EXTENDED LAGUERRE INEQUALITIES AND A CRITERION FOR REAL ZEROS

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Abstract. Let $f(z) = e^{-bz^2} f_1(z)$ where $b \geq 0$ and $f_1(z)$ is a real entire function of genus 0 or 1. We give a necessary and sufficient condition in terms of a sequence of inequalities for all of the zeros of $f(z)$ to be real. These inequalities are an extension of the classical Laguerre inequalities.

1. Introduction

The Laguerre-Pólya class, denoted $\mathcal{LP}$, is the collection of real entire functions obtained as uniform limits on compact sets of polynomials with real coefficients having only real zeros. It is known that a function f is in $\mathcal{LP}$ if and only if it can be represented as

$$f(z) = e^{-bz^2} f_1(z) \tag{1.1}$$

where $b \geq 0$ and where $f_1(z)$ is a real entire function of genus 0 or 1 having only real zeros. The basic theory of $\mathcal{LP}$ can be found in [6, Ch. 8] and [8, Ch. 5.4].

In this paper, we extend a theorem of Csordas, Patrick, and Varga on a necessary and sufficient condition for certain real entire functions to belong to the Laguerre-Pólya class. They proved the following:

Theorem 1.1. *Let*

$$f(z) = e^{-bz^2} f_1(z), \qquad (b \geq 0,\ f(z) \not\equiv 0),$$

where $f_1(z)$ is a real entire function of genus 0 or 1. Set

$$L_n[f](x) = \sum_{k=0}^{2n} \frac{(-1)^{k+n}}{(2n)!} \binom{2n}{k} f^{(k)}(x) f^{(2n-k)}(x) \tag{1.2}$$

for $x \in \mathbb{R}$ and $n \geq 0$. Then $f(z) \in \mathcal{LP}$ if and only if

$$L_n[f](x) \geq 0 \tag{1.3}$$

for all $x \in \mathbb{R}$ and all $n \geq 0$.

The forward direction is due to Patrick [7, Thm. 1]. The reverse direction was proved by Csordas and Varga [4, Thm. 2.9]. Theorem 1.1 is significant because it gives a nontrivial sequence of inequality conditions that hold for functions in the Laguerre-Pólya class. The case $n = 1$ reduces to the classical Laguerre inequality which says that if $f(z) \in \mathcal{LP}$, then

$$[f'(x)]^2 - f(x)f''(x) \geq 0$$

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for $x \in \mathbb{R}$. Consequently, inequalities like those in Theorem 1.1 are sometimes called Laguerre-type inequalities. Csordas and Escassut discuss the inequalities $L_n[f](x) \geq 0$ and related Laguerre-type inequalities in [3]. Other results on similar inequalities of Turán and Laguerre types can be found in [1] and [2].

2. An extension of Laguerre-type inequalities

In this section, we extend Theorem 1.1 and give new necessary and sufficient inequality conditions for a function to belong to the Laguerre-Pólya class.

First we generalize the operator L_n defined in Theorem 1.1. Let

$$g(z) = \sum_{\ell=0}^{M} c_\ell z^\ell = \prod_{j=1}^{M}(z+\alpha_j) \tag{2.1}$$

be a polynomial with complex roots. Define $\Phi(z,t)$ as the product

$$\Phi(z,t) = \prod_{j=1}^{M} f(z+\alpha_j t). \tag{2.2}$$

The coefficients of the Maclaurin series of $\Phi(z,t)$ with respect to t are functions of z, and we write:

$$\Phi(z,t) = \sum_{k=0}^{\infty} A_k(z) t^k, \tag{2.3}$$

where

$$A_k(z) = \frac{1}{k!}\Big[\frac{d^k}{dt^k}\Phi(z,t)\Big]_{t=0} = \frac{1}{k!}\Big[\frac{d^k}{dt^k}\prod_{j=1}^{M} f(z+\alpha_j t)\Big]_{t=0}. \tag{2.4}$$

Since $f(z)$ is entire, each $A_k(z)$ is entire. Another expression for $A_k(z)$ is given in (3.1). The choice $g(z) = 1 + z^2 = (z-i)(z+i)$ produces $A_{2k+1}(z) = 0$ and $A_{2k}(z) = L_k[f](z)$ as in (1.2) of Theorem 1.1. Thus, we may regard the sequence of functions $A_k(z)$ as a generalization of the sequence $L_k[f](z)$. We note that the zeros of $A_k(z)$ were studied by Dilcher and Stolarsky in [5]. In §3, we give several examples of $A_k(z)$ for interesting choices of $g(z)$.

Theorem 2.1. *Let $f(z) = e^{-bz^2} f_1(z)$, where $f_1(z) \not\equiv 0$ is a real entire function of genus 0 or 1 and $b \geq 0$. Assume $g(z)$ in (2.1) is an even polynomial with non-negative real coefficients having at least one non-real root. Then $f \in \mathcal{LP}$ if and only if*

$$A_k(x) \geq 0 \tag{2.5}$$

for all $x \in \mathbb{R}$ and all $k \geq 0$.

Corollary 2.2. *The choice $g(z) = 1 + z^2$ in Theorem 2.1 gives $f(z) \in \mathcal{LP}$ if and only if $L_k[f](x) \geq 0$ for all $x \in \mathbb{R}$ and all $k \geq 0$, as stated in Theorem 1.1.*

Proof of Theorem 2.1. Since $g(z)$ is an even polynomial, it follows that α_j is a root if and only if $-\alpha_j$ is a root with the same multiplicity. So,

$$\Phi(z,t) = \prod_{j=1}^{M} f(z+\alpha_j t) = \prod_{j=1}^{M} f(z-\alpha_j t) = \Phi(z,-t).$$

Hence, $A_k(z) \equiv 0$ for all odd k and we may write

$$\Phi(z,t) = \sum_{k=0}^{\infty} A_{2k}(z) t^{2k}.$$

Now assume $A_{2k}(x) \geq 0$ for all $x \in \mathbb{R}$ and all $k \geq 0$. Let $f(z) = e^{-bz^2} f_1(z)$ where $b \geq 0$ and $f_1(z)$ is a real entire function of genus 0 or 1, and assume $f(z)$ is not identically zero. Suppose, by way of contradiction, that $f(z)$ has a non-real root, say z_0. Let α_s be any fixed non-real root of $g(z)$ and write

$$z_0 = x_0 + \alpha_s t_0,$$

where both x_0 and t_0 are real. Then $f(z_0) = f(x_0 + \alpha_s t_0) = 0$, and

$$0 = \Phi(x_0, t_0) = \prod_{j=1}^{M} f(x_0 + \alpha_j t_0) = \sum_{k=0}^{\infty} A_{2k}(x_0) t_0^{2k}.$$

Assume $t \neq 0$. Then the nonnegativity of $A_{2k}(x_0)$ implies $A_{2k}(x_0) = 0$ for all k. This in turn implies $\Phi(x_0, t)$ is identically zero for all complex t. But that is false since $f(z)$ is a nonzero entire function. Therefore, $t_0 = 0$. Then $z_0 = x_0 + \alpha_s t_0 = x_0$ is also real, contradicting the choice of z_0. Thus, all the roots of $f(z)$ are real and $f(z) \in \mathcal{LP}$.

Conversely, assuming $f(z) \in \mathcal{LP}$, we will show that $A_{2k}(x) \geq 0$ for all $x \in \mathbb{R}$ and all $k \geq 0$. We will show this when $f(z)$ is a polynomial and the result for arbitrary $f(z) \in \mathcal{LP}$ will follow by taking limits. Let

$$f(z) = \prod_{i=1}^{n} (z + r_i)$$

where $r_1, \ldots, r_n$ are real. Calculating $\Phi(z,t)$ gives

$$\begin{aligned} \Phi(z,t) &= \prod_{j=1}^{M} f(z + \alpha_j t) = \prod_{j=1}^{M} \prod_{i=1}^{n} (z + \alpha_j t + r_i) \\ &= \prod_{i=1}^{n} \prod_{j=1}^{M} \big((z + r_i) + \alpha_j t\big) = \prod_{i=1}^{n} \sum_{\ell=0}^{M} c_\ell (z + r_i)^\ell t^{M-\ell}, \end{aligned} \tag{2.6}$$

where $g(z) = \prod_{j=1}^{M} (z + \alpha_j) = \sum_{\ell=0}^{M} c_\ell z^\ell$. Since $g(z)$ is an even polynomial, $c_\ell = 0$ for odd ℓ and

$$\Phi(z,t) = \prod_{i=1}^{n} \sum_{\ell=0}^{M/2} c_{2\ell} (z + r_i)^{2\ell} t^{M-2\ell} = \sum_{k=0}^{nM/2} A_{2k}(z) t^{2k}. \tag{2.7}$$

From (2.7), $A_{2k}(z)$ is the sum of products of terms of the form $c_{2\ell}(z+r_i)^{2\ell}$. Because $c_{2\ell} \geq 0$ and $(z + r_i)^{2\ell}$ is a square, it follows that $A_{2k}(x) \geq 0$ for real x.

Now let $f(z) \in \mathcal{LP}$ be an arbitrary function that is not a polynomial. Then there exist polynomials $f_n(z) \in \mathcal{LP}$ such that

$$\lim_{n \to \infty} f_n(z) = f(z)$$

uniformly on compact sets. The derivatives also satisfy

$$\lim_{n \to \infty} f_n^{(k)}(z) = f^{(k)}(z)$$

uniformly on compact sets. If we write

$$\Phi_n(z,t) = \prod_{j=1}^{M} f_n(z+\alpha_j t) = \sum_{k=0}^{\infty} A_{n,2k}(z) t^{2k},$$

we see from (2.4) that

$$\lim_{n\to\infty} A_{n,2k}(z) = A_{2k}(z)$$

uniformly on compact sets. Since $A_{n,2k}(x) \geq 0$ for real x, the limit also satisfies this inequality. Thus, for arbitrary $f(z) \in \mathcal{LP}$, $A_{2k}(x) \geq 0$ for $x \in \mathbb{R}$ and $k \geq 0$, completing the proof of the theorem. □

3. Discussion and Examples

The function $A_k(z)$ in (2.4) is described in terms of the kth derivative of a product of entire functions. Either by using the generalized product rule for derivatives or by expanding each $f(z+\alpha_j t)$ as a series and multiplying series, one obtains the following formula for $A_k(z)$:

$$A_k(z) = \sum_{\lambda \vdash k} \frac{m_\lambda(\alpha_1,\ldots,\alpha_M)}{\lambda_1! \cdots \lambda_r!} f(z)^{M-r} \prod_{j=1}^{r} f^{(\lambda_j)}(z), \tag{3.1}$$

where $\lambda \vdash k$ means that the sum is over all unordered partitions λ of k,

$$\lambda = (\lambda_1,\ldots,\lambda_r) \qquad k = \lambda_1 + \cdots + \lambda_r,$$

where r is the length of the partition λ, and where $m_\lambda(\alpha_1,\ldots,\alpha_M)$ is the monomial symmetric function of M variables for the partition λ evaluated at the roots $\alpha_1,\ldots,\alpha_M$.

The coefficients c_ℓ of $g(z) = \sum_{\ell=0}^{M} c_\ell z^\ell = \prod_{j=1}^{M}(z+\alpha_j)$ are elementary symmetric function of $\alpha_1,\ldots,\alpha_M$. The monomial symmetric functions $m_\lambda(\alpha_1,\ldots,\alpha_M)$ appearing in (3.1) can therefore be calculated in terms of $c_0,\ldots,c_M$ without direct reference to $\alpha_1,\ldots,\alpha_M$. We see that if $c_0,\ldots,c_M$ are real, then $m_\lambda(\alpha_1,\ldots,\alpha_M)$ is also real. However, in general $m_\lambda(\alpha_1,\ldots,\alpha_M)$ is not necessarily positive even if all the c_ℓ are positive. So, in the setting of Theorem 2.1, the type of summation appearing in (3.1) will typically involve both addition and subtraction, and the nonnegativity of $A_k(x)$ for real x is not directly obvious from this representation.

Example 1. Let $g(z) = 4 + z^4$ and let $f(z) \in \mathcal{LP}$. Then $\Phi(z,t)$ is

$$f(z+(1+i)t)f(z+(1-i)t)f(z+(-1+i)t)f(z+(-1-i)t) = \sum_{k=0}^{\infty} A_{2k}(z)t^{2k}.$$

A small calculation shows that

$$\begin{aligned}\tfrac{3}{2}A_4(x) = -f(z)^3 f^{(4)}(z) + 3f(z)^2 f''(z)^2 + 6f'(z)^4 \\ + 4f(z)^2 f^{(3)}(z) f'(z) - 12 f(z) f'(z)^2 f''(z).\end{aligned}$$

According to Theorem 2.1 this expression is nonnegative for all $x \in \mathbb{R}$.

Example 2. Let $f(z) = \prod_{i=1}^{n}(z+r_i)$ where $r_1,\ldots,r_n \in \mathbb{R}$ and let

$$g(z) = z^{2m} + 1 = \prod_{j=1}^{2m}(z + \omega^{2j-1})$$

where $\omega = \exp(2\pi i/4m)$ and $m \in \mathbb{N}$. Calculating as in (2.6) gives

$$\begin{aligned}\Phi(z,t) &= \prod_{i=1}^{n}\left((z+r_i)^{2m}+t^{2m}\right)\\ &= f(z)^{2m}\prod_{i=1}^{n}\left(1+\frac{t^{2m}}{(z+r_i)^{2m}}\right)\\ &= f(z)^{2m}\sum_{k=0}^{n} e_k\left(\tfrac{1}{(z+r_1)^{2m}},\dots,\tfrac{1}{(z+r_n)^{2m}}\right)(t^{2m})^k,\end{aligned}$$

where e_k is the kth elementary symmetric function of n variables evaluated at $(z+r_1)^{-2m},\dots,(z+r_n)^{-2m}$. Thus, if $x \in \mathbb{R}$,

$$A_{2mk}(x) = f(x)^{2m} e_k\left(\tfrac{1}{(x+r_1)^{2m}},\dots,\tfrac{1}{(x+r_n)^{2m}}\right)$$

is expressed as a sum of squares of real numbers and is therefore nonnegative. Dilcher and Stolarsky studied the zeros of $A_{2mk}(x)$. (See Prop. 2.3 and §3 of [5]).

Example 3. This example illustrates how certain modifications to Theorem 2.1 are possible. Let $f(z)$ be a polynomial with *negative* roots. Then $f(z) = \prod_{i=1}^{n}(z+r_i)$ where each $r_i > 0$. Let

$$g(z) = 1+z+z^2 = (z+e^{\pi i/3})(z+e^{-\pi i/3}).$$

Although $g(z)$ is not even as in the hypothesis of the theorem, its coefficients are nonnegative. Then

$$\begin{aligned}\Phi(z,t) &= f(z+te^{\pi i/3})f(z+te^{-\pi i/3})\\ &= \underbrace{f(z)^2}_{A_0(z)} + \underbrace{f(z)f'(z)}_{A_1(z)}\,t + \underbrace{\tfrac{1}{2!}\left(2f'(z)^2 - f(z)f''(z)\right)}_{A_2(z)}\,t^2\\ &\quad + \underbrace{\tfrac{1}{3!}\left(3f'(z)f''(z) - 2f(z)f'''(z)\right)}_{A_3(z)}\,t^3\\ &\quad + \underbrace{\tfrac{1}{4!}\left(6f''(z)^2 - 4f'(z)f^{(3)}(z) - f(z)f^{(4)}(z)\right)}_{A_4(z)}\,t^4 + \cdots\end{aligned}$$

On the other hand, calculating as in (2.6) gives

$$\begin{aligned}\Phi(z,t) &= \prod_{i=1}^{n}\left((z+r_i)^2+(z+r_i)t+t^2\right)\\ &= f(z)^2\prod_{i=1}^{n}\left(1+\frac{t}{z+r_i}+\frac{t^2}{(z+r_i)^2}\right)\\ &= f(z)^2\sum_{k=0}^{2n}\Bigg(\sum_{\substack{\lambda\vdash k\\ \lambda_j\le 2}} m_\lambda\left(\tfrac{1}{z+r_1},\dots,\tfrac{1}{z+r_n}\right)\Bigg)t^k,\end{aligned}$$

where the inner sum is over all unordered partitions λ of k whose parts satisfy $\lambda_j \le 2$ and where m_λ is the monomial symmetric function in n variables for the partition λ evaluated at $(z+r_1)^{-1},\dots,(z+r_n)^{-1}$. From the last expression, we see that each

$$A_k(x) \ge 0$$

for all $x \geq 0$ and all $k \geq 0$.

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