TOTALLY p-ADIC ALGEBRAIC NUMBERS OF DEGREE 4

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Abstract. We generalize work of Stacy [6], to obtain upper bounds independent of p for the minimal height of a totally p-adic algebraic number of degree 4. We also compute actual values of this minimal height for small primes p.

1. Introduction

For an algebraic number α with minimal polynomial f_α, we define the Mahler measure and the logarithmic Weil height.

Definition 1. Let $f = \sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x]$ be a degree n polynomial with complex roots $\alpha_1, \dots, \alpha_n$. The *Mahler measure* of f is defined as,

$$M(f) = |a_n| \prod_{i=1}^{n} \max(1, |\alpha_i|).$$

Definition 2. Let α be a nonzero algebraic number with minimal polynomial $f_\alpha \in \mathbb{Z}[x]$ over $\mathbb{Z}$ of degree n. The *logarithmic Weil height* of α is

$$h(\alpha) = \frac{\log(M(f_\alpha))}{n}.$$

In 1975, Schinzel [5] (see also [3]) proved that for any totally real algebraic number α (i.e., an α such that every root of its minimal polynomial over $\mathbb{Q}$ is real) with $\alpha \neq 0, \pm 1$,

$$h(\alpha) \geq \frac{1+\sqrt{5}}{2}.$$

As a generalization of the concept of a totally real algebraic number, we make the following definition.

Definition 3. Let p be a prime number. An algebraic number is *totally p-adic* if all of the roots of its minimal polynomial over $\mathbb{Q}$ lie in $\mathbb{Q}_p$.

Equivalently, we note that an algebraic number α is totally p-adic if and only if p splits completely in $\mathbb{Q}(\alpha)$.

Bombieri and Zannier [2] proved the existence of a lower bound for the height of nonzero totally p-adic non-roots of unity when the degree of α is allowed to be arbitrary. Pottmeyer [4, Theorems 1.1 and 1.2] proves that for such α, we have

$$h(\alpha) > \frac{\log(p/2)}{p+1}, \quad \text{for odd primes } p$$

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and

$$h(\alpha) > \frac{\log(2)}{4}, \qquad \text{for } p = 2.$$

In this paper, we limit ourselves to nonzero totally p-adic algebraic numbers of fixed degree n that are not roots of unity, and make the following definition:

Definition 4. Let p be a prime number, and let n be a positive integer. Then $\tau_{n,p}$ is the minimum height of a totally p-adic nonzero algebraic number of degree n that is not a root of unity.

In 2020, Stacy [6] proved the following bound on $\tau_{3,p}$.

Theorem 1. *For any prime number $p > 3$,*

$$\tau_{3,p} \leq 0.703762.$$

This bound holds for all primes $p > 3$; Stacy then proceeds to find the actual values of $\tau_{3,p}$ for small primes $p > 3$.

In this paper, we extend Stacy's results to algebraic numbers of degree 4, proving that

Theorem 2. *For any prime number p,*

$$\tau_{4,p} \leq \frac{\log(5)}{4} \approx 0.40236.$$

In addition, we give a table of actual values for $\tau_{4,p}$ for small primes p, together with the minimal polynomial of a totally p-adic algebraic number α of degree 4 having minimal height; i.e., a totally p-adic α with $h(\alpha) = \tau_{4,p}$.

2. $(\mathbb{Z}/2\mathbb{Z})^3$-extensions of $\mathbb{Q}$

In this section, we will show that given an extension $L/\mathbb{Q}$ with Galois group $(\mathbb{Z}/2\mathbb{Z})^3$ and generators for the degree 4 subfields of L, we obtain a bound B depending only on L and the choice of generators (but not on p), such that for all primes p that are unramified in $L/\mathbb{Q}$, we have $\tau_{4,p} < B$. To get an explicit value for B we will choose a specific field L in section 3.

For the remainder of this section, let $L/\mathbb{Q}$ be any Galois extension with Galois group $(\mathbb{Z}/2\mathbb{Z})^3$. We note that L has seven quartic subfields, each of which has Galois group $(\mathbb{Z}/2\mathbb{Z})^2$; we will denote these fields by $K_1, \ldots, K_7$.

Theorem 3. *Let $L/\mathbb{Q}$ be a Galois extension with $\mathrm{Gal}(L/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^3$. For any prime number p that is unramified in $L/\mathbb{Q}$, at least one of the quartic subfields K_i is generated by an algebraic number α_i that is totally p-adic.*

Proof. For each quartic subfield K_i of L, choose a generator $\alpha_i \in K_i$ such that $K_i = \mathbb{Q}(\alpha_i)$ and α_i is not a root of unity.

If we let p be a rational prime that is not ramified in $L/\mathbb{Q}$, then the Frobenius element of p will have order either 1 or 2 (as, indeed, is true of all elements of the $\mathrm{Gal}(L/\mathbb{Q})$).

If the order of the Frobenius of p is 1, then p splits completely in L, and hence in all of the K_i. Hence, each α_i is totally p-adic.

On the other hand, if the order of Frobenius of p is 2, then the fixed field of the Frobenius will be one of the K_i. Since $L/\mathbb{Q}$ is an abelian extension, every unramified prime p splits completely in the fixed field of its Frobenius, so p splits completely in K_i, and the generator α_i is totally p-adic. □

Subfield	Defining Polynomial	Mahler Measure	Height
$\mathbb{Q}(\sqrt{-2}, \sqrt{-3})$	$3x^4 - 2x^2 + 3$	3.000	0.274653
$\mathbb{Q}(\sqrt{-2}, \sqrt{3})$	$3x^4 + 2x^2 + 3$	3.000	0.274653
$\mathbb{Q}(\sqrt{-1}, \sqrt{2})$	$x^4 + 2x^2 + 4x + 2$	3.414	0.306971
$\mathbb{Q}(\sqrt{-1}, \sqrt{3})$	$x^4 + 2x^3 + 2x^2 - 2x + 1$	3.732	0.329236
$\mathbb{Q}(\sqrt{2}, \sqrt{3})$	$x^4 - 4x^2 + 1$	3.732	0.329236
$\mathbb{Q}(\sqrt{2}, \sqrt{-3})$	$x^4 + 2x^2 + 4$	4.000	0.346574
$\mathbb{Q}(\sqrt{-1}, \sqrt{6})$	$5x^4 + 4x^3 + 4x + 5$	5.000	0.402360

Table 1. Quartic Subfields of $\mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{3})$

Choosing the α_i as in the proof of the theorem, we see that for an arbitrary p that is unramified in L, there is at least one α_i that is totally p-adic, so we see that $\tau_{4,p} \leq h(\alpha_i) \leq \max_i(h(\alpha_i))$. We thus get the following bound:

Theorem 4. *Given α_i generating the K_i as described above, we have that for p unramified in L,*

$$\tau_{4,p} \leq \max(h(\alpha_i)).$$

In order to get an explicit bound (independent of p) on the value of $\tau_{4,p}$, we see that we only need to find a specific Galois extension L as above, determine generating elements α_i for each K_i, and find their heights. If we can find α_i of minimal height in K_i, we will obtain a better bound.

3. Bounding $\tau_{4,p}$

Proof of Theorem 2. To bound $\tau_{4,p}$, we choose the field $L = \mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{3})$. One checks easily that $\mathrm{Gal}(L/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^3$, and that $L/\mathbb{Q}$ is ramified only at 2 and 3. For each of the seven degree 4 subfields, we list a defining polynomial, the Mahler measure of the polynomial, and the height of a root of the polynomial (rounded to six decimals) in Table 1.

To compute this table, we used GP/Pari [7] to compute a monic defining polynomial and integral basis for each field. We then computed the minimal polynomial of all small integer linear combinations of the integral basis to find a monic defining polynomial with small Mahler measure. Doing this yielded monic defining polynomials for the seven fields that all had Mahler measure at most 9. Finally, using the techniques of the next section, we created a list of all irreducible degree 4 polynomials of Mahler measure less than 9, and for each field, we searched the list to find a non-cyclotomic polynomial with minimum Mahler measure defining the field, which we then included in the table. By Theorem 4 and Table 1, we then see that for $p \neq 2, 3$, we have $\tau_{4,p} \leq \log(5)/4 \approx 0.4023595$.

Using this same list of polynomials, we found the polynomial $f_2 = 2x^4 - x^3 + 2x^2 + x + 4$, which has Mahler measure 4, and note that 2 splits completely in the root field of f_2. Hence, a root α of f_2 is totally p-adic and has height $\log(4)/4 \approx 0.346574$. We thus find that $\tau_{4,2} \leq 0.346574$. Similarly, for $p = 3$, we find the polynomial $f_3 = 3x^4 - 4x^3 + 4x^2 - 4x + 3$ with Mahler measure 3; it defines a field in which 3 splits completely. The height of a root of f_3 is $\log(3)/4 \approx 0.2746531$, so we find that $\tau_{4,3} \leq 0.2746531$. Hence, we have proven that for all p,

$$\tau_{4,p} \leq \frac{\log(5)}{4} \approx 0.4023595,$$

completing the proof of Theorem 2. $\square$

By examining the list of polynomials of Mahler measure at most 5, we find that there is no other $(\mathbb{Z}/2\mathbb{Z})^3$-field such that all of its quartic subfields have a defining polynomial with Mahler measure at most 5. Hence, the bound that we find is the best possible bound obtainable by this method.

4. Actual values of $\tau_{4,p}$ for small p

In order to compute actual values of $\tau_{4,p}$ for small p, we follow a procedure similar to that described by Stacy in [6] for $\tau_{3,p}$.

We begin by bounding the coefficients of a polynomial in terms of its Mahler measure:

Theorem 5. [1, pg. 25] *Let $f = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0$, and let $M(f)$ be the Mahler measure of f. Then*

$$|a_{n-r}| \leq \binom{n}{r} M(f).$$

This implies that for a given bound B, all irreducible degree n polynomials with Mahler measure at most B lie in the finite set of polynomials satisfying the bounds $|a_{n-r}| \leq \binom{n}{r}B$. Using GP/Pari [7], we can examine each polynomial with coefficients satisfying these bounds and eliminate those with Mahler measure greater than B and those that are reducible. In addition, we can use symmetry to reduce the number of polynomials we need to examine; since $f(x)$ and $\pm f(-x)$ have the same Mahler measure and define the same field, we only need to examine polynomials with $a_n > 0$ and $a_{n-1} \geq 0$.

Using these bounds we created three lists of quartic polynomials. The first consists of all irreducible (over $\mathbb{Z}$) quartic polynomials with Mahler measure at most 9 (subject to our symmetry constraints). This search took approximately 6 hours. The list contains 286,546 polynomials. Using this list as described in the previous section, we reduced the bound on $\tau_{4,p}$ from $\log(9)/4$ to $\log(5)/4$.

We then create a list of all irreducible quartic polynomials with Mahler measure at most 5 (subject to our symmetry constraints). This list contains 13,408 polynomials. Since we have proven that $\tau_{4,p} \leq \log(5)/4$ for all primes p, we are guaranteed that for every prime p, some polynomial in this list has a totally p-adic root. Computing this list from scratch took approximately 10 minutes; we could also have computed it just by eliminating polynomials with Mahler measure larger than 5 from the first list.

Finally, we reduced the size of this second list by eliminating polynomials that define isomorphic number fields. If two polynomials define isomorphic number fields, their roots are totally p-adic for the same set of primes p, so we only need to keep the polynomial with smaller Mahler measure (although we do not keep cyclotomic polynomials, whose roots are roots of unity). With this reduction, we obtained a list of 4,562 irreducible polynomials defining nonisomorphic quartic fields, with the property that for any prime p the roots of at least one of these polynomials are totally p-adic. Our method of construction guarantees that for

p	$\tau_{4,p}$	Defining polynomial f	Mahler Measure of f
2	0.346574	$2x^4 + x^3 + 2x^2 - x + 4$	4.00000
3	0.274653	$3x^4 + x^3 + 5x^2 + x + 3$	3.00000
5	0.274653	$2x^4 + 3$	3.00000
7	0.274653	$3x^4 - 4x^2 + 3$	3.00000
11	0.173287	$2x^4 - 3x^2 + 2$	2.00000
13	0.235153	$2x^4 + x^2 - 2$	2.56155
17	0.232996	$x^4 + 2x^3 - x^2 + x + 2$	2.53954
19	0.173287	$2x^4 + x^2 + 2$	2.00000
23	0.158244	$x^4 + x^3 - x + 1$	1.88320
29	0.120303	$x^4 + x^2 - 1$	1.61803
31	0.173287	$2x^4 - x^2 + 2$	2.00000
37	0.173287	$x^4 + x^3 - x - 2$	2.00000
41	0.173287	$x^4 + 2x^2 + 2$	2.00000
43	0.135884	$x^4 + x^3 + 2x^2 + 2x + 1$	1.72208
47	0.173287	$2x^4 + x^2 + 2$	2.00000
53	0.173287	$2x^4 + x^2 + 2$	2.00000
59	0.156051	$x^4 + x^3 - 2x - 1$	1.86676
61	0.207861	$x^4 + 2x^3 + 2x + 1$	2.29663
67	0.173287	$x^4 + x^2 + x + 2$	2.00000
71	0.156051	$x^4 + x^3 - 2x - 1$	1.86676
73	0.173287	$x^4 + 2$	2.00000
79	0.173287	$x^4 + x^3 - x^2 + 2$	2.00000
83	0.080571	$x^4 + x - 1$	1.38028
89	0.120303	$x^4 + x^2 - 1$	1.61803
97	0.173287	$2x^4 + x^3 + x + 2$	2.00000
101	0.120303	$x^4 + x^2 - 1$	1.61803
103	0.135884	$x^4 + x^3 + 2x^2 + 2x + 1$	1.72208
107	0.173287	$2x^4 - x^2 + 2$	2.00000
109	0.173287	$2x^4 + 2x^3 + x^2 + 2x + 2$	2.00000
113	0.144611	$x^4 + x^3 + x^2 - x - 1$	1.78326
127	0.158244	$x^4 + x^3 - x + 1$	1.88320
131	0.173287	$2x^4 + x^3 + x + 2$	2.00000
137	0.158244	$x^4 + x^3 - x + 1$	1.88320
139	0.135884	$x^4 + x^3 + 2x^2 + 2x + 1$	1.72208
149	0.173287	$2x^4 + 2x^3 + x^2 + 2x + 2$	2.00000
151	0.158244	$x^4 + x^3 - x + 1$	1.88320
157	0.110534	$x^4 + x^2 + x + 1$	1.55603
163	0.173287	$x^4 + x + 2$	2.00000
167	0.173287	$x^4 + x^3 + x + 2$	2.00000

TABLE 2. Actual values of $\tau_{4,p}$ for $p < 170$ (rounded to six decimals)

each prime p, this list contains a noncyclotomic polynomial of minimal possible Mahler measure whose roots are totally p-adic.

For a given prime p, we can search through this third list (we could also search through the second list, but less efficiently) to find the smallest polynomial (in terms of Mahler measure) in which p splits completely. Then the height of a root of this polynomial will be the actual value of $\tau_{4,p}$.

In Table 2, we display the results of this computation for each prime $p < 170$. For each such prime, we give the actual value of $\tau_{4,p}$ (rounded to six decimal places), as well as a polynomial that achieves this value. We note that our list of polynomials guarantees that for any p (no matter how large), we will rapidly be able to compute the actual value of $\tau_{4,p}$.

The smallest possible value of $\tau_{4,p}$ is 0.080572, and is achieved at primes that split completely in the field defined by $x^4 - x - 1$ (the polynomial of smallest Mahler measure in our sorted list). The largest value of $\tau_{4,p}$ for $2 < p < 10^{10}$ is $\log(3)/4 \approx 0.274654$, (note that $\tau_{4,2}$ is larger than this value). For $\tau_{4,p}$ to be larger than $\log(3)/4$, the prime p would have to fail to split completely in all of the 333 distinct quartic fields having defining polynomials with Mahler measure at most 3; heuristically, about 1 in $3.5 \cdot 10^{11}$ primes should satisfy this condition (assuming that the probability of a prime splitting completely in a quartic field is the reciprocal of the degree of its Galois closure). Since there are only about $4.5 \cdot 10^8$ primes below 10^{10}, it is not surprising that we found none (except $p = 2$) with $\tau_{4,p} > \log(3)/4$.

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