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FOURIER TRANSFORMS HAVING ONLY REAL ZEROS

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ABSTRACT. Let $G(z)$ be a real entire function of order less than 2 with only real zeros. Then we classify certain distribution functions F such that the Fourier transform $H(z) = \int_{-\infty}^{\infty} G(it)e^{izt}\, dF(t)$ has only real zeros.

1. INTRODUCTION

Pólya [13] suggested that determining the class of functions whose Fourier transforms have only real zeros would be a 'rather artificial question' if it were not for the Riemann Hypothesis. For $\Re(s) > 1$, the Riemann zeta function is defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. It has an analytic continuation, and the function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

is entire. The Riemann Hypothesis states that all the zeros of $\xi(s)$ satisfy $\Re(s) = 1/2$. A proof of the Riemann Hypothesis would be a major advance for analytic number theory. Let $\Xi(z) = \xi\left(\frac{1}{2} + iz\right)$. It is well known (see Titchmarsh [18], chapter 10) that

$$\boldsymbol{\Xi}(z) = \int_{-\infty}^{\infty} \boldsymbol{\Phi}(x)e^{izx}dx$$

where

$$\boldsymbol{\Phi}(x) = \sum_{n=1}^{\infty}\left(4n^4\pi^2e^{9x/2} - 6n^2\pi e^{5x/2}\right)\exp\left(-n^2\pi e^{2x}\right).$$

In other words, the Riemann Hypothesis is true if and only if the Fourier transform $\Xi(z)$ has only real zeros.

Pólya wrote several papers (such as [11, 12, 13, 14], which can all be found in [15]) in which he studied the reality of the zeros of various Fourier transforms. A particularly interesting result is the following:

Proposition 1 (Pólya [14])**.** *Let f be an integrable function of a real variable t such that $f(t) = \overline{f(-t)}$ and $f(t) = O(e^{-|t|^b})$ for $t \to \pm\infty$ and $b > 2$. Assume that*

$$\int_{-\infty}^{\infty} f(t)e^{izt}dt$$

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has only real zeros. Let ϕ *be a real entire function having only real zeros, and assume that* ϕ *has a Weierstrass product of the form*

$$\phi(z) = cz^m e^{\alpha z - \beta z^2} \prod_k \left(1 - \frac{z}{\alpha_k}\right) e^{z/\alpha_k}$$

where c, α, α_k *are real,* $\beta \geq 0$, *and* m *is a nonnegative integer. (In other words,* ϕ *belongs to the Laguerre-Pólya class.) Then*

$$\int_{-\infty}^{\infty} \phi(it) f(t) e^{izt}\, dt$$

has only real zeros.

In this paper we are concerned with constructing Fourier transforms with only real zeros in which the measure is *not* assumed to be the ordinary Lebesgue measure dt. The main result is the following theorem:

Theorem 1. *Suppose* G *is an entire function of order* < 2 *that is real on the real axis and has only real zeros. Let* $\{a_k\}$ *be a nonincreasing sequence of positive real numbers, let* $\{X_k\}$ *be the sequence of independent random variables such that* X_k *takes values* ± 1 *with equal probability, and let* F_n *be the distribution function of the normalized sum* $Y_n = (a_1 X_1 + \cdots + a_n X_n)/s_n$ *where* $s_n^2 = a_1^2 + \cdots + a_n^2$. *The functions* F_n *converge pointwise to a continuous distribution* $F = \lim_{n\to\infty} F_n$. *Let* H *be the Fourier transform of* $G(it)$ *with respect to the measure* dF. *That is,*

$$H(z) = \int_{-\infty}^{\infty} G(it) e^{izt}\, dF(t).$$

Then H *is an entire function of order* ≤ 2 *that is real on the real axis. If* H *is not identically zero, then* H *has only real zeros.*

Theorem 1 includes cases not covered in Proposition 1 because the distribution function $F(t)$, although continuous, need not be differentiable. If $F(t)$ is differentiable, we may write

$$\int_{-\infty}^{\infty} G(it) e^{izt}\, dF(t) = \int_{-\infty}^{\infty} G(it) e^{izt} F'(t)\, dt.$$

However, since not all functions $f(t)$ in Proposition 1 are of the form $f(t) = F'(t)$ for the types of distributions in Theorem 1, Proposition 1 covers cases not included in Theorem 1. So, while there is some overlap between Proposition 1 and Theorem 1, neither implies the other.

The proof of Theorem 1 is given in §2. Before proceeding with the proof we mention that the proof relies on a result about sums of exponential functions. Let $h_n(z)$ be the function of a complex variable z defined by

$$h_n(z) = \sum G(\pm i a_1 \pm \cdots \pm i a_n) e^{iz(\pm b_1 \pm \cdots \pm b_n)}$$

where the summation is over all 2^n possible sign combinations, the same sign combination being used in both the argument of G and in the exponent. The numbers $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are positive, and G is as in Theorem 1. The author shows in [3] that all the zeros of the exponential sum $h_n(z)$ are real. Interestingly, the proof of this fact is similar to the proof of the Lee-Yang Circle Theorem from statistical mechanics (cf. [8] Appendix II or [17] Chapter 5). It should be pointed out that this result of the author is related to de Branges' Hilbert spaces of entire functions [5]. Let $a_k = b_k$, $s_n^2 = a_1^2 + \cdots + a_n^2$, and

$$H_n(z) = 2^{-n} \sum G\big((\pm ia_1 \pm \cdots \pm ia_n)/s_n\big) e^{iz(\pm a_1 \pm \cdots \pm a_n)/s_n}.$$

All of the zeros of $H_n(z)$ are real. Theorem 1 is established by showing that the limit

$$H(z) = \lim_{n\to\infty} H_n(z)$$

is uniform on compact sets.

2. Proof of Theorem 1

The proof of Theorem 1 is presented in this section as a sequence of lemmas. We begin with some notation.

The Laguerre-Pólya class $\mathcal{LP}$ of functions consists of the entire functions having only real zeros with a Weierstrass factorization of the form

$$az^q e^{\alpha z - \beta z^2} \prod (1 - z/\alpha_n) e^{z/\alpha_n}$$

where a, α, β are real, $\beta \geq 0$, q is a nonnegative integer, and the α_n are nonzero real numbers such that $\sum \alpha_n^{-2} < \infty$. We shall be most interested in the subset $\mathcal{LP}^*$ of the Laguerre-Pólya class consisting of all elements of $\mathcal{LP}$ of order < 2. Thus, β is necessarily 0 for functions in $\mathcal{LP}^*$.

The distribution function T for a random variable Y is $T(x) = \Pr(Y \leq x)$. We will consider the following types of random variables and their distribution functions: Let $\{a_k\}$ be a nonincreasing sequence of positive real numbers. Let $\{X_k\}$ be a sequence of independent random variables such that X_k takes values ± 1 with equal probability. Let Y_n be the sum

$$Y_n = \frac{a_1 X_1 + \cdots + a_n X_n}{s_n}$$

where $s_n^2 = a_1^2 + \cdots + a_n^2$. F_n will denote the distribution function of Y_n, and F will denote the limit $F = \lim_{n\to\infty} F_n$. In Theorem 1 the distribution F has variance 1. However, F could be rescaled to have any other positive value for its variance. The following lemma describes this F.

Lemma 1. *The sequence F_n converges pointwise to a continuous distribution F. If the sequence s_n is unbounded, F is the normal distribution. If the sequence s_n is bounded, F is not the normal distribution.*

Proof. This is proved in Lemma 1 of [2]. □

Lemma 2 (Pólya [12], Hilfssatz II)**.** *Let a be a positive constant, let b be real, and let $G(z)$ be an entire function of genus 0 or 1 that for real z takes real values, has at least one real zero, and has only real zeros. Then the function*

$$e^{ib} G(z + ia) + e^{-ib} G(z - ia)$$

has only real zeros.

Proof. Pólya's original statement is Hilfssatz II in [12]. Pólya's argument is reiterated as Proposition 2 in [4]. □

We need the following important fact about $H_n(z)$.

Lemma 3. *Suppose $G \in \mathcal{LP}^*$. Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be positive real numbers. The exponential sum*

$$h_n(z) = \sum G(\pm ia_1 \pm \cdots \pm ia_n) e^{iz(\pm b_1 \pm \cdots \pm b_n)}$$

obtained by summing over all sign combinations, the same combination being used in the argument of G as in the exponent, is in $\mathcal{LP}^*$.

Proof. It is clear that $h_n(z)$ is real for real z and has order 1. The fact that $h_n(z)$ has real zeros is proved in [3] by a method similar to that of the Lee-Yang Circle Theorem (found in Appendix II in [8]). □

If $s_n^2 = a_1^2 + \cdots + a_n^2$ and if we use the notation of Riemann-Steiltjes integration, an immediate corollary to Lemma 3 is the following:

Corollary 4. *The function*

$$H_n(z) = 2^{-n} \sum G\big((\pm ia_1 \pm \cdots \pm ia_n)/s_n\big) e^{iz(\pm a_1 \pm \cdots \pm a_n)/s_n} = \int_{-\infty}^{\infty} G(it) e^{it} \, dF_n(t)$$

is in $\mathcal{LP}^*$.

In Lemmas 6 and 7 we will show that the integrals $\int_{-\infty}^{\infty} G(it)e^{it} \, dF_n(t)$ converge uniformly to $\int_{-\infty}^{\infty} G(it)e^{it} \, dF_n(t)$ for z in compact sets. The proof of Lemma 6 will require the following 1994 result of Pinelis, which is an improvement of a conjecture by Eaton [6]:

Lemma 5 (Pinelis [10], Corollary 2.6)**.** *Let X_k be independent random variables taking values ± 1 with equal probability. Let $s_n^2 = a_1^2 + \cdots + a_n^2$, and let*

$$Y_n = \frac{a_1 X_1 + \cdots + a_n X_n}{s_n}.$$

Then

$$\Pr(|Y_n| > u) < 2c(1 - \Phi(u))$$

where $c = 2e^3/9$, $\Phi(u) = \int_{-\infty}^{u} \phi(t) \, dt$, and $\phi(t) = (2\pi)^{-1/2} e^{-t^2/2}$.

Lemma 6. *Let $\epsilon > 0$ be given, suppose $G \in \mathcal{LP}^*$, and let K be a compact subset of $\mathbb{C}$. Then there is a positive number A (depending on ϵ and K) such that*

$$\int_{|t|>A} |G(it)e^{izt}| \, dF_n(t) < \epsilon$$

for all n and all $z \in K$.

Proof. Let λ denote the order of G. By hypothesis, $\lambda < 2$. Choose δ with $\max(1, \lambda) < \delta < 2$. Then choose $A > 0$ large enough so that $|G(it)e^{izt}| < e^{|t|^\delta}$ for all $z \in K$ and $|t| > A$. Such an A exists as follows: Choose δ' with $\lambda < \delta' < \delta$. Then for sufficiently large A, $|G(it)| < e^{|t|^{\delta'}}$ whenever $|t| > A$. Since K is compact, there is an R so that $|z| < R$ for all $z \in K$. For sufficiently large A and $|t| > A$,

$$|G(it)e^{izt}| \leq |G(it)| e^{|z||t|} < e^{|t|^{\delta'} + R|t|} < e^{|t|^\delta}.$$

Thus, A exists as claimed. Then

$$\int_{A<|t|<B} |G(it)e^{izt}|dF_n(t) < 2\int_A^B e^{t^\delta}dF_n(t).$$

After integration by parts the right-hand side becomes

$$\begin{aligned}&2e^{B^\delta}(F_n(B)-1) - 2e^{A^\delta}(F_n(A)-1) - 2\int_A^B (F_n(t)-1)d(e^{t^\delta})\\ &< 2e^{A^\delta}(1-F_n(A)) + 2\int_A^B (1-F_n(t))\delta t^{\delta-1}e^{t^\delta}dt.\end{aligned}$$

According to Lemma 5 for $t \geq A$ and if A is sufficiently large, then

$$1 - F_n(t) \leq \beta\int_t^\infty e^{-u^2/2}du < \frac{e^{-t^2/2}}{2}$$

where $\beta = 4e^3/(9\sqrt{2\pi})$. This gives

$$\int_{A<|t|<B} |G(it)e^{izt}|dF_n(t) < e^{A^\delta - A^2/2} + \int_A^B \delta t^{\delta-1}e^{t^\delta - t^2/2}dt$$

and

$$\int_{A<|t|} |G(it)e^{izt}|dF_n(t) < e^{A^\delta - A^2/2} + \int_A^\infty \delta t^{\delta-1}e^{t^\delta - t^2/2}dt.$$

For sufficiently large A the last integral is bounded above by $e^{A^\delta - A^2/2}$. So,

$$\int_{A<|t|<B} |G(it)e^{izt}|dF_n(t) < 2e^{A^\delta - A^2/2}.$$

The right-hand side of the last inequality can be made arbitrarily small for sufficiently large A. Therefore, we obtain $\int_{|t|>A} |G(it)e^{izt}|dF_n(t) < \epsilon$ as desired. □

Lemma 7. *Let K be a compact subset of $\mathbb{C}$. Then*

$$\int_{-A}^A G(it)e^{izt}\,dF_n(t) \to \int_{-A}^A G(it)e^{izt}\,dF(t)$$

uniformly as $n \to \infty$ for $z \in K$.

Proof. By the Helly-Bray Theorem (see [9, p. 182] or [7, p. 298]) it is immediate that convergence occurs pointwise. We must, however, verify uniform convergence for $z \in K$.

Let $\epsilon > 0$ be given, and write $g_z(t) = G(it)e^{izt}$. Choose κ such that $\kappa > |g_z(t)|$ and $\kappa > |g_z'(t)|$ for all $z \in K$ and $t \in [-A, A]$. Integration by parts yields

$$\int_{-A}^A G(it)e^{itz}\,dF(t) = g_z(A)F(A) - g_z(-A)F(-A) - \int_{-A}^A F(t)g_z'(t)\,dt.$$

Then

$$\begin{aligned}&\left|\int_{-A}^{A} G(it)e^{itz}\,dF(t) - \int_{-A}^{A} G(it)e^{itz}\,dF_n(t)\right|\\ &\le |g_z(A)|\,|F(A)-F_n(A)| + |g_z(-A)|\,|F(-A)-F_n(-A)|\\ &\qquad + \int_{-A}^{A} |F(t)-F_n(t)|\,|g_z'(t)|\,dt\\ &\le \kappa|F(A)-F_n(A)| + \kappa|F(-A)-F_n(-A)| + \kappa 2A \max_{t\in[-A,A]} |F(t)-F_n(t)|.\end{aligned}$$

Since the functions F_n and F are distributions functions such that F_n converges to the continuous distribution F pointwise on $[-A,A]$, F_n converges to F uniformly on $[-A,A]$. Thus for sufficiently large n and all $t\in[-A,A]$,

$$|F_n(t)-F(t)| < \min\left(\frac{\epsilon}{6A\kappa}, \frac{\epsilon}{3\kappa}\right).$$

Therefore, for all sufficiently large n and all $z\in K$,

$$\left|\int_{-A}^{A} G(it)e^{itz}\,dF(t) - \int_{-A}^{A} G(it)e^{itz}\,dF_n(t)\right| < \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} = \epsilon.$$

This shows that the convergence $\int_{-A}^{A} G(it)e^{itz}\,dF_n(t) \to \int_{-A}^{A} G(it)e^{itz}\,dF(t)$ is uniform as claimed. □

Lemma 8. *Suppose $G\in\mathcal{LP}^*$. Then $H(z) = \int_{-\infty}^{\infty} G(it)e^{izt}\,dF(t)$ is an entire function that is real for real z, and if it does not vanish identically, then it has only real zeros.*

Proof. Let $H_n(z) = \int_{-\infty}^{\infty} G(it)e^{izt}\,dF_n(t)$. By Lemmas 6 and 7, $H_n(z)$ converges uniformly to $H(z)$ on compact sets. Since $H_n(z)$ is real for real z, its limit $H(z)$ is real for real z. By Hurwitz's Theorem (see [1, Thm. 2, p. 178]), if $H(z)$ is not identically zero, its zeros are limit points of the zeros of the $H_n(z)$. Since, for each n, $H_n(z)$ has only real zeros, $H(z)$ also has only real zeros. □

Lemma 9. *Suppose $G\in\mathcal{LP}^*$. The order of $H(z) = \int_{-\infty}^{\infty} G(it)e^{izt}\,dF(t)$ is ≤ 2.*

Proof. Choose δ with $\lambda<\delta<2$ where λ is the order of G. Let M be large enough so that $|G(z)| < Me^{|z|^\delta}$ for all z. By applying Hölder's Inequality (see [16, p. 63]) we obtain

$$\begin{aligned}\left|\int_{-\infty}^{\infty} G(it)e^{-izt}dF(t)\right| &\le \int_{-\infty}^{\infty} Me^{|t|^\delta+|z||t|}dF(t)\\ &\le M\left(\int_{-\infty}^{\infty} e^{2|t|^\delta}dF(t)\right)^{1/2}\left(\int_{-\infty}^{\infty} e^{2|z||t|}dF(t)\right)^{1/2}.\end{aligned}$$

By Lemma 5 both integrals in the product converge. The first integral is independent of z. We will determine a bound for the second integral. Integration by parts and an application of Lemma 5 give

$$\begin{aligned}\int_{-\infty}^{\infty} e^{2|z||t|}dF(t) &= 2\int_{0}^{\infty} e^{2|z|t}dF(t)\\ &= 2e^{2|z|t}\big(F(t)-1\big)\Big|_0^{\infty} + 2\int_0^{\infty} 2|z|e^{2|z|t}\big(1-F(t)\big)\,dt\\ &= 1+4|z|\int_0^{\infty}|z|e^{2|z|t}\big(1-F(t)\big)\,dt\\ &\le 1+K|z|\int_0^{\infty} e^{2|z|t-t^2/2}dt \quad \text{where } K=\tfrac{16e^3}{9\sqrt{2\pi}}.\end{aligned}$$

Since

$$\int_0^{\infty} e^{2|z|t-t^2/2}dt \le e^{2|z|^2}\int_{-\infty}^{\infty} e^{-(t-2|z|)^2/2}dt = \sqrt{2\pi}\, e^{2|z|^2},$$

we see that $\left|\int_{-\infty}^{\infty} G(it)e^{-izt}dF(t)\right|$ is bounded by a constant times $|z|e^{2|z|^2}$. Thus, the order of $\int_{-\infty}^{\infty} G(it)e^{-izt}dF(t)$ is ≤ 2.

This completes the proof of Theorem 1. □

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