function of $s$ in the same region by analytic continuation.

Since $k(t) = (1 + t/4)^{-s}$, by Lemma 3.1 we have that $c(R)$ is analytic for $\mathrm{Re}\, s > 0$ except for a simple pole at $s = 1/2$. There are only a finite number of elliptic conjugacy classes $\{R\}$. The term $c(I)$ is a constant.

Since $g(0) = 2\sqrt{\pi}\Gamma(s-\frac{1}{2})\Gamma(s)^{-1}$,

$$h(0) = 2\sqrt{\pi}4^s \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \int_1^\infty (u + \frac{1}{u} + 2)^{\frac{1}{2}-s} \frac{du}{u}$$

and

$$g(\ln \frac{a}{d}) = 2\sqrt{\pi}4^{s-\frac{1}{2}} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \left(\frac{a}{d} + \frac{d}{a} + 2\right)^{\frac{1}{2}-s},$$

the sum of first two terms on the right side of the identity (3.6) is analytic for $\mathrm{Re}\, s > 0$ except for a pole at $s = 1/2$.

By Stirling's formula the identity

$$\frac{\Gamma'(z)}{\Gamma(z)} = \ln z + O(1) \tag{3.9}$$

holds uniformly when $|\arg z| \le \pi - \delta$ for a small positive number $\delta$. It follows from (3.8) and (3.9) that the third term on the right side of the identity (3.6) is analytic for $\mathrm{Re}\, s > 0$ except for a possible pole at $s = 1/2$.

By Theorem 4.4.1 of Kubota [9], each Eisenstein series $E_i(z, s)$ has a meromorphic continuation to the whole $s$-plane, and the identity

$$\sum_{j=1}^{\nu} \varphi_{ij}\left(\frac{1}{2} + ir\right) \varphi_{ij}\left(\frac{1}{2} - ir\right) = 1$$

hold for all real $r$ and for $i = 1, 2, \cdots, \nu$. It follows that functions $\varphi_{ij}(s)$, $i, j = 1, 2, \cdots, \nu$, are analytic on the line $\mathrm{Re}\, s = 1/2$. Let $Y$ be a fixed large positive number. By (3.3) we have

$$
\begin{aligned}
&\mathrm{Re}\sum_{j=1}^{\nu}\varphi'_{ij}\left(\frac{1}{2}+ir\right)\varphi_{ij}\left(\frac{1}{2}-ir\right)\\
&=2\ln Y+\frac{\varphi_{ii}(1/2-ir)Y^{2ir}-\varphi_{ii}(1/2+ir)Y^{-2ir}}{2ir}\\
&-\int_{D_Y}E_i\left(z,\frac{1}{2}+ir\right)E_i\left(z,\frac{1}{2}-ir\right)dz+o(1).
\end{aligned}
\tag{3.10}
$$

Let $\mathfrak{a}_i = u_i/w_i$ be a cusp given in (3.1), and let $\eta = 1/2 + ir$. By the argument following Lemma 3.6 of [3], we have

$$
\begin{aligned}
E_i(z,\eta)=\delta_{\mathfrak{a}_i\infty}y^{\eta}&+\frac{\sqrt{\pi}\Gamma(\eta-\frac{1}{2})}{\Gamma(\eta)}\varphi_{\mathfrak{a}_i\infty,0}(\eta)y^{1-\eta}\\
&+\frac{2\pi^{\eta}\sqrt{y}}{\Gamma(\eta)}\sum_{m\neq 0}|m|^{\eta-\frac{1}{2}}K_{\eta-\frac{1}{2}}(2|m|\pi y)\varphi_{\mathfrak{a}_i\infty,m}(\eta)e(mx)
\end{aligned}
\tag{3.11}
$$

where

$$
\begin{aligned}
&\varphi_{\mathfrak{a}_i\infty,0}(\eta)\\
&=\frac{\varphi(w_i)}{\varphi((w_i,N/w_i))}\left(\frac{(w_i,N/w_i)}{w_iN}\right)^{\eta}\prod_{p|N}\frac{1}{1-p^{-2\eta}}\prod_{p|\frac{N}{w_i}}(1-p^{1-2\eta})\frac{\zeta 2\eta-1}{\zeta(2\eta)}
\end{aligned}
$$

and

$$
\varphi_{\mathfrak{a}_i\infty,m}(\eta)=\left(\frac{(w_i,N/w_i)}{w_iN}\right)^{\eta}\sum_{(c,N/w_i)=1}\frac{1}{c^{2\eta}}\sum_{\substack{d \bmod (cw_i),\,(d,cw_i)=1\\ cd\equiv u_i \bmod (w_i,N/w_i)}}e\left(-\frac{md}{cw_i}\right).
$$

It follows from the functional identity of the Riemann zeta-function that

$$
\begin{aligned}
\frac{\Gamma(\eta-\frac{1}{2})}{\Gamma(\eta)}\varphi_{\mathfrak{a}_i\infty,0}(\eta)=&\pi^{2\eta-\frac{3}{2}}\frac{\Gamma(1-\eta)}{\Gamma(\eta)}\frac{\zeta(2-2\eta)}{\zeta(2\eta)}\frac{\varphi(w_i)}{\varphi((w_i,N/w_i))}\\
&\times\left(\frac{(w_i,N/w_i)}{w_iN}\right)^{\eta}\prod_{p|N}\frac{1}{1-p^{-2\eta}}\prod_{p|\frac{N}{w_i}}(1-p^{1-2\eta}).
\end{aligned}
$$

By using Stirling's formula

$$
|\Gamma(\sigma+it)|\sim\sqrt{2\pi}e^{-\pi|t|/2}|t|^{\sigma-1/2}
$$

for any fixed real value of $\sigma$ as $t \to \infty$, we obtain that

$$\frac{\Gamma(\eta - \frac{1}{2})}{\Gamma(\eta)} \varphi_{\mathfrak{a}_i \infty, 0}(\eta) y^{1-\eta} \ll \sqrt{y} \ln^2(|r| + 1)). \tag{3.12}$$

By (1.1) and by partial integration, we find that

$$\frac{K_{\eta-1/2}(2|m|\pi y)}{\Gamma(\eta)} = \frac{2^{\eta-1/2}}{(2|m|\pi y)^{\eta-1/2}\sqrt{\pi}} \int_0^\infty \frac{\cos(2|m|\pi y t)}{(1+t^2)^\eta} dt \ll \frac{1+|r|^3}{|m|^3 y^3}. \tag{3.13}$$

Then it follows from (3.8), (3.10)-(3.13), Lemma 4.7 of [3] together with the proof of Theorem 1 in [10], and Cauchy's inequality that the fourth term on the right side of the identity (3.6) is analytic for $\operatorname{Re} s > 0$ except for a possible pole at $s = 1/2$.

Also, it follows from (3.4) and Lemma 8.2 of Strömbergsson [20] that the fourth term on the right side of the identity (3.6) is analytic for $\operatorname{Re} s > 0$ except for a possible pole at $s = 1/2$.

Therefore, by (3.7) we have proved that the series

$$\sum_{\{P\}} c(P)$$

represents an analytic function of $s$ in the half-plane $\operatorname{Re} s > 0$ except for having a possible pole at $s = 1/2$ and for having simple poles at $s = 1, \frac{1}{2} \pm i\kappa_j$, $j = 1, 2, \cdots$. $\square$

## 4. Proof of the Main Theorem

A quadratic form $ax^2 + bxy + cy^2$, which is denoted by $[a, b, c]$, is said to be primitive if $(a, b, c) = 1$ and $b^2 - 4ac = d \in \Omega$. Two quadratic forms $[a, b, c]$ and $[a', b', c']$ are equivalent if an element $\gamma \in SL_2(\mathbb{Z})$ exists such that

$$\begin{pmatrix} a' & b'/2 \\ b'/2 & c' \end{pmatrix} = \gamma^t \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \gamma,$$

where $\gamma^t$ is the transpose of $\gamma$. This relation partitions quadratic forms into equivalence classes, and two such forms from the same class have the same discriminant. The number of classes $h_d$ of primitive indefinite quadratic forms of a given discriminant $d$ is finite, and is called the class number of indefinite quadratic forms. Two quadratic forms $[a, b, c]$ and $[a', b', c']$ are $\Gamma_0(N)$-conjugate if an element $\gamma \in \Gamma_0(N)$ exists such that

$$\begin{pmatrix} a' & b'/2 \\ b'/2 & c' \end{pmatrix} = \gamma^t \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \gamma.$$

**Lemma 4.1.** *Let $P$ be a hyperbolic element of $\Gamma_0(N)$. Then there exists a primitive indefinite quadratic form $[a, b, c]$ of discriminant $d$ such that*

$$P = \begin{pmatrix} \frac{v - bNu(a,N)^{-1}}{2} & -cNu(a,N)^{-1} \\ aNu(a,N)^{-1} & \frac{v + bNu(a,N)^{-1}}{2} \end{pmatrix}$$

*with*

$$v^2 - \{dN^2(a,N)^{-2}\}u^2 = 4.$$

*If $\lambda_P$ is an eigenvalue of $P$, then*

$$\lambda_P - \frac{1}{\lambda_P} = \pm \frac{Nu}{(a,N)}\sqrt{d}.$$

*Let $(v_1, u_1)$ with $v_1, u_1 > 0$ be the fundamental solution of Pell's equation $v^2 - d_1 u^2 = 4$, where $d_1 = dN^2/(a,N)^2$. Then $\Gamma_P$ is generated by the hyperbolic element*

$$P_1 = \begin{pmatrix} \frac{1}{2}\left(v_1 - b\frac{Nu_1}{(a,N)}\right) & -c\frac{Nu_1}{(a,N)} \\ a\frac{Nu_1}{(a,N)} & \frac{1}{2}\left(v_1 + b\frac{Nu_1}{(a,N)}\right) \end{pmatrix}$$

*of $\Gamma_0(N)$. Let*

$$P_0 = \begin{pmatrix} \frac{v_0 - bu_0}{2} & -cu_0 \\ au_0 & \frac{v_0 + bu_0}{2} \end{pmatrix},$$

*where the pair $(v_0, u_0)$ is the fundamental solution of Pell's equation $v^2 - du^2 = 4$. If $P$ is $\Gamma_0(N)$-conjugate to a hyperbolic element $P' \in \Gamma_0(N)$, then $P_0$ is $\Gamma_0(N)$-conjugate to $P'_0$, where $P'_0$ is associated with $P'$ similarly as $P_0$ is associated with $P$.*

*Proof.* Let

$$P = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be a hyperbolic element of $\Gamma_0(N)$. By Lemma 3.3 of [1], fixed points $r_1$, $r_2$ of $P$ are not rational numbers, and they satisfy the equation $Cr^2 + (D - A)r - B = 0$. This implies that $\Gamma_P$ is the subgroup of elements in $\Gamma_0(N)$ having $r_1$, $r_2$ as fixed points. Let $a = C/\mu, b = (D - A)/\mu$ and $c = -B/\mu$, where $\mu = (C, D - A, -B)$. Then $[a, b, c]$ is a primitive quadratic form with $r_1$, $r_2$ being the roots of the equation $ar^2 + br + c = 0$. By Sarnak [13], the subgroup of elements in $SL_2(\mathbb{Z})$ having $r_1$, $r_2$ as fixed points consists of matrices of the form

$$\begin{pmatrix} \frac{v-bu}{2} & -cu \\ au & \frac{v+bu}{2} \end{pmatrix}$$

with $v^2 - du^2 = 4$ where $d = b^2 - 4ac$, and it is generated by the primitive hyperbolic element

$$P_0 = \begin{pmatrix} \frac{v_0-bu_0}{2} & -cu_0 \\ au_0 & \frac{v_0+bu_0}{2} \end{pmatrix}$$

where the pair $(v_0, u_0)$ is the fundamental solution of Pell's equation $v^2 - du^2 = 4$.

Since $P$ and $P_0$ have the same fixed points, we have $A = D - bC/a$ and $B = -cC/a$. Since $P$ belongs to $\Gamma_0(N)$ and $AD - BC = 1$, $C$ satisfies

$$\begin{cases} aD^2 - bDC + cC^2 = a \\ a|C, \ N|C. \end{cases} \tag{4.2}$$

Let $\lambda_P$ be an eigenvalue of $P$. Then it is a solution of the equation $\lambda^2 - (A + D)\lambda + 1 = 0$. By using $A = D - bC/a$ and $B = -cC/a$, we obtain that

$$\lambda_P - \frac{1}{\lambda_P} = \pm\frac{C}{a}\sqrt{d} \tag{4.3}$$

and

$$\lambda_P + \frac{1}{\lambda_P} = 2D - \frac{b}{a}C. \tag{4.4}$$

Conversely, let a pair $(C, D)$ be a solution of the equation (4.2). Define $A = D - bC/a$ and $B = -cC/a$. Then the matrix

$$P = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is a hyperbolic element of $\Gamma_0(N)$ having the same fixed points as $P_0$, and its eigenvalues satisfy (4.3) and (4.4).

Next, let $v = 2D - b\frac{C}{a}$ and $u = \frac{C}{a}$. Then the equation (4.2) becomes $v^2 - du^2 = 4$ with $N|au$. Since $N|au$, this equation can be written as

$$v^2 - \frac{dN^2}{(a, N)^2}u^2 = 4.$$

We also have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \frac{v-bNu(a,N)^{-1}}{2} & -cNu(a,N)^{-1} \\ aNu(a,N)^{-1} & \frac{v+bNu(a,N)^{-1}}{2} \end{pmatrix}. \tag{4.5}$$

Since $\Gamma_P$ is cyclic, a solution $v_1, u_1' > 0$ of Pell's equation $v^2 - du^2 = 4$ exists such that $\Gamma_P$ is generated by

$$P_1 = \begin{pmatrix} \frac{v_1-bu_1'}{2} & -cu_1' \\ au_1' & \frac{v_1+bu_1'}{2} \end{pmatrix}$$

and such that $P_1$ is the smallest positive integer power of $P_0$ among all powers of $P_0$ belonging to $\Gamma_0(N)$. Note that the eigenvalues of $P_1$ are

$$\frac{v_1 \pm \sqrt{d}u_1'}{2}.$$

Since $\Gamma_P$ is generated by $P_1$, $(v_1, u_1')$ is the minimal solution of the equation $v^2 - du^2 = 4$ with $N|au_1'$ in the sense that $(v_1 + \sqrt{d}u_1')/2$ is of the smallest value among all such solutions. Since $N|au_1'$, we have $\frac{N}{(a,N)}|u_1'$. Write

$$u_1' = \frac{Nu_1}{(a,N)}.$$

Then the pair $(v_1, u_1)$ with $v_1, u_1 > 0$ must be the fundamental solution of the Pell equation $v^2 - d_1u^2 = 4$, where $d_1 = dN^2(a,N)^{-2}$.

Assume that $P$ is $\Gamma_0(N)$-conjugate to $P'$. Let $P_0'$ be the primitive hyperbolic element of $SL_2(\mathbb{Z})$ corresponding to $[a', b', c']$ of discriminant $d'$, which is associated with $P'$ similarly as $P_0$ is associated with $P$. Note that eigenvalues of $P$ and $P'$, which are greater than one, are

$$\frac{v + \frac{Nu}{(a,N)}\sqrt{d}}{2}$$

and

$$\frac{v' + \frac{Nu'}{(a',N)}\sqrt{d'}}{2},$$

respectively. Since $P$ and $P'$ are conjugate, the two eigenvalues must be equal, and hence we must have $v = v'$. If $P' = \gamma^{-1}P\gamma$ for some element $\gamma \in \Gamma_0(N)$, then by using the expression (4.5) for $P$ and $P'$ we find that

$$\frac{Nu}{(a,N)}\gamma^t \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \gamma = \frac{Nu'}{(a',N)} \begin{pmatrix} a' & b'/2 \\ b'/2 & c' \end{pmatrix}.$$

Since $[a, b, c]$ and $[a', b', c']$ are primitive, we have

$$\frac{Nu}{(a, N)} = \frac{Nu'}{(a', N)},$$

and hence we must have $d = d'$. Thus, $[a, b, c]$ and $[a', b', c']$ are two $\Gamma_0(N)$-conjugate primitive indefinite quadratic forms. Since two forms $[a, b, c]$ and $[a', b', c']$ of the same discriminant are equivalent in $\Gamma_0(N)$ if and only if an element $\gamma \in \Gamma_0(N)$ exists such that $\gamma^{-1} P_0 \gamma = P_0'$, $P_0$ is $\Gamma_0(N)$-conjugate to $P_0'$.

This completes the proof of the lemma.

**Corollary.** *If $[a, b, c]$ and $[a', b', c']$ are two $\Gamma_0(N)$-conjugate primitive indefinite quadratic forms, then we have $(a, N) = (a', N)$.*

*Proof.* Assume that an element $\gamma \in \Gamma_0(N)$ exists such that

$$\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \gamma = \begin{pmatrix} a' & b'/2 \\ b'/2 & c' \end{pmatrix}.$$

Then we have

$$a' = a\alpha^2 + b\alpha\gamma + c\gamma^2.$$

Since $N|\gamma$, this identity implies that $(a, N)|(a', N)$. Similarly, we can obtain that $(a', N)|(a, N)$, and therefore we have $(a, N) = (a', N)$. $\square$

We recall that two quadratic forms $[a, b, c]$ and $[a', b', c']$ are equivalent in $\Gamma_0(N)$ if an element $\gamma \in \Gamma_0(N)$ exists such that

$$\begin{pmatrix} a' & b'/2 \\ b'/2 & c' \end{pmatrix} = \gamma^t \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \gamma.$$

This relation partitions the quadratic forms into equivalence classes, and two such forms from the same class have the same discriminant. The number of such classes of a given discriminant $d$ is finite, and is denoted by $H_d$.

**Lemma 4.2.** *Let $[a_j, b_j, c_j]$, $j = 1, 2, \cdots, H_d$, be a set of representatives for classes of primitive indefinite quadratic forms of discriminant $d$, which are not equivalent under $\Gamma_0(N)$. Then we have*

$$\sum_{\{P\}:\ \text{hyperbolic } P \in \Gamma_0(N)} c(P)$$
$$= 4\sqrt{\pi} \frac{\Gamma(s - \frac{1}{2})}{N\Gamma(s)} \sum_{d \in \Omega} \sum_{j=1}^{H_d} \sum_{u} \frac{(a_j, N) \ln \epsilon_{d_1}}{u\sqrt{d}} \left(1 + \frac{d(Nu)^2}{4(a_j, N)^2}\right)^{\frac{1}{2} - s}$$

*for* $\operatorname{Re} s > 1$*, where* $d_1 = dN^2(a_j, N)^{-2}$ *and where the summation on* $u$ *is taken over all the positive integers* $u$ *such that* $4 + dN^2(a_j, N)^{-2}u^2$ *is the square of an integer.*

*Proof.* It follows from (2.1) and Theorem 3.2 that

$$\sum_{\{P\}} c(P) = 2\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \sum_{\{P\}} \frac{\ln NP_1}{\lambda_P - 1/\lambda_P}\left(1 + \frac{(\lambda_P - 1/\lambda_P)^2}{4}\right)^{\frac{1}{2}-s}$$

for $\operatorname{Re} s > 1$, where $\lambda_P > 1$ is an eigenvalue of $P$ and $P_1$ is given in Lemma 4.1. Let $P$ be associated with a primitive indefinite quadratic form $[a_j, b_j, c_j]$ as in Lemma 4.1. Then by Lemma 4.1 we have

$$\lambda_P - \frac{1}{\lambda_P} = \frac{Nu}{(a_j, N)}\sqrt{d}.$$

By Lemma 4.1, we also have

$$\sqrt{NP_1} = \frac{v_1 + \sqrt{d_1}u_1}{2} = \epsilon_{d_1}.$$

If $P'$ is a hyperbolic of $\Gamma_0(N)$, and is associated with a primitive indefinite quadratic form $[a_{j'}, b_{j'}, c_{j'}]$ as in Lemma 4.1, then $P$ and $P'$ are $\Gamma_0(N)$-conjugate only if $[a_j, b_j, c_j]$ and $[a_{j'}, b_{j'}, c_{j'}]$ are $\Gamma_0(N)$-conjugate by Lemma 4.1. Next, let $T$ be a hyperbolic element of $\Gamma_0(N)$. Assume that $T$ is associated with a primitive indefinite quadratic form $[a, b, c]$ as in Lemma 4.1. If the discriminant of $[a, b, c]$ is not equal to the discriminant of $[a_j, b_j, c_j]$ which is associated with $P$, then $P$ and $T$ are not $\Gamma_0(N)$-conjugate by the argument in the last paragraph of the proof of Lemma 4.1. The stated identity then follows. □

**Lemma 4.3.** *Let* $k$ *be a divisor of* $N$*. Then the number of indefinite primitive quadratic forms* $[a, b, c]$ *with* $(a, N) = k$ *of discriminant* $d$*, which are not equivalent under* $\Gamma_0(N)$*, is equal to*

$$h_{d_1} \prod_{p^{2l} | (d,k)} p^l \cdot \prod_{p|k} \left(1 + \left(\frac{d}{p}\right)\right)$$

*where* $d_1 = dN^2/k^2$*, where the first product runs over all distinct primes* $p$ *with* $p^{2l}$ *being the greatest even* $p$*-power factor of* $(d, k)$ *and the second product is taken over all distinct primes* $p$ *dividing* $k$*.*

*Proof.* Let $[a,b,c]$ and $[a',b',c']$ be two indefinite primitive quadratic forms of discriminant $d$ with $(a,N)=k=(a',N)$. If they are equivalent under $\Gamma_0(N)$, then an element $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)\in\Gamma_0(N)$ exists such that

$$\frac{b'}{2}=\alpha(a\beta+\frac{b}{2}\delta)+\gamma(\frac{b}{2}\beta+c\delta).$$

This implies that $b'\equiv b \pmod{2k}$. In particular, if $b'\not\equiv b \pmod{2k}$, then $[a,b,c]$ and $[a',b',c']$ are not $\Gamma_0(N)$-equivalent.

Assume that $\varrho$ is an integer with $1\leqslant\varrho\leqslant 2k$. Denote by $\Lambda_{k,d,\varrho}$ the set of representatives of indefinite primitive quadratic forms $[a,b,c]$ of discriminant $d$ with $(a,N)=k$ and $b\equiv\varrho \pmod{2k}$, which are not equivalent under $\Gamma_0(N)$. Let $\mathcal{L}^0_{N,d_1,\varrho N/k}$ be the set of representatives of indefinite quadratic forms $[aN,b,c]$ of discriminant $d_1$ with $(a,b,c)=1,(N,b,c)=N/k$ and $b\equiv\varrho N/k \pmod{2N}$, which are not equivalent under $\Gamma_0(N)$.

A map from $\Lambda_{k,d,\varrho}$ to $\mathcal{L}^0_{N,d_1,\varrho N/k}$ is defined by

$$T:[a,b,c]\to[aN/k,bN/k,cN/k].$$

We claim that $T$ is bijective. By the definition of $\Lambda_{k,d,\varrho}$, we see that $T$ is injective. Conversely, if $[a_1N,b_1,c_1]$ is an element of $\mathcal{L}^0_{N,d_1,\varrho N/k}$, then we have $(a_1,b_1,c_1)=1,(N,b_1,c_1)=N/k, b_1\equiv\varrho N/k \pmod{2N}$ and $b_1^2-4Na_1c_1=d_1$. Since $(N,b_1,c_1)=N/k$, we have $N/k|c_1$. Let $a=a_1k, b=b_1k/N$ and $c=c_1k/N$. Then we have $b\equiv\varrho \pmod{2k}$ and $d=b^2-4ac$. Since $(a_1,N/k)=1$, we have $(a,N)=k$. We claim that $(a,b,c)=1$, that is, $(a_1N,b_1,c_1)=N/k$. Since $(a_1,b_1,c_1)=1$ and $(N,b_1,c_1)=N/k$, we have $(a_1N,b_1,c_1)=N/k$. Therefore, we have $(a,b,c)=1$. Thus, $[a,b,c]$ is an element of $\Lambda_{k,d,\varrho}$, and $T$ maps it into the element $[a_1N,b_1,c_1]$ of $\mathcal{L}^0_{N,d_1,\varrho N/k}$. Therefore, $T$ is surjective. Thus, we have proved that $T$ is a bijection if the set $\Lambda_{k,d,\varrho}$ is not empty. By Proposition, p. 505 of Gross, Kohnen and Zagier [4], the number of elements contained in $\mathcal{L}^0_{N,d_1,\varrho N/k}$ is $h_{d_1}$, and hence the set $\Lambda_{k,d,\varrho}$ contains $h_{d_1}$ elements if it is not empty.

If $\varrho_1\not\equiv\varrho_2 \pmod{2k}$, then the set of $\Gamma_0(N)$-equivalence classes represented by elements in $\Lambda_{k,d,\varrho_1}$ is disjoint from the set of $\Gamma_0(N)$-equivalence classes represented by elements in $\Lambda_{k,d,\varrho_2}$. Now, we want to count the number of non-empty sets $\Lambda_{k,d,\varrho}$. That is, we want to count the number of solutions $\varrho$ of the equation

$$\varrho^2\equiv d \pmod{4k},\ 1\leqslant\varrho\leqslant 2k. \tag{4.6}$$

If $(d,k)=1$, by Theorem 3.4 of Chapter 12, Hua [7] the number of solution of the equation (4.6) is equal to

$$\prod_{p|k}\left(1+\left(\frac{d}{p}\right)\right).$$

If there exists a prime number $q$ satisfying $q|k, q^2 \nmid k$ and $q^2|d$, then we have $q|\varrho$, and the equation (4.6) can be written as

$$\left(\frac{\varrho}{q}\right)^2 \equiv \frac{d}{q^2} \ (\mathrm{mod}\ \frac{4k}{q}),\ 1 \leqslant \frac{\varrho}{q} \leqslant \frac{2k}{q}. \tag{4.7}$$

Dividing out $\varrho, d$ and $k$ all such prime numbers $q$ as in (4.7), we can reduce the second case to the first case when $(d,k)=1$. Then, by using properties of the Legendre symbol we obtain that the number of solution of the equation (4.6) in the second case is still equal to

$$\prod_{p|k}\left(1+\left(\frac{d}{p}\right)\right).$$

Next, we consider the case when there exists a prime number $q$ satisfying $q^2|k$ and $q^2|d$. Then we have $q|\varrho$, and the equation (4.6) can be written as

$$\left(\frac{\varrho}{q}\right)^2 \equiv \frac{d}{q^2} \ (\mathrm{mod}\ \frac{4k}{q^2}),\ 1 \leqslant \frac{\varrho}{q} \leqslant \frac{2k}{q}.$$

Dividing out $\varrho, d$ and $k$ all such prime numbers $q$ as in (4.7), we can reduce this case to the first case when $(d,k)=1$. Then, by using properties of the Legendre symbol we obtain that the number of solution of the equation (4.6) in this case is still equal to

$$\prod_{p^{2l}|(d,k)} p^l \cdot \prod_{p|k}\left(1+\left(\frac{d}{p}\right)\right)$$

where $p^{2l}$ is the greatest even power of $p$ dividing $(d,k)$.

Finally, we consider the case when there exists a prime number $q$ satisfying $q|k, q|d$ and $q^2 \nmid d$. We have again $q|\varrho$. The equation (4.6) can be written as

$$q\left(\frac{\varrho}{q}\right)^2 \equiv \frac{d}{q} \ (\mathrm{mod}\ \frac{4k}{q}),\ 1 \leqslant \frac{\varrho}{q} \leqslant \frac{2k}{q}. \tag{4.8}$$

We can assume that $q \neq 2$. Otherwise, if $q = 2$ then we must have $q^2|d$. If (4.8) is solvable, then we have $(q, 4k/q) = 1$. Otherwise, we must have $q^2|d$. Hence a number $x_q$ exists such that $qx_q \equiv 1 \pmod{4k/q}$. Then the equation (4.8) can be written as

$$\left(\frac{\varrho}{q}\right)^2 \equiv x_q \frac{d}{q} \pmod{\frac{4k}{q}},\ 1 \leqslant \frac{\varrho}{q} \leqslant \frac{2k}{q}. \tag{4.9}$$

Note that we have

$$\left(\frac{x_q d/q}{p}\right) = \left(\frac{q}{p}\right)\left(\frac{d/q}{p}\right) = \left(\frac{d}{p}\right) \tag{4.10}$$

for any prime number $p|(k/q)$. Dividing out all such primes $q$ as in (4.9) and using (4.10), we obtain that the number of solution of the equation (4.6) in the final case is equal to

$$\prod_{p|k}\left(1 + \left(\frac{d}{p}\right)\right).$$

This completes the proof of the lemma. $\square$

**Theorem 4.4.** *We have*

$$\sum_{\{P\}\text{: hyperbolic } P\in\Gamma_0(N)} c(P) = 4\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \sum_{\substack{m|N \\ m \text{ square-free}}} \sum_{k|N} \frac{\mu((m,k))}{(m,k)}$$
$$\times \sum_{d\in\Omega}\sum_{u}\left(\frac{d}{m}\right)\prod_{p^{2l}|(d,N/k)} p^l \cdot \frac{h_d \ln \epsilon_d}{u\sqrt{d}}\left(1 + \frac{dk^2u^2}{4}\right)^{\frac{1}{2}-s}$$

*for* $\mathrm{Re}\, s > 1$, *where the summation on* $u$ *is taken over all the positive integers* $u$ *such that* $\sqrt{4 + dk^2u^2} \in \mathbb{Z}$.

*Proof.* Let $[a_j, b_j, c_j]$, $j = 1, 2, \cdots, H_d$, be a set of representatives for classes of primitive indefinite quadratic forms of discriminant $d$, which are not equivalent under $\Gamma_0(N)$. By Lemma 4.2, we have

$$\sum_{\{P\}} c(P) = 4\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\sum_{d\in\Omega}\sum_{j=1}^{H_d}\sum_{u}\frac{\ln \epsilon_{d_1}}{u\sqrt{d_1}}\left(1 + \frac{d_1u^2}{4}\right)^{\frac{1}{2}-s}$$

for $\operatorname{Re} s > 1$, where $d_1 = dN^2(a_j, N)^{-2}$ and the summation on $u$ is taken over all the positive integers $u$ such that $\sqrt{4 + d_1 u^2} \in \mathbb{Z}$. Let $k = N/(a_j, N)$. By using Lemma 4.3, we can write the above identity as

(4.11)

$$\sum_{\{P\}} c(P) = 4\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \sum_{k|N} \sum_{d\in\Omega} \sum_{u} \prod_{p^{2l}|(d,N/k)} p^l \cdot \prod_{p|\frac{N}{k}} \left(1 + \left(\frac{d}{p}\right)\right) \times \frac{h_{d_1} \ln \epsilon_{d_1}}{u\sqrt{d_1}} \left(1 + \frac{d_1 u^2}{4}\right)^{\frac{1}{2}-s}$$

for $\operatorname{Re} s > 1$, where $d_1 = dk^2$. By using Dirichlet's class number formula

$$h_{d_1} \ln \epsilon_{d_1} = \sqrt{d_1} L(1, \chi_{d_1})$$

and by using the identity (See Theorem 11.2 of Chapter 12, Hua [7])

$$L(1, \chi_{d_1}) = L(1, \chi_d) \prod_{p|k} \left(1 - \left(\frac{d}{p}\right) p^{-1}\right),$$

we can write (4.11) as

$$\begin{aligned}\sum_{\{P\}} c(P) =& 4\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \sum_{k|N} \sum_{d\in\Omega} \sum_{u} \prod_{p^{2l}|(d,N/k)} p^l \cdot \prod_{p|\frac{N}{k}} \left(1 + \left(\frac{d}{p}\right)\right) \\ &\times \prod_{p|k} \left(1 - \left(\frac{d}{p}\right) p^{-1}\right) \frac{h_d \ln \epsilon_d}{u\sqrt{d}} \left(1 + \frac{dk^2 u^2}{4}\right)^{\frac{1}{2}-s}\end{aligned}$$

for $\operatorname{Re} s > 1$, where the summation on $u$ is taken over all the positive integers $u$ such that $\sqrt{4 + dk^2 u^2} \in \mathbb{Z}$. Since

$$\prod_{p|\frac{N}{k}} \left(1 + \left(\frac{d}{p}\right)\right) \prod_{p|k} \left(1 - \left(\frac{d}{p}\right) p^{-1}\right) = \sum_{\substack{m|N \\ m \text{ square-free}}} \left(\frac{d}{m}\right) \frac{\mu((m,k))}{(m,k)}$$

where $\mu$ is the Möbius function, we have

(4.12)

$$\begin{aligned}\sum_{\{P\}} c(P) =& 4\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \sum_{\substack{m|N \\ m \text{ square-free}}} \sum_{k|N} \frac{\mu((m,k))}{(m,k)} \\ &\times \sum_{d\in\Omega} \sum_{u} \left(\frac{d}{m}\right) \prod_{p^{2l}|(d,N/k)} p^l \cdot \frac{h_d \ln \epsilon_d}{u\sqrt{d}} \left(1 + \frac{dk^2 u^2}{4}\right)^{\frac{1}{2}-s}.\end{aligned}$$

Next, we show that

$$\sum_{d\in\Omega}\sum_{u}\left(\frac{d}{m}\right)\prod_{p^{2l}|(d,N/k)}p^l\cdot\frac{h_d\ln\epsilon_d}{u\sqrt{d}}\left(1+\frac{dk^2u^2}{4}\right)^{\frac{1}{2}-s}$$

is absolutely convergent for $\sigma=\operatorname{Re}s>1$. Since

$$\begin{aligned}&|\sum_{d\in\Omega}\sum_{u}\left(\frac{d}{m}\right)\prod_{p^{2l}|(d,N/k)}p^l\cdot\frac{h_d\ln\epsilon_d}{u\sqrt{d}}\left(1+\frac{dk^2u^2}{4}\right)^{\frac{1}{2}-s}|\\ &\ll\sum_{d\in\Omega}\sum_{u}\frac{h_d\ln\epsilon_d}{u\sqrt{d}}\left(1+\frac{dk^2u^2}{4}\right)^{\frac{1}{2}-\sigma}.\end{aligned}\tag{4.13}$$

It is proved in Li [11] that the right side of (4.13) is convergent for $\sigma>1$, and hence, the right side of the stated identity is absolutely convergent for $\operatorname{Re}s>1$.

This completes the proof of the theorem. $\square$

*Proof of Theorem 1.* By the argument given at the end of Li [11], we see that

$$\begin{aligned}&\sum_{d\in\Omega,u}\left|\frac{h_d\ln\epsilon_d}{\sqrt{d}u}(1+\frac{dk^2u^2}{4})^{\frac{1}{2}-s}-k^{1-2s}\frac{h_d\ln\epsilon_d}{(du^2)^s}\right|\\ &\ll\sum_{d\in\Omega,u}\frac{(du^2)^{\frac{1}{2}+\epsilon-1-\sigma}}{u^{1+2\epsilon}}<\infty\end{aligned}$$

for $\sigma=\operatorname{Re}s>0$. Then it follows from Theorem 4.4 and Theorem 3.4 that the function $L_N(s)$ represents an analytic function in the half-plane $\operatorname{Re}s>0$ except for having a possible pole at $s=1/2$ and for having simple poles at $s=1,\frac{1}{2}\pm i\kappa_j$, $j=1,2,\cdots$.

This completes the proof of the theorem.

## 5. Proof of Corollary 2

**Lemma 5.1.** *Let*

$$\begin{aligned}L(s)=&\sum_{d\in\Omega}\sum_{\substack{u>0\\ v^2-du^2=4}}\left(1+\left(\frac{d}{p}\right)\right)\frac{h_d\ln\epsilon_d}{(du^2)^s}\\ &+\sum_{d\in\Omega}\sum_{\substack{u>0\\ v^2-dp^2u^2=4}}\frac{h_{dp^2}\ln\epsilon_{dp^2}}{(dp^2u^2)^s}-\sum_{d\in\Omega}\sum_{\substack{u>0\\ v^2-du^2=4}}\frac{h_d\ln\epsilon_d}{(du^2)^s}\end{aligned}$$

*for* $\mathrm{Re}\, s > 1$. *Then* $L(s)$ *is analytic for* $\mathrm{Re}\, s > 1$ *and has an analytic continuation to the half-plane* $\mathrm{Re}\, s > 1/2$ *except for having simple poles at* $s = \frac{1}{2} + \sqrt{1/4 - \lambda_j}$, $j = 1, 2, \cdots, S$.

*Proof.* It is proved in Li [11] that the series

$$D(s) = \sum_{d \in \Omega} \frac{h_d \ln \epsilon_d}{d^s} \sum_{\substack{u>0 \\ v^2 - du^2 = 4}} \frac{1}{u^{2s}},$$

represents an analytic function of $s$ in the half-plane $\mathrm{Re}\, s > 1/2$ except for having a simple poles at $s = 1$. Let

$$h(r) = 2\sqrt{\pi}\Gamma(s - \frac{1}{2})\Gamma^{-1}(s) 4^{s - \frac{1}{2}} \int_0^\infty (u + \frac{1}{u} + 2)^{\frac{1}{2} - s} u^{ir-1} du$$

for $\mathrm{Re}\, s > \frac{1}{2}$. By (2.3), (3.4), (3.5), Lemma 3.5, Lemma 4.1, and Lemma 4.2 of Li [11], we have that

$$D(s) - h(-i/2)$$

is analytic in the half-plane $\mathrm{Re}\, s > 1/2$. Let $L_N(s)$ be given as in Theorem 1 with $N = p$. Then, by (2.3), (3.13), (3.14), Theorem 3.12, Lemma 4.1, and Theorem 4.6 of Conrey and Li [1], we have that

$$L_N(s) - h(-i/2)$$

is an analytic function of $s$ in the half-plane $\mathrm{Re}\, s > 1/2$ except for simple poles at $s = \frac{1}{2} + \sqrt{1/4 - \lambda_j}$, $j = 1, 2, \cdots, S$. It follows that

$$L(s) = L_N(s) - D(s)$$

is an analytic function of $s$ in the half-plane $\mathrm{Re}\, s > 1/2$ except for simple poles at $s = \frac{1}{2} + \sqrt{1/4 - \lambda_j}$, $j = 1, 2, \cdots, S$.

This completes the proof of the lemma. $\square$

By using the formula (See Theorem 11.2 of Chapter 12, Hua [7])

$$h_{dp^2} \ln \epsilon_{dp^2} = \left(p - \left(\frac{d}{p}\right)\right) h_d \ln \epsilon_d,$$

we find that

$$\begin{aligned} L(s) &= \sum_{d\in\Omega} \sum_{\substack{u>0 \\ v^2-du^2=4}} \left(\frac{d}{p}\right) \frac{h_d \ln \epsilon_d}{(du^2)^s} + \sum_{d\in\Omega} \sum_{\substack{u>0 \\ v^2-dp^2u^2=4}} \frac{h_{dp^2} \ln \epsilon_{dp^2}}{(dp^2u^2)^s} \\ &= \sum_{d\in\Omega} \sum_{\substack{u>0,\, p\nmid u \\ v^2-du^2=4}} \left(\frac{d}{p}\right) \frac{h_d \ln \epsilon_d}{(du^2)^s} + p^{1-2s} \sum_{d\in\Omega} \sum_{\substack{u>0 \\ v^2-dp^2u^2=4}} \frac{h_d \ln \epsilon_d}{(du^2)^s}. \end{aligned}$$

Therefore, we can write

(5.1)

$$L(s) = \sum_{u=1, p\nmid u}^{\infty} \frac{1}{u^{2s}} \sum_{\substack{d\in\Omega \\ v^2-du^2=4}} \left(\frac{d}{p}\right) \frac{h_d \ln \epsilon_d}{d^s} + p^{1-2s} \sum_{u=1}^{\infty} \frac{1}{u^{2s}} \sum_{\substack{d\in\Omega \\ v^2-dp^2u^2=4}} \frac{h_d \ln \epsilon_d}{d^s}$$

for $\mathrm{Re}\, s > 1$.

**Lemma 5.2.** *Let $u$ be the product of a power of 2 and a power of an odd prime. Then all integers $d \in \Omega$, which satisfy $v^2 - du^2 = 4$ for some integers $v$, are given by*

$$d = \begin{cases} n^2u^2 \pm 4n, & \text{if } 2 \nmid u; \\ n^2u^2 \pm 4n, & \text{if } 2|u \text{ and } 4 \nmid u; \\ n^2u^2 \pm 4n \quad \text{or} \quad (n+\frac{1}{4})^2u^2 - (4n+1), & \\ \text{or} \quad (n+\frac{3}{4})^2u^2 + (4n+3), & \text{if } 4|u \text{ and } 8 \nmid u; \\ n^2u^2 \pm 4n \text{ or } (n+\frac{1}{4})^2u^2 + (4n+1), & \text{if } 8|u \end{cases}$$

*for $n = 1, 2, 3, \cdots$.*

*Proof.* The idea is to solve Pell's equation $(v-2)(v+2) = du^2$ for integers $v$.

If $2 \nmid u$, then we must have $u^2|v-2$ or $u^2|v+2$. Otherwise, if $u$ contains a prime factor $q$ dividing both $v-2$ and $v+2$, then $q|4$, and hence $q = 2$. This is a contradiction. Therefore, we have $v = \pm 2 + nu^2$ for $n = 1, 2, 3, \cdots$. From Pell's equation $v^2 - 4 = du^2$, we deduce that $d = n^2u^2 \pm 4n$ for $n = 1, 2, 3, \cdots$, which belong to the set $\Omega$.

If $2|u$ and $4 \nmid u$, then we have $v = \pm 2 + \frac{1}{2}mu^2$ for $m = 1, 2, \cdots$. Since we want $d$ to be in the set $\Omega$, $m$ has to be an even integer. Therefore, we have $d = n^2u^2 \pm 4n$ for $n = 1, 2, 3, \cdots$.

If $4|u$, then we have $v = \pm 2 + \frac{1}{4}mu^2$ for $m = 1, 2, \cdots$. Note that $d = \frac{1}{16}m^2u^2 \pm m$ in this case. *First*, we consider the case when $8 \nmid u$. Then

$d \equiv m^2 \pm m \pmod 4$. If $m$ is even, then $d \equiv \pm m \pmod 4$, and hence $m \equiv 0 \pmod 4$. Thus, we have $d = n^2u^2 \pm 4n$ for $n = 1, 2, 3, \cdots$. If $m$ is odd, then $d \equiv 1 \pm m \pmod 4$. When $m = 4n + 1$ for $n = 1, 2, 3, \cdots$, we have $d = (n + \frac{1}{4})^2u^2 - (4n + 1)$. When $m = 4n + 3$ for $n = 1, 2, 3, \cdots$, we have $d = (n + \frac{3}{4})^2u^2 + (4n + 3)$. *Next*, we consider the case when $8|u$. Then $d \equiv \pm m \pmod 4$. This implies that $m = 4n$ or $m = 4n + 1$ with $d = \frac{1}{16}m^2u^2 + m$, that is, $d = n^2u^2 \pm 4n$ or $d = (n + \frac{1}{4})^2u^2 + (4n + 1)$ $n = 1, 2, 3, \cdots$.

This completes the proof of the lemma. $\square$

By using the Pólya-Vinogradov inequality [2], we obtain that

$$\left| \sum_{k=d+1}^{\infty} \left(\frac{d}{k}\right) k^{-1} \right| \leqslant c\frac{\ln d}{\sqrt{d}}$$

where $c$ is an absolute constant (cf. Siegel [19]). ¿From Dirichlet's class number formula

$$h_d \ln \epsilon_d = \sqrt{d} \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{d}{k}\right),$$

we deduce that

$$h_d \ln \epsilon_d = \sqrt{d} \sum_{k=1}^{d} \frac{1}{k} \left(\frac{d}{k}\right) + O(\ln d) \tag{5.2}$$

where the constant implied by the symbol $O$ is an absolute constant.

*Proof of Corollary 2.* By an argument similar to that made in the proof of Lemma 5.2, we find that all integers $d \in \Omega$ satisfying $v^2 - du^2 = 4$ are among the integers $d$'s given by

(5.3)

$$du_1^2 = \begin{cases} n^2u_2^2 \pm 4n, & \text{if } 2 \nmid u_2; \\ n^2u_2^2 \pm 4n, & \text{if } 2|u_2 \text{ and } 4 \nmid u_2; \\ n^2u_2^2 \pm 4n \quad \text{or} \quad (n+\frac{1}{4})^2u_2^2 - (4n+1), & \\ \text{or} \quad (n+\frac{3}{4})^2u_2^2 + (4n+3), & \text{if } 4|u_2 \text{ and } 8 \nmid u_2; \\ n^2u_2^2 \pm 4n \text{ or } (n+\frac{1}{4})^2u_2^2 + (4n+1), & \text{if } 8|u_2 \end{cases}$$

for $n = 1, 2, 3, \cdots$ and for all decompositions $u = u_1u_2$ with $u_2 \geqslant u_1$. Similarly, we can find all integers $d \in \Omega$ satisfying $v^2 - dp^2u^2 = 4$. Let

$\sigma = \operatorname{Re} s$, and let $d(u)$ denote the number of positive divisors of $u$. Then by (5.1)-(5.3) we have

$$L(s) - L_p(s) \ll \sum_{u=1}^{\infty} \frac{d(u)}{u^{2\sigma}} \sum_{n=1}^{\infty} \frac{\ln n}{n^{2\sigma}} < \infty$$

for $\sigma > 1/2$. Note that $\left(\frac{d}{p}\right) = \left(\frac{v^2-4}{p}\right)$ if $v^2 - du^2 = 4$ and $p \nmid u$. Therefore, the stated result follows from Lemma 5.1.

This completes the proof of the corollary. $\square$

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