EVEN GALOIS REPRESENTATIONS AND THE COHOMOLOGY OF $\mathrm{GL}(2,\mathbb{Z})$

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Abstract. Let ρ be an even two-dimensional representation of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which is induced from a character χ of odd order of the absolute Galois group of a real quadratic field K. After imposing some additional conditions on χ, we attach ρ to a Hecke eigenclass in the cohomology of $\mathrm{GL}(2,\mathbb{Z})$ with coefficients in a certain infinite-dimensional vector space V over an arbitrary field of characteristic not equal to 2. The space V is defined purely algebraically starting from the field K.

1. Introduction

One type of noncommutative reciprocity law connects a Galois representation (i.e. a continuous homomorphism $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(n,\mathbb{F})$ for some field $\mathbb{F}$, where $G_{\mathbb{Q}}$ is the absolute Galois group of $\mathbb{Q}$) with a system of eigenvalues of a Hecke algebra of some reductive $\mathbb{Q}$-group acting on an $\mathbb{F}$-vector space V. The connection consists of an equality, for almost all rational prime numbers ℓ, between the characteristic polynomial of the image of a Frobenius element at ℓ under ρ and a "Hecke polynomial" constructed according to a simple recipe from the eigenvalues of the Hecke operators at ℓ. In such a case, it is standard terminology to say that ρ is "attached" to the system of Hecke operators, or to an eigenvector in V that supports the system.

We say that ρ is odd if the characteristic of $\mathbb{F}$ equals 2 or if the image of complex conjugation is conjugate to a diagonal matrix with alternating 1's and -1's on the diagonal. When ρ is odd, there are many profound theorems and conjectures concerning these reciprocity laws. For example, if $n = 2$, ρ is odd, and $\mathbb{F}$ is a finite field, Serre's conjecture [17] (now a theorem of Khare and Wintenberger [12, 13] and Kisin[14]) states that ρ is attached to a holomorphic modular form that is an eigenform of the Hecke operators. Other papers, such as [3, 4, 11] conjecture the existence of analogous attachments for general values of n, with modular forms replaced by elements of arithmetic cohomology groups. In all of these conjectures, ρ is odd.

Conversely, work of Scholze [16] proves that any eigenclass of the Hecke operators in the cohomology of a congruence subgroup Γ of $\mathrm{GL}(n,\mathbb{Z})$ with coefficients in a finite-dimensional admissible module M over a field $\mathbb{F}$ has an attached Galois representation. For a field $\mathbb{F}$ of characteristic 0, this theorem was already proven in [10] by Harris, Lan, Taylor and Thorne. "Admissible" means that if $\mathbb{F}$ has characteristic 0 then M is an algebraic representation, and if $\mathbb{F}$ has positive characteristic, then the matrices used to define the Hecke operators act on M via reduction modulo some fixed integer.

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Caraiani and Le Hung [7] showed that for a Hecke eigenclass $z \in H^*(\Gamma, M)$ for a congruence subgroup Γ of $\mathrm{GL}(n, \mathbb{Z})$ and an admissible module M, the representation guaranteed to exist by Scholze's theorem is odd if the characteristic of $\mathbb{F}$ is positive or if z is cuspidal.

It is natural then to ask about what kind of cohomological attachment to expect if ρ is not odd. Of course, any module M occurring in such a reciprocity law could not be admissible. One reason to suppose there may be some kind of theorem along these lines is that the work in [3] gives examples of odd 3-dimensional Galois representations each of which is a sum of an even 2-dimensional Galois representation σ and a character, and which appear to be attached to the cohomology of a congruence subgroup of $\mathrm{GL}(3, \mathbb{Z})$ with admissible coefficients (over a finite field.) Some kind of cohomological attachment for σ may explain these phenomena.

We have no idea what may be the case for a general Galois representation. In this paper, we set $n = 2$, and say that ρ is even if it is not odd. Our main theorem asserts the attachment of certain even Galois representations to Hecke eigenclasses in $H^*(\mathrm{GL}(2, \mathbb{Z}), M)$ for some "natural" (infinite dimensional) module M. We have to be careful with the exact definition of "attachment", which we will explain, and then we can state the theorem.

Let f be a modular form of weight $k \geq 0$ on the upper half plane, with level $\Gamma_1(N)$ and nebentype θ, and suppose that f is an eigenform for the Hecke operators T_ℓ and $T_{\ell,\ell}$ for all $\ell \nmid N$. Denote the eigenvalue of T_ℓ by a_ℓ, and the eigenvalue of $T_{\ell,\ell}$ by A_ℓ. When $k \geq 2$, and f is holomorphic, there is a Galois representation ρ such that for all $\ell \nmid N$,

$$\det(I - \rho(\mathrm{Frob}_\ell)X) = 1 - a_\ell X + \ell A_\ell X^2,$$

where (in this case), A_ℓ is easily seen to be equal to $\ell^{k-2}\theta(\ell)$.

The cases when $k = 0$ or 1 are different. If $k = 1$ and f is holomorphic, or if $k = 0$ and f is a Maass form where the eigenvalue of the Laplacian is $1/4$, there is an attached Galois representation (this is only conjectural in the Maass form case) with finite image. In both cases, the motivic weight of f is 0, and the characteristic polynomial of Frob_ℓ equals

$$1 - a_\ell X + \theta(\ell) X^2.$$

These forms of the Hecke polynomials depend on the usual normalization of the Hecke operators. The correct definition of attachment to use in our situation is analogous to the case of a Maass form.

Definition 1.1. Let V be a Hecke module over the field $\mathbb{F}$, occurring in the homology of $\mathrm{GL}(2, \mathbb{Z})$ with a non-admissible coefficient module. Let $v \in V$ be an eigenvector for the Hecke operators T_ℓ and $T_{\ell,\ell}$ for almost all primes. Let a_ℓ be the eigenvalue of T_ℓ acting on v, and A_ℓ the eigenvalue of $T_{\ell,\ell}$ acting on v. Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(2, \mathbb{F})$ be an even Galois representation. We say that ρ is *attached* to v if, for almost all ℓ,

$$\det(I - \rho(\mathrm{Frob}_\ell)X) = 1 - a_\ell X + A_\ell X^2.$$

Our main theorem (Theorem 11.1) then takes the following form.

Theorem. *Let K be a real quadratic field of discriminant d, and let $\mathbb{F}$ be a field of characteristic 0 or a finite field of odd characteristic. In the first case, set $p = 1$ and in the second case let p be the characteristic of $\mathbb{F}$. Let $\chi : G_K \to \mathbb{F}^\times$ be a character with finite image. Let L be the fixed field of the kernel of χ and choose $N \in \mathbb{Z}$ so* *that L/K is unramified outside primes of K dividing N. Then we may consider χ as a character on the fractional ideals of K relatively prime to N. Define the subgroup $K(M,q) \subseteq K^\times$ as in Definition 3.7. Assume that*

(1) χ is trivial on the principal fractional ideals of K generated by elements of $K(M,q)$.
(2) χ is trivial on the principal fractional ideals of K generated by elements of $\mathbb{Q}^\times$ that are prime to pdN.
(3) $[L:K]$ is odd.
(4) $L/\mathbb{Q}$ is Galois.

Then $\rho : G_\mathbb{Q} \to \mathrm{GL}(2,\mathbb{F})$ given by $\rho = \mathrm{Ind}_{G_K}^{G_\mathbb{Q}} \chi$ is an even Galois representation and is attached to a Hecke eigenclass in $H^1(\mathrm{GL}(2,\mathbb{Z}), \mathfrak{M}(M,q)^*)$, for a certain module $\mathfrak{M}(M,q)$ defined in Definition 4.2.

The module $\mathfrak{M}(M,q)$ that we use is naturally defined in terms of the field K. It is a countably infinite-dimensional module related to the kind of module that we we used in [2] to study reducible cases of the Serre-type conjecture for GL(3)/$\mathbb{Q}$.

The conditions imposed on χ guarantee that L is a totally real field, which is why the induced representation ρ can be even. Examples of characters χ satisfying the conditions of the theorem include unramified characters of G_K and characters of G_K cutting out subfields of ring class fields of K that are Galois over $\mathbb{Q}$ and of odd degree over K.

The Galois representations in the theorem are known to be attached to Maass forms. The results of [6] (building on [15]) thus also serve to attach the Galois representations that we study to cohomology groups. However, there are very significant differences between the construction in [6] and our construction. First, [6] uses wholly different coefficient modules than we do. Their modules are real vector spaces and their techniques are analytic. In contrast, our coefficient modules are duals of countably infinite-dimensional vector spaces over any field of characteristic not equal to 2, and our methods are purely algebraic. Second, the results of [6] hold for arbitrary Maass forms, not only those that are expected to have arithmetic significance. Third, we hope to be able to use our theorem, combined with techniques similar to those of [2], to prove a Serre type conjecture for an odd 3-dimensional Galois representation which has a 2-dimensional constituent equivalent to an induced representation as treated in the theorem. We do not think that the results of [6] would be helpful in this regard, because for Serre type conjectures it is essential for the coefficient module to be a vector space over a finite field.

The proof of our theorem goes as follows. Viewing K as a two-dimensional $\mathbb{Q}$-vector space, we construct a $\mathrm{GL}(2,\mathbb{Q})$-module $\mathfrak{M}$ consisting of formal $\mathbb{F}$-linear combinations of elements of $\mathfrak{X}$, where $\mathfrak{X}$ is the set of homothety classes of column vectors in K^2 where the ratio of the two entries is not in $\mathbb{Q} \cup \infty$. The homotheties involved here are restricted to multiplication by the elements of a carefully chosen subgroup of $K^\times$ (see sections 2, 3, and 4). The stabilizer $\Gamma_x \subset \mathrm{GL}(2,\mathbb{Z})$ of a homothety class $x \in \mathfrak{X}$ is an infinite abelian group generated by $\{\pm I_2\}$ and the image γ_x of a certain unit in the ring of integers of K under a certain embedding of K into $\mathrm{GL}(2,\mathbb{Q})$ as a non-split torus.

Because of the factor $\{\pm I_2\}$ in the stabilizers, we must assume that the characteristic of $\mathbb{F}$ is not equal to 2.

Incidentally, the matrix γ_x also stabilizes a closed geodesic in the quotient of the upper half plane modulo $\mathrm{GL}(2,\mathbb{Z})$. Our initial idea was to work with the fundamental classes of these closed geodesics in some appropriate homology group. In the end we work directly with certain classes in $H_1(G,\mathfrak{M})$. (The construction of these homology classes is similar to a construction used in [1], but we do not see any deep connection between the two situations.)

Because $K^\times$ is abelian, it also acts on $\mathfrak{M}$, commuting with the $\mathrm{GL}(2,\mathbb{Q})$-action. We replace $\mathfrak{M}$ by a related submodule $\mathfrak{M}(M,q)$ on which a certain large subgroup of $\mathbb{Q}^\times$ acts via a quadratic character q related to the quadratic field $K/\mathbb{Q}$. The purpose of this is to make q equal the central character of the coefficient module $\mathfrak{M}(M,q)$, which then causes the coefficient of X^2 in the Hecke polynomial to be correct.

In section 5, we obtain an isomorphism of $H_1(\mathrm{GL}(2,\mathbb{Z}),\mathfrak{M})$ with a direct sum of $H_1(\Gamma_x,\mathbb{F})$'s, where x runs over a set of representatives of the $\mathrm{GL}(2,\mathbb{Z})$-orbits in $\mathfrak{X}$. This uses Shapiro's lemma and is an algebraic version of the fundamental classes of the corresponding closed geodesics.

To complete section 5, we work out how the Hecke operators act on this direct sum. We use the method of partial Hecke operators described in [2] to get a tractable formula for the action of a Hecke operator on $H_1(\mathrm{GL}(2,\mathbb{Z}),\mathfrak{M}(M,q))$. This rather delicate analysis is what allows us to compare the Hecke operator at ℓ with a Laplace operator on the ℓ-adic Bruhat-Tits tree below.

We call attention to the connection proved in Lemma 5.8 between two constants, d_j and e_j, which are indexes of certain subgroups of units. When we compare the Hecke operator at ℓ with the Laplace operator on functions of ℓ-adic lattices, at the beginning of the proof of Theorem 8.8, this connection appears in a surprising and crucial fashion.

A class in $H_1(\mathrm{GL}(2,\mathbb{Z}),\mathfrak{M}(M,q))$ has finite support, and the Hecke operators expand that support. So there will not be any Hecke eigenvectors in the homology group. Instead, we find Hecke eigenvectors in the dual space $H^1(\mathrm{GL}(2,\mathbb{Z}),\mathfrak{M}(M,q)^*)$.

A key idea is to interpret elements of the dual space $H^1(\mathrm{GL}(2,\mathbb{Z}),\mathfrak{M}(M,q)^*)$ of the homology as functions on the space of lattices in K, which we explain in section 6. In order to construct suitable such functions we introduce, in section 7, the Bruhat-Tits graph $\mathcal{T}_\ell$ for $\mathrm{GL}(2,\mathbb{Q}_\ell)$ or a certain double cover $\mathcal{T}_\ell^2$ of $\mathcal{T}_\ell$, depending on whether ℓ is split or inert in K.

We then relate the Hecke operators at ℓ to a Laplacian on $\mathcal{T}_\ell$ (or $\mathcal{T}_\ell^2$). We must go to a double cover in the inert case to make this work compatibly with the central character q on $\mathfrak{M}(M,q)$. Then, in sections 8 and 9, we construct functions on lattices that have the desired Hecke eigenvalues.

These functions on lattices are infinite products over the rational primes of the local functions we construct on the graphs. Lattices which are fractional ideals in K play a special role and we call them "idealistic" lattices. The construction of the local functions depends on the distinction between idealistic and non-idealistic lattices.

To define the desired cohomology class, the infinite product has to satisfy a certain global invariance property (proved in Section 10), which follows from the fact that χ is a global character on ideals. This global requirement means that we cannot simply define the local functions on lattices any way we want to obtain any random set of Hecke eigenvalues. Instead, the situation is rather rigid.

Finally, in section 11 we prove the main theorem (Theorem 11.1) stated above.

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2. Lattices and Homotheties in K

In this section and the next two sections we construct the coefficient modules we use in the homology of $\mathrm{GL}(2,\mathbb{Z})$ which appear in our main theorem as stated in the introduction.

Fix a real quadratic field K, its ring of integers $\mathfrak{O}$ and an element $\omega \in \mathfrak{O}$ such that $\mathfrak{O} = \mathbb{Z}[\omega]$. Let d be the discriminant of $K/\mathbb{Q}$. Let ϵ be a fundamental unit, i.e. a unit whose image modulo ± 1 generates $\mathfrak{O}^\times/\{\pm 1\}$.

Consider K as a two-dimensional vector space over $\mathbb{Q}$. By a lattice in K, we will mean a free $\mathbb{Z}$-module of rank 2 contained in K. Such a module has as a $\mathbb{Z}$-basis two $\mathbb{Q}$-linearly independent elements.

Let Y be the set of all column vectors ${}^t(a,b) \in K^2$ with $b \neq 0$ and $a/b \notin \mathbb{Q}$. If we let $\bar{\omega} = {}^t(\omega, 1) \in Y$, then every element of Y is of the form $\gamma\bar{\omega}$ for some $\gamma \in \mathrm{GL}(2,\mathbb{Q})$. In addition, given $y, y' \in Y$, there is a unique $\gamma \in \mathrm{GL}(2,\mathbb{Q})$ with $y = \gamma y'$. There is a natural action of $K^\times$ by scalar multiplication on Y, which we write as a right action.

Definition 2.1. Let $y = {}^t(a,b) \in Y$. Define Λ_y to be the $\mathbb{Z}$-lattice in K generated by a and b (i.e. the set of all integer linear combinations of a and b).

Note that for $\alpha \in K^\times$, we have $\Lambda_{y\alpha} = \alpha\Lambda_y$.

Definition 2.2. Let $H \subseteq K^\times$ be a multiplicative subgroup of $K^\times$. Two lattices Λ_1 and Λ_2 in K will be said to be *homothetic* if there is some $\alpha \in K^\times$ such that $\Lambda_1 = \alpha\Lambda_2$. If $\alpha \in H$, we will say that the lattices are *H-homothetic*.

Homothety and H-homothety of lattices are equivalence relations on the set of all lattices in K.

Definition 2.3. Let H be a multiplicative subgroup of $K^\times$. Define Y/H to be the quotient of Y with respect to the right action of scalar multiplication by H. The left action of $\mathrm{GL}(2,\mathbb{Q})$ on Y then gives a left action of $\mathrm{GL}(2,\mathbb{Q})$ on Y/H.

Lemma 2.4. *There is a bijection between* $\mathrm{GL}(2,\mathbb{Z})$*-orbits of elements of* Y/H *and the set* $\mathcal{H}$ *of* H*-homothety classes of lattices in* K*.*

Proof. Define a map $f : Y/H \to \mathcal{H}$ by setting $f(x)$ equal to the H-homothety class of Λ_y for any $y \in Y$ representing $x \in Y/H$. This map is constant on $\mathrm{GL}(2,\mathbb{Z})$-orbits, and is easily seen to induce a bijection between $\mathrm{GL}(2,\mathbb{Z})$-orbits and H-homothety classes of lattices in K. □

Lemma 2.5. *Let* Λ *be a lattice in* K*. Then there is a positive integer* m *such that* $\epsilon^m\Lambda = \Lambda$*.*

Proof. Note that if Λ and Λ' are $K^\times$-homothetic, the lemma will be true for Λ if and only if it is true for Λ', with the same value of m (since $K^\times$ is commutative.) Hence, we may, without loss of generality, assume that Λ is contained in $\mathfrak{O}$. Since Λ is a rank two $\mathbb{Z}$-submodule of $\mathfrak{O}$, it must have finite index in $\mathfrak{O}$. We may thus choose an $N \in \mathbb{Z}$ such that $N\mathfrak{O} \subseteq \Lambda \subseteq \mathfrak{O}$. Since $\mathfrak{O}/N\mathfrak{O}$ is finite and multiplication by ϵ permutes its elements, there is some positive $m \in \mathbb{Z}$ such that $\delta = \epsilon^m$ acts trivially on $\mathfrak{O}/N\mathfrak{O}$, and hence on $\Lambda/N\mathfrak{O}$. Then δ must take Λ to itself, so $\delta\Lambda \subseteq \Lambda$. We must also have $\delta^{-1}\Lambda \subseteq \Lambda$, so $\Lambda \subseteq \delta\Lambda \subseteq \Lambda$, and therefore $\delta\Lambda = \Lambda$. □

Definition 2.6. Let H be a subgroup of $K^\times$ containing -1. Given $x \in Y/H$, we define Γ_x to be the stabilizer of x in $\mathrm{GL}(2,\mathbb{Z})$, and $\hat{\Gamma}_x$ to be the quotient $\Gamma_x/\{\pm I\}$.

Remark 2.7. Note that for $y \in Y$, $(-I)y = y(-1)$, so since $-1 \in H$, we have $-I \in \Gamma_x$ for any $x \in Y/H$.

Theorem 2.8. *Let H be a subgroup of $K^\times$ such that $H \cap \mathfrak{O}^\times$ is infinite and $-1 \in H$, and let $x \in Y/H$ be represented by $y \in Y$. Then Γ_x is generated by $\{-I, g\}$, where $g \in \Gamma_x$ satisfies*

$$gy = y\epsilon^m$$

and m is smallest possible positive integer such that $\epsilon^m \in H$ and $\Lambda_y = \Lambda_{y\epsilon^m}$. Further, $\hat{\Gamma}_x$ is cyclic, generated by the image of g.

Proof. Let x be represented by $y = {}^t(a,b) \in Y$. Choose the smallest positive m such that $\epsilon^m\Lambda_y = \Lambda_y$ and $\epsilon^m \in H$. Then $\Lambda_y = \Lambda_{y\epsilon^m}$, so $y\epsilon^m$ is a basis of Λ_y. Hence, there is some $g \in \mathrm{GL}(2,\mathbb{Z})$ such that $gy = y\epsilon^m$. Since $\epsilon^m \in H$, we see that $gx = x$.

We now show that every element in Γ_x is (up to a sign) a power of g. Let $\eta \in \Gamma_x$. Then, since $\eta x = x$, there is some $\alpha \in H$ such that $\eta y = y\alpha$. Now α is an eigenvalue of η, and $\eta \in \mathrm{GL}(2,\mathbb{Z})$, so $\alpha \in \mathfrak{O}^\times$. Hence, $\alpha = \pm\epsilon^r$. By the division algorithm and the minimality of m, we see that $\alpha = \pm(\epsilon^m)^k$ for some k. Hence, $\eta = \pm g^k$. □

Certain elements $x \in Y/H$ will be quite important to us; for these elements, the value of m in the previous proof is determined solely by H.

Definition 2.9. Let H be a multiplicative subgroup of $K^\times$. If $x \in Y/H$ can be represented by $y \in Y$ such that Λ_y is a fractional ideal in K, then we say that x is *idealistic.*

Note that determining whether x is idealistic does not depend on the choice of $y \in Y$ representing x.

Corollary 2.10. *Let H be a subgroup of $K^\times$ such that $H \cap \mathfrak{O}^\times$ is infinite and $-1 \in H$. If $x \in Y/H$ is idealistic, the value of m in Theorem 2.8 is equal to the smallest positive integer k such that $\epsilon^k \in H$.*

For any subgroup H of $K^\times$ such that $H \cap \mathfrak{O}^\times$ is infinite and $-1 \in H$, then since $\hat{\Gamma}_x$ is cyclic, there is a canonical isomorphism

$$I_x : H_1(\hat{\Gamma}_x, \mathbb{F}) \to \hat{\Gamma}_x \otimes_{\mathbb{Z}} \mathbb{F}.$$

Since the characteristic of $\mathbb{F}$ is not equal to 2, the Hochschild-Serre spectral sequence for the exact sequence $0 \to \{\pm 1\} \to \Gamma_x \to \hat{\Gamma}_x \to 0$ yields an isomorphism

$$\phi_x : H_1(\Gamma_x, \mathbb{F}) \to H_1(\hat{\Gamma}_x, \mathbb{F}).$$

Definition 2.11. Let H be any subgroup of $K^\times$ such that $H \cap \mathfrak{O}^\times$ is infinite and $-1 \in H$, and let $x \in Y/H$.

(a) Define m_x to be the integer m described in Theorem 2.8.
(b) Define g_x to be the generator of $\hat{\Gamma}_x$ described in Theorem 2.8.

(c) Define z_x to be the generator of $H_1(\Gamma_x,\mathbb{F})$ such that $I_x(\phi_x(z_x)) = g_x \otimes 1$.

Corollary 2.12. *Let H be a subgroup of $K^\times$ such that $H \cap \mathfrak{O}^\times$ is infinite and $-1 \in H$. If $x, x' \in Y/H$ are in the same* GL$(2,\mathbb{Z})$*-orbit, then $m_x = m_{x'}$.*

Proof. If $x = \gamma x'$ for $\gamma \in \mathrm{GL}(2,\mathbb{Z})$, then we may choose $y, y' \in Y$ representing x, x', respectively such that $y = \gamma y'$. Then $\Lambda_y = \Lambda_{y'}$. From the description of m in Theorem 2.8, we see that $m_x = m_{x'}$. □

Lemma 2.13. *If $\gamma \in \mathrm{GL}(2,\mathbb{Z})$, then conjugation by γ from Γ_x to $\Gamma_{\gamma x}$ induces a map $\hat{\Gamma}_x \to \hat{\Gamma}_{\gamma x}$ that takes g_x to $g_{\gamma x}$, and induces a map on homology which takes z_x to $z_{\gamma x}$.*

Proof. Let y represent x. Then γy represents γx. There is a unique lift $g \in \mathrm{GL}(2,\mathbb{Z})$ of g_x, such that $gy = y\epsilon^{m_x}$. Then

$$(\gamma g \gamma^{-1})\gamma y = (\gamma y)\epsilon^{m_x},$$

so, since $m_x = m_{\gamma x}$, we see that $(\gamma g_x \gamma^{-1}) = g_{\gamma x}$. □

3. $H(M,q)$-HOMOTHETY

In this section we define a subgroup of $K^\times$, and study homothety under this subgroup. We also define a space of homothety classes that we will use to define the coefficient module for our homology groups.

Definition 3.1. Let $\mathbb{F}$ be a field of characteristic not equal to 2. If the characteristic of $\mathbb{F}$ is 0, we will define $p = 1$, otherwise we define p to be the characteristic of $\mathbb{F}$. Let $d = \mathrm{disc}(K)$, let N be an arbitrary natural number, and let M be a positive integer dividing pdN.

Definition 3.2. Denote by $\mathbb{Z}_{(pdN)}$ the ring consisting of elements of $\mathbb{Q}$ with denominators prime to pdN.

Definition 3.3. With p, d, M, N defined as above, define

$$K^\times_{(pdN)} = \{c \in K^\times : (c) \text{ is relatively prime to } (pdN)\}.$$

and

$$K(M) = \{c \in K^\times_{(pdN)} : c \equiv 1 \pmod{M}.\}$$

By $c \equiv 1 \pmod{M}$, we mean that $v_\pi(c-1) \geq v_\pi(M)$, for each prime π of K dividing M, where v_π is the valuation on K associated with the prime π. Note that $K^\times_{(pdN)}$ and $K(M)$ are multiplicative subgroups of $K^\times$, and that $K(M) \subseteq K^\times_{(pdN)}$ with finite index.

Next, we define a map from K to the ring $M(2,\mathbb{Q})$ of 2×2 matrices with entries in $\mathbb{Q}$.

Definition 3.4. Let $y = {}^t(a,b) \in Y$. We define an injective ring homomorphism $r_y : K \to M(2,\mathbb{Q})$ by

$$r_y(c)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ac \\ bc \end{pmatrix}$$

for $c \in K$.

Let $\theta : \mathbb{Z} \to \mathbb{F}$ be the quadratic Dirichlet character cutting out $K/\mathbb{Q}$. Note that since K is a real quadratic field, $\theta(-1) = 1$. We now extend θ to a character of $K^\times_{(pdN)}$.

Definition 3.5. Define $q : K^{\times}_{(pdN)} \to \mathbb{F}^{\times}$ to be the composition of the following multiplicative maps:

(1) The map taking $a \in K^{\times}_{(pdN)}$ to the principal fractional ideal $(a) \subset K$,
(2) The map taking a fractional ideal to its prime factorization,
(3) The map taking a product of powers of prime ideals to the subproduct of powers of inert prime ideals,
(4) The map taking an inert prime ideal (ℓ) to $\theta(\ell)$.

We note that q is a homomorphism. By the following lemma, it extends θ.

Lemma 3.6. *If $r \in \mathbb{Z}$ is relatively prime to pdN, then $q(r) = \theta(r)$.*

Proof. This follows from the fact that $q(\ell) = \theta(\ell)$ for primes $\ell \nmid pdN$ of $\mathbb{Q}$, and that $q(-1) = 1 = \theta(-1)$. $\square$

Definition 3.7. With p, d, N, M from Definition 3.1,

(a) Define $K(M, q)$ to be the kernel of $q|_{K(M)}$.
(b) Define $Q(q)$ to be the kernel of q on $\mathbb{Z}^{\times}_{(pdN)}$.
(c) Define $H(M, q)$ to be the subgroup of $K^{\times}$ generated by $Q(q)$ and $K(M, q)$.

Lemma 3.8. *$H(M, q) \cap \mathfrak{O}^{\times}$ is infinite and contains -1.*

Proof. Every unit in the ring of integers of K is in the kernel of q, and has a power that is congruent to 1 mod M. Further, $-1 \in Q(q)$. $\square$

Definition 3.9. Let $\mathfrak{X} = Y/H(M, q)$ and let $\mathfrak{M} = \mathbb{F}\mathfrak{X}$.

Since $\mathbb{F}Y$ is a $(\mathrm{GL}(2, \mathbb{Q}), K^{\times})$-bimodule, we obtain an isomorphism

$$\mathbb{F}Y \otimes_{H(M,q)} \mathbb{F} \cong \mathbb{F}\mathfrak{X} \otimes \mathbb{F} = \mathbb{F}\mathfrak{X}.$$

Because $K^{\times}$ is commutative, this is an isomorphism of $(\mathrm{GL}(2, \mathbb{Q}), K^{\times})$-bimodules.

Lemma 3.10. *Let $x \in \mathfrak{X}$ be represented by $y \in Y$. Then*

(a) $\Gamma_x = \{r_y(c) : c \in H(M, q)\} \cap \mathrm{GL}(2, \mathbb{Z})$.
(b) If $g \in \Gamma_x$, then $g = r_y(c)$ for some $c \in \mathfrak{O}^{\times}$.

Proof. (a) Suppose $g \in \Gamma_x$. Then we have that $gy = yc$ for some $c \in H(M, q)$. Since $yc = r_y(c)y$, and the entries of y are a $\mathbb{Q}$-basis of K, we see that $g = r_y(c)$. Hence, g is in the given intersection, and any g in the given intersection fixes x.

(b) Let $g \in \Gamma_x$. Then $g = r_y(c)$, for some $c \in H(M, q)$, and the characteristic polynomial of g is the same as that of multiplication by c on K. Since $g \in \mathrm{GL}(2, \mathbb{Z})$, we see that $c \in \mathfrak{O}^{\times}$. $\square$

Denote by $\widehat{\omega}$ the image in $\mathfrak{X}$ of $\bar{\omega} = {}^t(\omega, 1)$ and recall the definition of m_x from Definition 2.11.

Definition 3.11. Define $i^M = m_{\widehat{\omega}}$.

Lemma 3.12. *Let $x \in \mathfrak{X}$.*

(i) For any x, $i^M \mid m_x$.
(ii) If x is idealistic, then $m_x = i^M$.

Proof. (i) For any $x \in \mathfrak{X}$ represented by $y \in Y$, let $\phi_x : \Gamma_x \to H(M,q)$ be the injective homomorphism defined by $\phi_x(g) = r_y^{-1}(g)$. Then the image of ϕ_x is a subgroup of $\mathfrak{O}^\times$ containing $\{\pm 1\}$, and this subgroup has index m_x in $\mathfrak{O}^\times$.

For an element $c \in \mathfrak{O}^\times$, we see (by Lemma 3.10(a)) that $r_y(c) \in \Gamma_x$ implies that $c \in H(M,q)$, which in turn implies that $r_{\widehat{\omega}}(c) \in \Gamma_{\widehat{\omega}}$. Hence, $r_y^{-1}(\Gamma_x) \subseteq r_{\widehat{\omega}}^{-1}(\Gamma_{\widehat{\omega}})$. Hence, the index in $\mathfrak{O}^\times$ of the first (m_x) is a multiple of the index of the second ($m_{\widehat{\omega}} = i^M$).

(ii) Since x is idealistic, Λ_y is a fractional ideal of K. Hence, $r_y(c) \in \mathrm{GL}(2,\mathbb{Z})$ for all $c \in \mathfrak{O}^\times$. Hence,

$$r_y^{-1}(\Gamma_x) = \mathfrak{O}^\times \cap H(M,q) = r_{\widehat{\omega}}^{-1}(\Gamma_{\widehat{\omega}}).$$

Since these subgroups are equal, $m_x = i^M$. □

Definition 3.13. For $x \in \mathfrak{X}$, set $m'_x = m_x/i^M$.

Lemma 3.14. *Let $r \in \mathbb{Z}_{(pdN)}^\times$, let $\alpha \in H(M,q)$, and let $y \in Y$. If $r_y(r\alpha) \in \mathrm{GL}(2,\mathbb{Z})$, then $q(r) = 1$.*

Proof. Since the characteristic polynomial of $r\alpha$ is the same as the characteristic polynomial of $r_y(r\alpha)$, we see that $r\alpha$ is a unit in $\mathfrak{O}$. Hence, $q(r\alpha) = q(r)q(\alpha) = 1$. Since $q(\alpha) = 1$ by the definition of $H(M,q)$, we see that $q(r) = 1$. □

4. Defining the coefficient module $\mathfrak{M}(M,q)$

The elements of $\mathfrak{X}$ consist of $H(M,q)$-homothety classes of elements of Y. In order to study them more fully, we fix once and for all a certain element of $\mathbb{Z}_{(pdN)}^\times$.

Definition 4.1. Let $\xi \in \mathbb{Z}_{(pdN)}^\times$ be any element of $\mathbb{Z}_{(pdN)}^\times$ with $q(\xi) = -1$.

Since ξ is not in $H(M,q)$, we see that for any $x \in \mathfrak{X}$ the two elements $x\xi$ and x are distinct. In addition, since $\xi^2 \in H(M,q)$, right multiplication by ξ induces an involution on the elements of $\mathfrak{X}$, and in fact, this involution is independent of the choice of ξ.

Recall that $\mathfrak{M} = \mathbb{F}\mathfrak{X}$ is the $\mathbb{F}$-vector space consisting of formal $\mathbb{F}$-linear combinations of elements of $\mathfrak{X}$; i.e. the set of all elements of the form $\sum_{x\in\mathfrak{X}} c_x x$ with $c_x \in \mathbb{F}$. Also, $\mathfrak{M}$ is a $(\mathrm{GL}(2,\mathbb{Q}), K^\times)$-bimodule. Since the action of ξ on the right is an involution on $\mathfrak{X}$, it induces an involution on $\mathfrak{M}$; the eigenvalues of this involution are all either 1 or -1.

Definition 4.2. Let $\mathfrak{M}(M,q)$ be the eigenspace in $\mathfrak{M}$ of ξ with eigenvalue -1.

It is clear that $\mathbb{Z}_{(pdN)}^\times$ acts on $\mathfrak{M}(M,q)$ via the character q.

Lemma 4.3. *Let $x \in \mathfrak{X}$. Then x and $x\xi$ are in different $\mathrm{GL}(2,\mathbb{Z})$-orbits.*

Proof. Suppose x is represented by $y \in Y$, and $x\xi = \gamma x$ for some $\gamma \in \mathrm{GL}(2,\mathbb{Z})$. Then there is some $\alpha \in H(M,q)$ such that $y\xi\alpha = \gamma y$, which implies that $\gamma = r_y(\xi\alpha) \in \mathrm{GL}(2,\mathbb{Z})$. However, this contradicts Lemma 3.14. □

We now consider a collection $\mathfrak{A}$ of representatives of the $\mathrm{GL}(2,\mathbb{Z})$-orbits in $\mathfrak{X}$.

Definition 4.4. Let $\mathfrak{A}$ be a collection of $\mathrm{GL}(2,\mathbb{Z})$-orbit representatives in $\mathfrak{X}$, chosen so that if $x \in \mathfrak{A}$, then $x\xi \in \mathfrak{A}$. Choose a subset $\mathcal{A} \in \mathfrak{A}$ such that for every $x \in \mathfrak{A}$, exactly one of $\{x, x\xi\}$ is in $\mathcal{A}$.

The following lemma then follows easily.

Lemma 4.5. *Let $\mathcal{A}$ be the subset of $\mathfrak{A}$ defined above. Then the set*

$$\{gx : x \in \mathfrak{A}\}$$

is a basis for $\mathfrak{M}$, and the set

$$\{g(x - x\xi) : x \in \mathcal{A}, g \in \mathrm{GL}(2, \mathbb{Z})\}$$

is a basis of $\mathfrak{M}(M, q)$.

From this fact, the following lemma follows immediately.

Lemma 4.6. *The map from*

$$\bigoplus_{x \in \mathfrak{A}} \mathbb{F}[\mathrm{GL}(2, \mathbb{Z})] \otimes_{\mathbb{F}\Gamma_x} \mathbb{F} \to \mathfrak{M}$$

taking $g \otimes 1$ in the summand corresponding to $x \in \mathfrak{A}$ to gx is a $\mathrm{GL}(2, \mathbb{Z})$-invariant isomorphism.

5. Homology with coefficients in $\mathfrak{M}(M, q)$ and Hecke operators

In this section, we fix an element $x_0 \in \mathfrak{X}$ and a set $\mathfrak{A}$ of $\mathrm{GL}(2, \mathbb{Z})$-orbit representatives in $\mathfrak{X}$ as in Definition 4.4 that contains x_0. We have fixed an element $\xi \in \mathbb{Z}_{(pdN)}^\times$ with $q(\xi) = -1$. Using the set $\mathfrak{A}$, we will study the action of the Hecke operators on the homology of $\mathfrak{M}$, and ultimately on the homology of $\mathfrak{M}(M, q)$.

As a consequence of Lemma 4.6, we have that

$$\mathfrak{M} \cong \bigoplus_{x \in \mathfrak{A}} \mathrm{Ind}_{\Gamma_x}^{\mathrm{GL}(2,\mathbb{Z})} \mathbb{F}.$$

Hence, by Shapiro's lemma, we have

$$H_1(\mathrm{GL}(2, \mathbb{Z}), \mathfrak{M}) \cong \bigoplus_{x \in \mathfrak{A}} H_1(\Gamma_x, \mathbb{F}).$$

Recall that in Definition 2.11, for each $x \in \mathfrak{X}$, we have a canonical generator z_x for $H_1(\Gamma_x, \mathbb{F})$. Then $\{z_x : x \in \mathfrak{A}\}$ is a basis for $H_1(\mathrm{GL}(2, \mathbb{Z}), \mathfrak{M})$. Recall that ξ acts as an involution on $\mathfrak{M}$, and hence on $H_1(\mathrm{GL}(2, \mathbb{Z}), \mathfrak{M})$. From Lemma 4.6, we see that the action of ξ on $\mathfrak{M}$ swaps the summands corresponding to $x, x\xi \in \mathfrak{A}$. Hence, under Shapiro's isomorphism, we see that the action of ξ on $H_1(\mathrm{GL}(2, \mathbb{Z}), \mathfrak{M})$ is converted to an action on $\oplus_{x \in \mathfrak{A}} H_1(\Gamma_x, \mathbb{F})$ that swaps z_x and $z_{x\xi}$ (the groups Γ_x and $\Gamma_{x\xi}$ are identical, but z_x and $z_{x\xi}$ are in different summands). This proves:

Lemma 5.1. *The set*

$$\{z_x - z_{x\xi} = z_x - \xi z_x \ | x \in \mathcal{A}\}$$

is a basis for $H_1(\mathrm{GL}(2, \mathbb{Z}), \mathfrak{M}(M, q))$.

We will now use Section 3 of [2] to examine an individual Hecke operator T_s. (Warning: in [2] the modules are right-modules, but here we are using left-modules, so the formulas need to be adjusted accordingly.) What we call $\mathfrak{X}$ is there called X.

To follow the notation of [2], we set $\Gamma = \mathrm{GL}(2, \mathbb{Z})$, and let $S = \mathrm{GL}(2, \mathbb{Z}_{(pdn)})$. Recall that $\mathfrak{A}$ is a set of representatives of Γ-orbits in $\mathfrak{X}$. Let W be the S-sheaf (in the terminology of [2]) whose stalk at each $x \in \mathfrak{X}$ is $\mathbb{F}$. In other words, W is

$\mathfrak{M}$, and it is an S-module. As a Γ-module, W is isomorphic to a sum of induced representations

$$W = \bigoplus_{x \in \mathfrak{A}} \mathbb{F}[\Gamma] \otimes_{\Gamma_x} \mathbb{F},$$

where Γ_x is the stabilizer in Γ of x.

Let $s \in S$. We choose a set E of single coset representatives s_α for $\Gamma s \Gamma$, so that

$$\Gamma s \Gamma = \coprod_{s_\alpha \in E} \Gamma s_\alpha.$$

Then E is a finite set. We now adjust the elements $s_\alpha \in E$ in several stages to make the computation of Hecke operators easier.

We have chosen $\mathfrak{A}$ to contain x_0. The proof of Theorem 3.1 of [2] immediately shows that we can choose a finite set of elements $x_1, \dots, x_k$ of $\mathfrak{A}$ such that after possibly left-multiplying the s_α's by elements of Γ, we have a partition

$$E = \coprod_{i=1}^{k} R_i$$

where $R_i = \{s_\alpha \mid s_\alpha x_0 = x_i\}$. Now setting

$$G_i = \{g \in \Gamma s \Gamma \mid g x_0 = x_i\},$$

we see that since the x_i are in distinct GL(2, $\mathbb{Z}$)-orbits,

$$G_i = \coprod_{s_\alpha \in R_i} \Gamma_{x_i} s_\alpha.$$

The partial Hecke operator $T_{0i} = \Gamma_{x_i} \backslash G_i / \Gamma_{x_0}$ maps $H_*(\Gamma_{x_0}, M)$ to $H_*(\Gamma_{x_i}, M)$ for any S-module M.

Theorem 5.2. *Let ϕ be the isomorphism given by Shapiro's lemma:*

$$\phi : H_*(\mathrm{GL}(2, \mathbb{Z}), \mathfrak{M}) \to \bigoplus_{x \in \mathfrak{A}} H_*(\Gamma_x, \mathbb{F}).$$

If $z \in H_*(\Gamma_{x_0}, \mathbb{F})$, then

$$T_s(\phi^{-1}(z)) = \phi^{-1}\left(\sum_{i=1}^{k} T_{0i} z\right).$$

Proof. This is a restatement of Theorem 3.1 of [2], where we note that $W = \mathfrak{M}$. □

Now we rewrite each of the partial Hecke operators T_{0i} in terms of the homology of the stabilizers Γ_x of elements of $\mathfrak{A}$.

Fix i and note that for some finite set C_i,

$$G_i = \coprod_{t \in C_i} \Gamma_{x_i} t \Gamma_{x_0}$$

is a disjoint union of double cosets.

Let $T_t : H_1(\Gamma_{x_0}, \mathbb{F}) \to H_1(\Gamma_{x_i}, \mathbb{F})$ denote the Hecke operator corresponding to the double coset $\Gamma_{x_i} t \Gamma_{x_0}$.

Theorem 5.3. *With notation as above,*

(a) $T_{0i} = \sum_{t \in C_i} T_t$.

(b) After possibly left-multiplying the s_α*'s by elements of* $\mathrm{GL}(2,\mathbb{Z})$,

$$\Gamma_{x_i} t \Gamma_{x_0} = \coprod_{s_\alpha \in Q_{i,t}} \Gamma_{x_i} s_\alpha,$$

and E *is partitioned by the* $Q_{i,t}$*, as* i *and* t *vary.*

(c) After possibly left-multiplying one s_α *by an element of* $\mathrm{GL}(2,\mathbb{Z})$, $t \in Q_{i,t}$.

Proof. Let $F_\bullet$ be a resolution of $\mathbb{F}$ by free $\mathbb{F}\mathrm{GL}(2,\mathbb{Q})$-modules. We use f to stand for an arbitrary element of $F_\bullet$ and a to stand for an arbitrary function with finite support $a : F_\bullet \to \mathbb{F}$. For each $t \in C_i$, write its double coset as a disjoint union of single cosets:

$$\Gamma_{x_i} t \Gamma_{x_0} = \coprod_\beta \Gamma_{x_i} t_\beta.$$

The Hecke operator T_t maps the class of a cycle $\sum_f f \otimes_{\Gamma_{x_0}} a(f)$ to the class of $\sum_\beta \sum_f t_\beta f \otimes_{\Gamma_{x_i}} a(f)$. (Remember that Γ acts trivially on $\mathbb{F}$.)

There is a similar formula for the action of the Hecke operator T_{0i}. Since the set G_i is the union of the double cosets $\Gamma_{x_i} t \Gamma_{x_0}$, $t \in C_i$, it is the disjoint union of the single cosets $\Gamma_{x_i} t_\beta$ as t, β vary. Part (a) is now clear.

For (b), again write $\Gamma_{x_i} t \Gamma_{x_0} = \coprod_{t_\beta} \Gamma_{x_i} t_\beta$, for some choice of t_β. We know from the discussion preceding Theorem 5.2 that E is the disjoint union of the R_i and that

$$G_i = \coprod_{t \in C_i} \Gamma_{x_i} t \Gamma_{x_0} = \coprod_{s_\alpha \in R_i} \Gamma_{x_i} s_\alpha.$$

Therefore each coset $\Gamma_{x_i} t_\beta$ must equal $\Gamma_{x_i} s_\alpha$ for some s_α, and so we replace t_β with s_α. We let $Q_{i,t}$ be the set of s_α's corresponding to t. Then R_i is the disjoint union of the $Q_{i,t}$ as t varies, and (b) follows.

Finally, since $t \in \Gamma_{x_i} t \Gamma_{x_0}$, we see that $t \in \Gamma_{x_i} s_\alpha$ for some $s_\alpha \in Q_{i,t}$. We may replace that s_α by t, proving (c). □

Definition 5.4. Set

$$J = \sum_i |C_i|.$$

As i, t vary, enumerate the t's as $t_1, \ldots, t_J$. For each $j = 1, \ldots, J$, set

$$B_j = Q_{i,t}, \qquad U_j = T_{t_j}, \qquad \textit{and} \qquad x_j = x_i,$$

where (i,t) is the pair corresponding to j.

Set $\Gamma_0 = \Gamma_{x_0}$ and $\Gamma_j = \Gamma_{x_j}$.

Finally, set

$$d_j = [\Gamma_0 : t_j^{-1} \Gamma_j t_j \cap \Gamma_0], \quad \text{and} \quad e_j = [\Gamma_j : t_j \Gamma_0 t_j^{-1} \cap \Gamma_j].$$

Remark 5.5. Since all groups involved in the definitions of d_j and e_j contain $-I$, we note that we also have

$$d_j = [\hat\Gamma_0 : t_j^{-1} \hat\Gamma_j t_j \cap \hat\Gamma_0], \quad \text{and} \quad e_j = [\hat\Gamma_j : t_j \hat\Gamma_0 t_j^{-1} \cap \hat\Gamma_j].$$

Theorem 5.6. *(a)* $E = \{s_\alpha\} = \coprod B_j$ *and* $T_s = \sum_{j=1}^J U_j$ *where we suppress mention of the isomorphism* ϕ *from Theorem 5.2.*

(b) The number of elements in B_j *is* d_j.

(c) $U_j(z_{x_0}) = e_j z_{x_j}$*, where* z_x *is the generator of* $H_1(\Gamma_x, \mathbb{F})$ *chosen in Definition 2.11.*

(d) $U_j(z_{x_0} - \xi z_{x_0}) = e_j(z_{x_j} - \xi z_{x_j})$ *in* $H_1(\mathrm{GL}(2,\mathbb{Z}), \mathfrak{M}(M, q))$*.*

Proof. Part (a) follows from Theorem 5.2 and Theorem 5.3 (a) (b) and (d) combined with the new notation.

Part (b): The number of elements in B_j is the number of left Γ_{x_j}-cosets contained in $\Gamma_{x_j} t_j \Gamma_{x_0}$ (in our new notation $\Gamma_{x_j} t_j \Gamma_{x_0} = \Gamma_j t_j \Gamma_0$). A standard computation with cosets shows that the number of single cosets in this double coset equals the index $[\Gamma_0 : t_j^{-1} \Gamma_j t_j \cap \Gamma_0]$, which is d_j.

For part (c) we use the following lemma, which is standard, and follows easily from [5, Sec. III.9]:

Lemma 5.7. *Let* A *be an infinite cyclic group with generator* a*, and* $B \subset A$ *a subgroup of index* c*, and suppose that* A *acts trivially on* $\mathbb{F}$*. For an abelian group* G *acting trivially on* $\mathbb{F}$*, we identify* $H_1(G, \mathbb{F})$ *canonically with* $G \otimes_{\mathbb{Z}} \mathbb{F}$*.*

- *(i) The transfer map* $\mathrm{tr} : H_1(A, \mathbb{F}) \to H_1(B, \mathbb{F})$ *takes the generator* $a \otimes 1$ *to the generator* $a^c \otimes 1$*.*
- *(ii) The corestriction map* $i : H_1(B, \mathbb{F}) \to H_1(A, \mathbb{F})$ *(i.e. the map induced by the inclusion* $B \subset A$*) takes the generator* $a^c \otimes 1$ *to* c *times the generator* $a \otimes 1$*.*

Proof of part (c) continued: U_j is a Hecke operator. A standard fact about Hecke operators is that they can be written in terms of transfer, conjugation, and corestriction. (This is also easily checked on the chain level.) In our case, the following is true:

The map $U_j : H_1(\Gamma_0, \mathbb{F}) \to H_1(\Gamma_j, \mathbb{F})$ is given as the composition of the three maps

$$H_1(\Gamma_0, \mathbb{F}) \to H_1(\Gamma_0 \cap t_j^{-1} \Gamma_j t_j, \mathbb{F}) \xrightarrow{\phi_j} H_1(t_j \Gamma_0 t_j^{-1} \cap \Gamma_j, \mathbb{F}) \to H_1(\Gamma_j, \mathbb{F}),$$

where the first map is the transfer, the second map is the map induced on homology via conjugation by t_j on the group, and the third map is corestriction. We note that all of the groups involved in this diagram contain $\{\pm I\}$, and the maps described above each commute with the isomorphism on homology induced by the quotient map by $\{\pm I\}$. Hence, the composition above translates to the following,

$$H_1(\hat{\Gamma}_0, \mathbb{F}) \to H_1(\hat{\Gamma}_0 \cap t_j^{-1} \hat{\Gamma}_j t_j, \mathbb{F}) \xrightarrow{\phi_j} H_1(t_j \hat{\Gamma}_0 t_j^{-1} \cap \hat{\Gamma}_j, \mathbb{F}) \to H_1(\hat{\Gamma}_j, \mathbb{F}),$$

where all of the involved groups are cyclic, allowing us to use Lemma 5.7.

Let g_0 and g_j be the generators of $\hat{\Gamma}_0$ and $\hat{\Gamma}_j$ corresponding to the chosen generators z_{x_0} and z_{x_j} of the homology groups in Definition 2.11. Identifying elements of an abelian group with elements of the homology, we can say that by Lemma 5.7(i), the transfer takes g_0 to the generator $g_0^{d_j}$ of $\hat{\Gamma}_0 \cap t_j^{-1} \hat{\Gamma}_j t_j$.

Since $t_j x_0 = x_j$, conjugation by t_j takes $g_0^{d_j}$ to the generator $g_j^{e_j}$ of $t_j \hat{\Gamma}_0 t_j^{-1} \cap \hat{\Gamma}_j$. Then by Lemma 5.7 (ii), corestriction takes this to $g_j^{e_j}$.

Part (d) follows, since ξ commutes with the action of $\mathrm{GL}(2, \mathbb{Q})$ and so commutes with U_j. □

We now compute the values of e_j and d_j in terms of m_0 and m_j. Recall that $t_j x_0 = x_j$. For $j = 0, \ldots, J$, choose $y_j \in Y$ such that x_j is represented by y_j, and recall that Γ_j is the stabilizer of x_j in $\mathrm{GL}(2, \mathbb{Z})$ and ϵ is the fundamental unit of $\mathfrak{O}$

which we chose at the beginning of Section 2. Let $g_j \in \hat{\Gamma}_j$ and $m_j \in \mathbb{Z}$ be defined as in Definition 2.11 (with $H = H(M,q)$). Then g_j is a generator of $\hat{\Gamma}_j$ and $m_j > 0$.

Lemma 5.8. *With notation as above,*

$$e_j = \mathrm{LCM}(m_0, m_j)/m_j, \qquad d_j = \mathrm{LCM}(m_0, m_j)/m_0,$$

and $e_j m_j = d_j m_0$.

Proof. There are two lifts of $g_j \in \hat{\Gamma}_j$ to Γ_j; let $h_j \in \Gamma_j$ be the unique lift of g_j satisfying $h_j = r_{y_j}(\epsilon^{m_j})$ (the other lift will be $-h_j = r_{y_j}(-\epsilon^{m_j})$). We note that Γ_j is generated by $\{-I, h_j\}$, and any subgroup $G \subseteq \Gamma_j$ containing $-I$ is generated by $\{-I, h_j^k\}$, where k is the smallest positive integer such that $h_j^k \in G$. For such a subgroup $G \subseteq \Gamma_j$, we see easily that $[\Gamma_j : G] = k$. In particular, e_j is the smallest positive integer such that $h_j^{e_j} \in t_j \Gamma_0 t_j^{-1} \cap \Gamma_j$.

Now, $h_0 y_0 = y_0 \epsilon^{m_0}$. Since $t_j x_0 = x_j$, we have $t_j y_0 = y_j \alpha_j$ for some $\alpha_j \in H(M,q)$. Hence, $t_j^{-1} y_j = y_0 \alpha_j^{-1}$. It follows that $t_j h_0 t_j^{-1} y_j = y_j \epsilon^{m_0}$. In addition, $h_j y_j = y_j \epsilon^{m_j}$.

Let k be the smallest positive integer such that $h = (t_j h_0 t_j^{-1})^k \in \Gamma_j$. Then $\{-I, h\}$ will generate $\Gamma_j \cap t_j \Gamma_0 t_j^{-1}$. We note that k must be the smallest positive integer such that $(\epsilon^{m_0})^k$ is a power of ϵ^{m_j}, or in other words, the smallest positive integer such that $k m_0$ is a multiple of m_j. Hence, $k m_0 = \mathrm{LCM}(m_0, m_j)$, so $h = (t_j h_0 t_j^{-1})^{\mathrm{LCM}(m_0,m_j)/m_0}$, and we see that

$$h y_j = y_j \left(\epsilon^{\mathrm{LCM}(m_0,m_j)}\right) = y_j \left((\epsilon^{m_j})^{\mathrm{LCM}(m_0,m_j)/m_j}\right) = h_j^{\mathrm{LCM}(m_0,m_j)/m_j} y_j.$$

It follows that

$$h = h_j^{\mathrm{LCM}(m_0,m_j)/m_j},$$

and therefore,

$$e_j = \mathrm{LCM}(m_0, m_j)/m_j.$$

Reversing the roles of Γ_0 and Γ_j and switching t_j and t_j^{-1}, we obtain

$$d_j = \mathrm{LCM}(m_0, m_j)/m_0. \qquad \square$$

6. Elements of $H^1(\mathrm{GL}(2,\mathbb{Z}), \mathfrak{M}(M,q)^*)$ interpreted as functions on lattices

We now interpret the cohomology of the dual of $\mathfrak{M}(M,q)$ as a collection of functions on a space of lattices.

Definition 6.1. Let Φ be a function from lattices in K to $\mathbb{F}$. We will say that Φ is *q-homogeneous* if $\Phi(\alpha\Lambda) = q(\alpha)\Phi(\Lambda)$ for all $\alpha \in \mathbb{Z}_{(pdN)}^\times$ and all lattices Λ. We further define $\xi\Phi$ by the formula $(\xi\Phi)(\Lambda) = \Phi(\xi\Lambda)$.

If H is a subgroup of $K^\times$, we will say that Φ is *H-invariant* if $\Phi(\alpha\Lambda) = \Phi(\Lambda)$ for all $\alpha \in H$ and all lattices Λ.

Remark 6.2. Since q is trivial on $H(M,q)$, a function Φ can be both q-homogeneous and $H(M,q)$-invariant. Note that $K(M,q)$-invariance together with q-homogeneity implies $H(M,q)$-invariance, since $H(M,q) = Q(q)K(M,q)$. In addition, since K is a real quadratic field, $q(-1) = 1$. If this were not the case, the fact that $-\Lambda = \Lambda$ for any lattice Λ in K would force all q-homogeneous functions to be identically 0.

Lemma 6.3. *Choose a set $\mathfrak{A}$ of* $\mathrm{GL}(2,\mathbb{Z})$*-orbit representatives of $\mathfrak{X}$ as in Definition 4.4.*

(a) There is a natural ξ-equivariant isomorphism between $H^1(\mathrm{GL}(2,\mathbb{Z}),\mathfrak{M}^*)$ and the vector space of $\mathbb{F}$-valued functions on lattices in K that are $H(M,q)$-invariant.

(b) There is a natural isomorphism between $H^1(\mathrm{GL}(2,\mathbb{Z}),\mathfrak{M}(M,q)^*)$ and the vector space of $\mathbb{F}$-valued functions on lattices in K that are q-homogeneous and $H(M,q)$-invariant.

Proof. (a) The choice of $\mathfrak{A}$ yields an isomorphism of $\mathrm{GL}(2,\mathbb{Z})$-modules

$$f:\mathfrak{M}\to\bigoplus_{x\in\mathfrak{A}}\mathbb{F}\,\mathrm{GL}(2,\mathbb{Z})\otimes_{\mathbb{F}\Gamma_x}\mathbb{F}.$$

This induces an isomorphism (via Shapiro's Lemma)

$$\begin{aligned}H_1(\mathrm{GL}(2,\mathbb{Z}),\mathfrak{M})&\cong\bigoplus_{x\in\mathfrak{A}}H_1(\mathrm{GL}(2,\mathbb{Z}),\mathbb{F}\,\mathrm{GL}(2,\mathbb{Z})\otimes_{\mathbb{F}\Gamma_x}\mathbb{F})\\&\cong\bigoplus_{x\in\mathfrak{A}}H_1(\Gamma_x,\mathbb{F})\\&\cong\bigoplus_{x\in\mathfrak{A}}\mathbb{F}.\end{aligned}$$

Using the natural duality between $H_1(\mathrm{GL}(2,\mathbb{Z}),\mathfrak{M})$ and $H^1(\mathrm{GL}(2,\mathbb{Z}),\mathfrak{M}^*)$, we see that determining an element of $H^1(\mathrm{GL}(2,\mathbb{Z}),\mathfrak{M}^*)$ is the same as giving a function from $\mathfrak{A}$ to $\mathbb{F}$.

The space of lattices $\mathcal{L}$ in K is in bijection with $\mathrm{GL}(2,\mathbb{Z})\backslash Y$, where a lattice Λ corresponds to the $\mathrm{GL}(2,\mathbb{Z})$-orbit of $y={}^t(a,b)\in Y$ where a,b is a $\mathbb{Z}$-basis of Λ. The set $\mathfrak{A}$ consists of a set of representatives for the $\mathrm{GL}(2,\mathbb{Z})$-orbits in $\mathfrak{X}$ and therefore is in bijection with $\mathrm{GL}(2,\mathbb{Z})\backslash Y/H(M,q)$. This is the same as $\mathcal{L}/H(M,q)$, i.e. the set of $H(M,q)$-homothety classes of lattices.

Therefore there is a natural isomorphism between the vector space of functions from $\mathfrak{A}$ to $\mathbb{F}$ and the vector space of $H(M,q)$-invariant functions on lattices in K. For future reference, we can write down this isomorphism explicitly:

If Φ is any $H(M,q)$-invariant function on lattices in K, we get a function g on $\mathfrak{A}$ as follows: Given $x\in\mathfrak{A}$, lift x to $y\in Y$ and set $g(x)=\Phi(\Lambda_y)$, where Λ_y is the lattice spanned by the entries of y.

Conversely, given a function g on $\mathfrak{A}$ and a lattice Λ in K, Λ corresponds (by choosing a basis $\{a,b\}$, setting $y={}^t(a,b)\in Y$ and projecting modulo $H(M,q)$) to an element $x'\in\mathfrak{X}$, which lies in the $\mathrm{GL}(2,\mathbb{Z})$-orbit of a unique $x\in\mathfrak{A}$. Define $\Phi(\Lambda)=g(x)$.

It is clear that this isomorphism is ξ-equivariant, where we define $(\xi g)(x)=g(x\xi)$.

(b) The module $\mathfrak{M}(M,q)$ is the q-isotypic component of ξ acting on $\mathfrak{M}$. Therefore $\mathfrak{M}(M,q)^*$ is the q-isotypic component of ξ acting on $\mathfrak{M}^*$. It follows that $H^1(\mathrm{GL}(2,\mathbb{Z}),\mathfrak{M}(M,q)^*)$ is the q-isotypic component of ξ acting on $H^1(\mathrm{GL}(2,\mathbb{Z}),\mathfrak{M}^*)$.

Therefore the ξ-equivariant isomorphism from part (a) induces an isomorphism between $H^1(\mathrm{GL}(2,\mathbb{Z}),\mathfrak{M}(M,q)^*)$ and the vector space of $\mathbb{F}$-valued functions on lattices in K that are q-homogeneous and $H(M,q)$-invariant, since $\mathbb{Z}^\times_{(pdN)}$ is generated by ξ and $Q(q)$, and $H(M,q)=Q(q)K(M,q)$ (see definition 3.7). ∎

7. The Branched Bruhat-Tits graph and the Laplacian

In order to construct functions on lattices that are eigenfunctions of the Hecke operators, we will use a modification of the Bruhat-Tits building [8, 18], in which we lift the Bruhat-Tits building to a finite branched cover.

For each prime ℓ unramified in K, let K_ℓ denote $K \otimes \mathbb{Q}_\ell$. Then K_ℓ is a two-dimensional vector space over $\mathbb{Q}_\ell$.

Definition 7.1. If ℓ is inert, then K_ℓ is a quadratic field extension of $\mathbb{Q}_\ell$. We fix the integral basis $\{1, \omega\}$ of K, and we identify K_ℓ with $\mathbb{Q}_\ell^2$ by identifying 1 and ω with the standard basis elements $e_1, e_2 \in \mathbb{Q}_\ell^2$.

If ℓ splits in K, then $(\ell) = \lambda\lambda'$ for prime ideals λ, λ' in $\mathfrak{O}$ lying over ℓ. Each of the completions K_λ and K'_λ is then isomorphic to $\mathbb{Q}_\ell$. Restricting these isomorphisms to K, we obtain two distinct Galois conjugate embeddings $i_\lambda, i_{\lambda'} : K \to \mathbb{Q}_\ell$. We then identify $K_\ell = K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ with $\mathbb{Q}_\ell^2$ via the map taking

$$t \otimes 1 \mapsto (i_\lambda(t), i_{\lambda'}(t)).$$

We abbreviate the notation by writing $t \mapsto (t, t')$.

Definition 7.2. By a *lattice* in K_ℓ, we will mean a rank two $\mathbb{Z}_\ell$-submodule of K_ℓ.

If Λ is a lattice in K, then $\Lambda_\ell = \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ is a lattice in K_ℓ.

Definition 7.3. Let ℓ be a prime, and n a positive integer. Denote the elements of $\mathbb{Q}_\ell^\times$ with ℓ-adic valuation divisible by n by V_n. We note that V_n is a subgroup of index n of $\mathbb{Q}_\ell^\times$.

Definition 7.4. Let Λ_1 and Λ_2 be lattices in K_ℓ. We say that Λ_1 and Λ_2 are *n-homothetic* if $\Lambda_1 = \alpha\Lambda_2$ for some $\alpha \in V_n$. Then n-homothety is an equivalence relation, and we call an equivalence class an n-homothety class of lattices in K_ℓ.

Definition 7.5. Let n a positive integer, K a real quadratic field, and ℓ a prime unramified in K . The *branched Bruhat-Tits graph* $\mathcal{T}_\ell^n$ is the graph whose vertices are n-homothety classes of lattices in K_ℓ. Two vertices are joined by an edge if there are representative lattices Λ_1 and Λ_2 of the vertices such that $\Lambda_2 \subset \Lambda_1$ or $\Lambda_1 \subset \Lambda_2$ with index ℓ.

Remark 7.6. The Bruhat-Tits tree is a special case of the branched Bruhat-Tits graph in which $n = 1$. When $n = 1$, we may denote $\mathcal{T}_\ell^n$ by $\mathcal{T}_\ell$. When $n > 1$, we will typically write vertices of $\mathcal{T}_\ell^n$ with a superscript n, i.e. $t^n \in \mathcal{T}_\ell^n$.

Definition 7.7. Let Λ be a lattice in K_ℓ. Denote the vertex of $\mathcal{T}_\ell^n$ represented by Λ by $\varpi(\Lambda)$. Denote the vertex of $\mathcal{T}_\ell$ represented by Λ by $\pi(\Lambda)$. Given a vertex $t^n \in \mathcal{T}_\ell^n$, there is a unique vertex $s \in \mathcal{T}_\ell$ containing t^n; we write $s = \pi(t^n)$.

Note that for any lattice Λ with $\varpi(\Lambda) = t^n$, $\pi(t^n) = \pi(\Lambda)$. To keep our notation less cluttered, if Λ is a lattice in K_ℓ, we will often denote $\varpi(\Lambda)$ by Λ, as long as the context makes this usage clear.

Remark 7.8. We note that for any vertex $t \in \mathcal{T}_\ell$, there are exactly n vertices $t^n \in \mathcal{T}_\ell^n$ with $\pi(t^n) = t$. If Λ is a lattice in K_ℓ representing t, these n vertices of $\mathcal{T}_\ell^n$ are represented by

$$\Lambda, \ell\Lambda, \ldots, \ell^{n-1}\Lambda.$$

Definition 7.9. If t is a vertex of $\mathcal{T}_\ell$, we will call the set $\{t^n \in \mathcal{T}_\ell^n : \pi(t^n) = t\}$ the *fiber* of t and also the *fiber* of t^n for any t^n in that set .

Definition 7.10. A vertex t^n is *idealistic* if t^n is the n-homothety class of the completion I_ℓ of some fractional ideal I of K.

We now review some facts about completions Λ_ℓ of lattices in K. Let ℓ be a prime of $\mathbb{Q}$.

By [19, V.2, Corollary to Theorem 2], the operations of sum and intersection of lattices in K commute with completion at ℓ. In addition, by [19, V.3, Theorem 2], a lattice Λ in K is determined by its set of completions Λ_w for all finite places w of $\mathbb{Q}$. In fact

$$\Lambda = \bigcap_w K \cap \Lambda_w.$$

Finally, completion at a finite place w of finitely generated $\mathbb{Z}$-modules is an exact functor [9, Theorem 7.2].

Applying these facts to fractional ideals of K, we note that if I is an ideal of $\mathfrak{O}$ of norm prime to ℓ, then I_ℓ is an ideal of $\mathfrak{O}_\ell$ of index prime to ℓ, so $I_\ell = \mathfrak{O}_\ell$. In addition, multiplication of relatively prime ideals (i.e. intersection) commutes with completion at ℓ. Hence, for an ideal I, the completion I_ℓ depends only on the factors of I of ℓ-power norm.

Now suppose that $t^n \in \mathcal{T}_\ell^n$ is idealistic. Then we may assume that t^n is represented by an ideal I_ℓ, where I is an ideal in $\mathfrak{O}$ whose norm is a power of ℓ. If ℓ is inert in K, such an I must be principal, so I_ℓ is $\mathbb{Q}_\ell^\times$-homothetic to $\mathfrak{O}_\ell$. Hence, t^n is idealistic if and only if $\pi(t^n)$ is represented by $\mathfrak{O}_\ell$.

On the other hand, if ℓ splits in K, then $\ell\mathfrak{O} = \lambda\lambda'$, where λ, λ' are prime ideals of $\mathfrak{O}$ lying over ℓ. We then see that $t^n \in \mathcal{T}_\ell^n$ is idealistic if and only if $\pi(t^n)$ is represented by an ideal of the form λ^k or $(\lambda')^k$ for some $k \in \mathbb{Z}$. In particular, if $\pi(t_1^n) = \pi(t_2^n)$, then t_1^n and t_2^n are either both idealistic, or both nonidealistic.

Lemma 7.11. *Suppose $n > 1$. Let $\Lambda_1 \supset \Lambda_2$ be lattices in K_ℓ with $[\Lambda_1 : \Lambda_2] = \ell$. Let $t_1^n = \varpi(\Lambda_1) \in \mathcal{T}_\ell^n$. Then there are precisely two vertices $t_2^n, t_3^n \in \mathcal{T}_\ell^n$ with $\pi(t_2^n) = \pi(t_3^n) = \pi(\Lambda_2)$ which are connected by an edge to t_1^n. If we let t_2^n be represented by Λ_2, then t_3^n is represented by $\ell^{-1}\Lambda_2$.*

Proof. Clearly, if we take $t_2^n = \varpi(\Lambda_2)$ and $t_3^n = \varpi(\ell^{-1}\Lambda_2)$, we see that t_2^n and t_3^n are distinct and have the desired properties. It remains to show that there is no third vertex t_4^n, distinct from t_2^n and t_3^n, with $\pi(t_4^n) = \pi(\Lambda_2)$, and such that there is an edge between t_4^n and t_1^n.

Suppose that $\pi(t_4^n) = \pi(\Lambda_2)$ and there is an edge between t_1^n and t_4^n. Then either there is a lattice Λ_4 representing t_4^n such that $\Lambda_1 \supset \Lambda_4$ and $[\Lambda_1 : \Lambda_4] = \ell$ or there is a lattice Λ_4 representing t_4^n such that $\Lambda_1 \subset \Lambda_4$ and $[\Lambda_4 : \Lambda_1] = \ell$.

Now suppose Λ_4 is homothetic to Λ_2, say with $\Lambda_4 = \alpha\Lambda_2$, where $\alpha \in \mathbb{Q}_\ell^\times$.

If $\Lambda_1 \supset \Lambda_4$ has index ℓ, then by hypothesis $\ell^{-1}\Lambda_2 \supset \Lambda_1$ has index ℓ and $\Lambda_1 \supset \alpha\Lambda_2$ has index ℓ. Hence, multiplying by ℓ, we see that $\Lambda_2 \supset \ell\alpha\Lambda_2$ with index ℓ^2. This implies that $v_\ell(\alpha) = 0$, so that $\alpha\Lambda_2 = \Lambda_2$, so $t_4^n = t_2^n$.

On the other hand, if $\Lambda_1 \subset \Lambda_4$ with index ℓ, then $\Lambda_2 \subset \Lambda_1$ has index ℓ by hypothesis and $\Lambda_1 \subset \alpha\Lambda_2$ with index ℓ, so $\Lambda_2 \subset \alpha\Lambda_2$ has index ℓ^2. Hence $v_\ell(\alpha) = -1$, and we see that $\alpha\Lambda_2 = \ell^{-1}\Lambda_2$, so $t_4^n = t_3^n$. ☐

Corollary 7.12. *Let $n > 1$, let t^n be a vertex in $\mathcal{T}_\ell^n$, and let $t = \pi(t^n) \in \mathcal{T}_\ell$. Let $s \in \mathcal{T}_\ell$ be a neighbor of t. Then there are exactly two neighbors s_1^n and s_2^n of t^n in $\mathcal{T}_\ell^n$ with $\pi(s_1^n) = \pi(s_2^n) = s$. If Λ represents t^n, then exactly one of s_1^n and s_2^n*

is represented by a sublattice Λ' of Λ of index ℓ; the other is represented by $\ell^{-1}\Lambda'$, which contains Λ with index ℓ.

Definition 7.13. Let $n \geq 1$, let $t^n \in \mathcal{T}_\ell^n$ be a vertex represented by a lattice Λ in K_ℓ, and let s^n be a neighbor of t^n. If s^n is represented by a sublattice of index ℓ in Λ, we call s^n a *downhill* neighbor of t^n; if it is represented by a lattice containing Λ with index ℓ, we call s^n an *uphill* neighbor of t^n.

We note that if $n = 1$, any neighbor of t^n is both an uphill and a downhill neighbor of t^n.

Definition 7.14. Let $t^n \in \mathcal{T}_\ell^n$. We define the *tier* of t^n to be the distance between $\pi(t^n)$ and $\pi(\mathfrak{O}_\ell)$ in $\mathcal{T}_\ell$. A neighbor of t^n of higher tier than t^n will be called an *outer neighbor* of t^n: a neighbor of lower tier will be called an *inner neighbor.*

Remark 7.15. Each $t^n \in \mathcal{T}_\ell^n$ has precisely $\ell + 1$ downhill neighbors and $\ell + 1$ uphill neighbors. The use of uphill and downhill matches our intuition; if s^n is a downhill neighbor of t^n, then t^n is an uphill neighbor of s^n. (However, as in an Escher staircase, it is possible to go uphill several times and return to your starting point without going downhill.)

Each vertex of positive tier has precisely ℓ downhill outer neighbors, and 1 downhill inner neighbor. It also has precisely ℓ uphill outer neighbors, and 1 uphill inner neighbor.

A vertex of tier 0 has only outer neighbors; $\ell + 1$ of them are uphill, and $\ell + 1$ are downhill. The following definition names a particular vertex of tier 0.

Definition 7.16. Let $t_0^n \in \mathcal{T}_\ell^n$ be the vertex represented by the lattice $\mathfrak{O}_\ell$.

There is a natural action of the group $\mathrm{GL}(2, \mathbb{Q}_\ell)$ on $\mathbb{Q}_\ell^2$, namely matrix multiplication with elements of $\mathbb{Q}_\ell^2$ considered as column vectors. We transfer this action to K_ℓ via the identification that we have made between K_ℓ and $\mathbb{Q}_\ell^2$. The action of $g \in \mathrm{GL}(2, \mathbb{Q}_\ell)$ is invertible, and preserves $\mathbb{Q}_\ell$-linear combinations, so it maps bases of $\mathbb{Q}_l^2$ to bases, maps lattices to lattices, and preserves n-homothety of lattices. Hence, multiplication by g defines a bijection from $\mathcal{T}_\ell^n$ to $\mathcal{T}_\ell^n$. We now record some properties of this action.

Lemma 7.17. *The action of an element $\gamma \in \mathrm{GL}(2, \mathbb{Z}_\ell)$ on $\mathcal{T}_\ell^n$ permutes the vertices of $\mathcal{T}_\ell^n$, fixes vertices of tier 0, and preserves edges (including whether the edge is uphill or downhill) and the tier of each vertex.*

Proof. Since the action of $\gamma \in \mathrm{GL}(2, \mathbb{Z}_\ell)$ is invertible, it is clear that the map it induces on vertices is a bijection. In addition, if $\Lambda_1 \subset \Lambda_2$ are lattices in K_ℓ with $[\Lambda_2 : \Lambda_1] = \ell$, then $\gamma\Lambda_1 \subset \gamma\Lambda_2$ with index ℓ, so edges are preserved (including whether the edge is uphill or downhill).

Since the action of γ stabilizes $\mathbb{Z}_\ell^2$, which is identified with $\mathfrak{O}_\ell$, it fixes vertices of tier 0. Since it preserves neighbors, a simple inductive argument shows that it maps each vertex to a vertex of the same tier. □

Lemma 7.18. *Multiplication by the fundamental unit $\epsilon \in K \subset K_\ell$ induces a permutation on the vertices of $\mathcal{T}_\ell^n$ given (on the level of $\mathbb{Z}_\ell$-lattices) by multiplication by a matrix in $\mathrm{GL}(2, \mathbb{Z}_\ell)$.*

Proof. Suppose that ℓ is inert in K. In this case (see Definition 7.1), we have identified $K_\ell = \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell\omega$ with $\mathbb{Q}_\ell^2$. Since multiplication by ϵ is $\mathbb{Q}$-linear on K

it induces a $\mathbb{Q}_\ell$-linear map on K_ℓ. Hence, multiplication by ϵ is represented by a matrix in $\mathrm{GL}(2,\mathbb{Q}_\ell)$. Since multiplication by ϵ is an automorphism of $\mathfrak{O}_\ell$ and $\mathfrak{O}_\ell$ is identified with $\mathbb{Z}_\ell^2 \subset \mathbb{Q}_\ell^2$, this matrix has entries in $\mathbb{Z}_\ell$, and since ϵ has norm ± 1, the matrix must have determinant ± 1, so we see that the matrix is in $\mathrm{GL}(2,\mathbb{Z}_\ell)$.

Now suppose that ℓ is split. Referring to Definition 7.1 again, we have identified K_ℓ with $\mathbb{Q}_\ell^2$, where $c \in K$ is identified with $(c, c') \in \mathbb{Q}_\ell^2$. Hence, multiplication by ϵ is represented by the matrix $\mathrm{diag}(\epsilon, \epsilon')$, which is in $\mathrm{GL}(2,\mathbb{Z}_\ell)$. □

We will define functions on lattices in K as products of local functions on lattices in K_ℓ. To that end, we make the following definitions.

Definition 7.19. Let $F(\mathcal{T}_\ell^n)$ be the set of $\mathbb{F}$-valued functions on the vertices of $\mathcal{T}_\ell^n$.

Definition 7.20. The *Laplace operator* Δ_ℓ^n on $F(\mathcal{T}_\ell^n)$ is defined by

$$\Delta_\ell^n(f)(t^n) = \sum_{u^n} f(u^n),$$

where the sum runs over the $\ell+1$ downhill neighbors $u^n \in \mathcal{T}_\ell^n$ of $t^n \in \mathcal{T}_\ell^n$.

Now we concentrate on the Hecke operator at ℓ and describe how its coset representatives interact with lattices. We assume from now on that ℓ is unramified in K. We choose $s = \mathrm{diag}(\ell, 1)$ and refer to Definition 5.4 for the sets B_j and the integers d_j. List $B_j = \{s_{\beta,j} \mid \beta = 1, \ldots, d_j\}$. These definitions as well as the following definition depend on various choices, such as an element $x_0 \in \mathfrak{X}$, a lift of x_0 to Y, and a choice of $\mathrm{GL}(2,\mathbb{Z})$-orbit representatives $\mathfrak{A}$ containing x_0. We will suppress these dependencies so as not to overburden the notation.

Definition 7.21. Let x_0 be represented by $y = {}^t(a, b) \in Y$ with $a, b \in K$, and let Λ_y be the $\mathbb{Z}$-lattice in K generated by a and b. For any j and any $s_\alpha \in B_j$, define

$$s_\alpha \Lambda_y = \Lambda_{s_\alpha y}.$$

Note that $s_\alpha \Lambda_y$ depends not just on the lattice Λ_y, but also on the choice of basis $y = {}^t(a_0, b_0) \in Y$ for Λ_y.

Lemma 7.22. *Let $s = \mathrm{diag}(\ell, 1)$ and let*

$$\mathrm{GL}(2,\mathbb{Z}) s \mathrm{GL}(2,\mathbb{Z}) = \coprod_\alpha \mathrm{GL}(2,\mathbb{Z}) s_\alpha$$

with the s_α chosen and partitioned into the sets B_j as described in Definition 5.4. Let Λ_y be a lattice in K with a chosen basis $y \in Y$. Then

(i) $\mathcal{L} = \{s_\alpha \Lambda_y\}$ consists of the $\ell+1$ lattices of index ℓ contained in Λ_y.

(ii) $\mathcal{L}$ is partitioned into the subsets

$$\mathcal{L}_j = \{s_{\beta,j}\Lambda_y | s_{\beta,j} \in B_j\},$$

and $|\mathcal{L}_j| = d_j$.

(iii) The same is true of the completions at ℓ: $\mathcal{L}_\ell = \{(s_\alpha \Lambda_y)_\ell\}$ consists of the $\ell+1$ lattices of index ℓ contained in $(\Lambda_y)_\ell$, and these are partitioned into the subsets

$$\mathcal{L}_{\ell,j} = \{(s_{\beta,j}\Lambda_y)_\ell | s_{\beta,j} \in B_j\}$$

and $|\mathcal{L}_{\ell,j}| = d_j$.

Proof. (i) Since s_α is an integral matrix of determinant ℓ, it is clear that $s_\alpha\Lambda_y$ has index ℓ in Λ_y, and all sublattices of Λ_y of index ℓ arise this way.

(ii) Since $\{s_\alpha\}$ is partitioned by the sets B_j, it is clear that the lattices are partitioned as indicated.

(iii) If Λ has index ℓ in Λ_y, then the completion Λ_ℓ has index ℓ in $(\Lambda_y)_\ell$, since taking completions of finitely generated modules is an exact functor. Given two lattices $\Lambda \neq \Lambda'$, each having index ℓ in Λ_y, we note that for all places $w \neq \ell$, $\Lambda_w = \Lambda'_w = (\Lambda_y)_w$. Since a lattice is determined by its completions at all finite places, we must have $\Lambda_\ell \neq \Lambda'_\ell$.

□

Definition 7.23. Let $\phi^n_\ell \in F(\mathcal{T}^n_\ell)$, let $x_0 \in \mathfrak{X}$, and let $\mathfrak{A}$ be any set of $\mathrm{GL}(2,\mathbb{Z})$-orbit representatives of $\mathfrak{X}$ containing x_0, as in Definition 4.4. Define the sets B_j in terms of x_0 and $\mathfrak{A}$ as in Definition 5.4. If, for all choices of $\mathfrak{A}$ and for all $y \in Y$ representing x_0, and for all $j = 1, \ldots, J$, we have that ϕ^n_ℓ is constant on the set

$$\{(s_{\beta,j}\Lambda_y)_\ell | \beta = 1, \ldots, d_j\}$$

of vertices of $\mathcal{T}^n_\ell$, then we will say that ϕ^n_ℓ is *locally constant* relative to T_ℓ and x_0.

If ϕ^n_ℓ is locally constant relative to T_ℓ and all $x_0 \in \mathfrak{X}$, then we say that ϕ^n_ℓ is *locally constant* relative to T_ℓ, or just that ϕ^n_ℓ is *locally constant*.

Lemma 7.24. *Let $\phi \in F(\mathcal{T}^n_\ell)$ be a function on the vertices of $\mathcal{T}^n_\ell$. Assume that for every vertex $t^n \in \mathcal{T}^n_\ell$, ϕ is constant on the set of non-idealistic outer downhill neighbors u^n of t^n. Then ϕ is locally constant relative to T_ℓ.*

Proof. Assume that ϕ satisfies the conditions of the lemma. Let $x_0 \in \mathfrak{X}$, choose any collection $\mathfrak{A}$ of orbit representatives containing x_0 as in Definition 4.4, and choose any $y \in Y$ representing x_0. Partition the set $\{s_\alpha\}$ of coset representatives for the Hecke operator T_ℓ as in Definition 5.4.

Let t^n be the vertex of $\mathcal{T}^n_\ell$ represented by Λ_y. For each set B_j, we wish to show that ϕ is constant on the set $\{(s_{\beta,j}\Lambda_y)_\ell | s_{\beta,j} \in B_j\}$. Let $1 \leq j \leq J$, choose any $s_{\beta,j} \in B_j$, and let u^n be the downhill neighbor of t^n represented by $(\Lambda_j)_\ell$, where $\Lambda_j = s_{\beta,j}\Lambda_y$. Then Λ_j is $H(M,q)$-homothetic to a lattice with a basis representing x_j (where $x_j \in \mathfrak{A}$ is given in Definition 5.4). We now divide the proof into 3 cases.

Case 1: Suppose u^n is idealistic. Then Λ_j is a fractional ideal of K, and it is $H(M,q)$-homothetic to a fractional ideal with basis representing x_j. Hence, $m_j = i^M$, so $d_j = 1$ by Lemmas 3.12 and 5.8. Hence, there is only one vertex on which ϕ must be constant.

Case 2: Suppose that u^n is the unique downhill inner neighbor of t^n. Recall from Theorem 2.8 that the stabilizer Γ_{x_0} of x_0 in $\mathrm{GL}(2,\mathbb{Z})$ is generated by $-I$ and an element γ_0 that acts on Λ_y as multiplication by $\delta_0 = \epsilon^m$ where $m = m_{x_0}$. From Theorem 2.8, we see that

$$\gamma_0\Lambda_y = \delta_0\Lambda_y = \Lambda_y.$$

Since multiplication by δ_0 fixes Λ_y, it also fixes $(\Lambda_y)_\ell = t^n$. By Lemma 7.18, multiplication by δ_0 also fixes each element of the fiber of $t^n_0 = \varpi(\mathfrak{O}_\ell)$. Therefore it fixes the unique downhill path from t^n to the fiber of t^n_0. Hence, multiplication by δ_0 must fix u^n.

Now, both Λ_j and $\delta_0\Lambda_j$ are sublattices of Λ_y of index ℓ. Since both must represent u^n, we see that they are equal. Since $\delta_0\Lambda_j = \Lambda_j$, we see that $m_j | m_0$, so that $d_j = 1$. Hence, again, there is only one vertex on which ϕ must be constant.

Case 3: Suppose u^n is a nonidealistic outer downhill neighbor of t^n. By cases 1 and 2, no vertex in $\{(s_{\beta,j}\Lambda_y)_\ell | s_{\beta,j} \in B_j\}$ can be idealistic or a downhill inner neighbor of t_n. Hence, ϕ is constant (by hypothesis) on all the vertices in the desired set. □

8. Functions on lattices; comparison between the Laplacian and a Hecke operator

In this section, we study functions on lattices in K_ℓ; in particular, we will show that, given an eigenfunction of the Laplacian Δ_ℓ^n, we can construct an eigenfunction of the Hecke operator T_ℓ. We will use this construction later in the paper to construct eigenfunctions of Hecke operators by performing the easier task of constructing eigenfunctions of the Laplacian.

Definition 8.1. Let $q : K_{(pdN)}^\times \to \mathbb{F}^\times$ be the character defined in Definition 3.5, and let ℓ be a prime of $\mathbb{Z}$ that does not divide pdN. We say that a function $f \in F(\mathcal{T}_\ell^n)$ is *q-homogeneous* (or just *homogeneous*, if q is understood) if, for all lattices Λ in K,

$$f(\ell\Lambda_\ell) = q(\ell)f(\Lambda_\ell).$$

Definition 8.2. For all finite places w of $\mathbb{Q}$ unramified in K, let $n_w = 2$ if w is inert in K, let $n_w = 1$ if w splits in K. Fix a prime ℓ of $\mathbb{Q}$ not dividing pdN, and let W be the set of all finite places of $\mathbb{Q}$ not dividing ℓpdN. For $w \in W$, let $\phi_w \in F(\mathcal{T}_w^{n_w})$ denote a homogeneous function such that $\phi_w(\mathfrak{O}_w) = 1$. We view the functions ϕ_w as fixed by the context, and do not include them in the following notation for Φ. For any homogeneous $\phi_\ell \in F(\mathcal{T}_\ell^{n_\ell})$, define the function $\Phi(\phi_\ell)$ on lattices Λ in K by the formula

$$\Phi(\phi_\ell)(\Lambda) = \phi_\ell(\Lambda_\ell) \prod_{w\in W} \phi_w(\Lambda_w).$$

Lemma 8.3. *The infinite product in the definition makes sense and $\Phi(\phi_\ell)$ is q-homogeneous. The map $\phi_\ell \mapsto \Phi(\phi_\ell)$ is $\mathbb{F}$-linear.*

Proof. For any given Λ, we have that $\Lambda_w = \mathfrak{O}_w$ for almost all w, so the product is actually finite. The linearity of the map $\phi_\ell \mapsto \Phi(\phi_\ell)$ is clear. Now suppose $\alpha \in \mathbb{Z}_{(pdN)}^\times$ and Λ is a lattice. Then α is prime to pdN and factors as

$$\alpha = \ell^{f_\ell} \prod_{w\in W} w^{f_w}.$$

Then

$$\Phi(\phi_\ell)(\alpha\Lambda) = \phi_\ell(\alpha\Lambda_\ell) \prod_{w\in W} \phi_w(\alpha\Lambda_w) = \phi_\ell(\ell^{f_\ell}\Lambda_\ell) \prod_{w\in W} \phi_w(w^{f_w}\Lambda_w).$$

Since ϕ_ℓ and all the ϕ_w are homogeneous, this equals

$$q(\ell^{f_\ell})\phi_\ell(\Lambda_\ell) \left(\prod_{w\in W} q(w^{f_w})\right) \left(\prod_{w\in W} \phi_w(\Lambda_w)\right) = q(\alpha)\Phi(\phi_\ell)(\Lambda).$$ □

We now proceed to the main theorem of this section: the comparison between the Hecke operator and the Laplace operator.

By Lemma 2.5 and the fact that $H(M, q) \cap \mathfrak{O}^\times$ is infinite, we see that for any lattice $\Lambda \subseteq K$, there is a minimal positive integer m_Λ such that both $\epsilon^{m_\Lambda}\Lambda = \Lambda$

and $\epsilon^{m_\Lambda} \in H(M,q)$. If Λ_1 and Λ_2 are $K^\times$-homothetic lattices in K, it is clear that $m_{\Lambda_1} = m_{\Lambda_2}$. Set $m'_\Lambda = m_\Lambda / i^M$. By Theorem 2.8, if $\Lambda = \Lambda_y$ for $y \in Y$, and x is the image in $\mathfrak{X}$ of y then $m_\Lambda = m_x$. Therefore, by Lemma 3.12, $i^M | m_\Lambda$ and m'_Λ is a positive integer.

Definition 8.4. Let $\psi_\ell \in F(\mathcal{T}_\ell^{n_\ell})$. We define the *transform* of ψ_ℓ to be the function $\hat{\psi}_\ell \in F(\mathcal{T}_\ell^n)$ given by the formula

$$\hat{\psi}_\ell(t^n) = m'_\Lambda \psi_\ell(t^n),$$

where Λ is any lattice in $\mathfrak{O}$ of ℓ-power index, such that Λ_ℓ represents t^n.

Lemma 8.5. *Given $\psi_\ell \in F(\mathcal{T}_\ell^n)$, the transform $\hat{\psi}_\ell$ is well defined.*

Proof. We need to show that for $t^n \in \mathcal{T}_\ell^n$, the value of m_Λ does not depend on the lattice Λ chosen to represent t^n. Note that up to homothety by powers of ℓ^n, there is a unique lattice $\Lambda' \subseteq \mathfrak{O}_\ell$ representing t^n. By [19, V.2, Theorem 2] there is a unique lattice $\Lambda \subseteq \mathfrak{O}$ of ℓ-power index such that $\Lambda_\ell = \Lambda'$. Since Λ' is uniquely defined up to homothety by powers of ℓ^n, so too is Λ. Finally, since homothety does not change the value of m_Λ, we see that m_Λ does not depend on the choice of Λ, so m'_Λ does not. □

If $\psi_\ell(\mathfrak{O}_\ell) = 1$, then $\hat{\psi}_\ell(\mathfrak{O}_\ell) = 1$, since $m'_\mathfrak{O} = 1$.

Lemma 8.6. *Let $\ell \nmid pdN$ be prime. If $\psi_\ell \in F(\mathcal{T}_\ell^n)$ is homogeneous, then $\hat{\psi}_\ell$ is also homogeneous.*

Proof. If $t^n \in \mathcal{T}_\ell^n$ is represented by Λ_ℓ, with Λ a lattice of ℓ-power index in $\mathfrak{O}$, then ℓt^n is represented by $\ell\Lambda_\ell$. Since $m'_\Lambda = m'_{\ell\Lambda}$, we have

$$\hat{\psi}_\ell(\ell t^n) = m'_{\ell\Lambda}\psi_\ell(\ell t^n) = m'_\Lambda q(\ell)\psi_\ell(t^n) = q(\ell)\hat{\psi}_\ell(t^n).$$ □

We now fix a set $\mathfrak{A}$ of representatives of the $\mathrm{GL}(2,\mathbb{Z})$-orbits in $\mathfrak{X}$, as in Definition 4.4. Recall from Lemma 6.3 that this choice fixes an isomorphism between the cohomology group

$$H^1(\mathrm{GL}(2,\mathbb{Z}), \mathfrak{M}(M,q)^*)$$

and q-homogeneous, $H(M,q)$-invariant functions on lattices.

As in Lemma 6.3 and its proof, if $\Phi(\hat{\psi}_\ell)$ is $H(M,q)$-invariant and q-homogeneous, view $\Phi(\hat{\psi}_\ell)$ as an element of

$$H^1(\mathrm{GL}(2,\mathbb{Z}), \mathfrak{M}(M,q)^*) \cong H_1(\mathrm{GL}(2,\mathbb{Z}), \mathfrak{M}(M,q))^*.$$

That is to say, view $\Phi(\hat{\psi}_\ell)$ as an $\mathbb{F}$-valued functional on $H_1(\mathrm{GL}(2,\mathbb{Z}), \mathfrak{M}(M,q))$, via the pairing

$$\langle \Phi(\hat{\psi}_\ell), \bullet \rangle : H_1(\mathrm{GL}(2,\mathbb{Z}), \mathfrak{M}(M,q)) \to \mathbb{F}.$$

The relation between $\Phi(\hat{\psi}_\ell)$ as a function on lattices and as a functional on homology is as follows: for a basis element z_x of $H_1(\mathrm{GL}(2,\mathbb{Z}), \mathfrak{M})$ with $x \in \mathfrak{A}$, we have that

$$\langle \Phi(\hat{\psi}_\ell), z_x \rangle = \Phi(\hat{\psi}_\ell)(\Lambda_y),$$

where $y \in Y$ is any representative of x. Then, from the definition of $\hat{\psi}_\ell$, we have that

$$\langle \Phi(\hat{\psi}_\ell), z_x \rangle = \Phi(\hat{\psi}_\ell)(\Lambda_y) = m'_x \Phi(\psi_\ell)(\Lambda_y),$$

a fact that we will use in the proofs of Theorem 8.8 and Corollary 8.9.

Definition 8.7. Let z_{x_j} be the chosen generator of $H_1(\Gamma_{x_j}, \mathbb{F})$ from Definition 2.11, and set $z_j = z_{x_j}$ and $m_j = m_{x_j}$ for $j = 0, \ldots, J$.

Theorem 8.8. *Let $\ell \nmid pdN$ be prime. For each finite place $w \in W$, fix a q-homogeneous function $\phi_w \in F(\mathcal{T}_\ell^{n_w})$, as in Definition 8.2. Let $n = n_\ell$, and let $\psi_\ell \in F(\mathcal{T}_\ell^n)$ be q-homogeneous. Assume that $\Phi(\hat{\psi}_\ell)$ is $H(M, q)$-homothety invariant. It will be q-homogeneous by Lemma 8.3.*

If ψ_ℓ is locally constant relative to T_ℓ and x_0, then

$$\langle \Phi(\hat{\psi}_\ell) T_\ell, z_0 \rangle = m_0' \Phi(\Delta_\ell^n \psi_\ell)(\Lambda_y),$$

where $y \in Y$ represents $x_0 \in \mathfrak{A}$.

Proof. Choose a $y \in Y$ representing x_0. Then for $j = 1, \ldots, J$, set $y_j = t_j y$, so that y_j represents x_j. We will write Λ_j in place of Λ_{y_j}.

By Theorem 5.6, $T_\ell = \sum_{j=1}^J U_j$, and for $1 \le j \le J$, we have

$$U_j(z_0) = e_j z_j.$$

Then

$$\begin{aligned} \langle \Phi(\hat{\psi}_\ell) T_\ell, z_0 \rangle &= \langle \Phi(\hat{\psi}_\ell), T_\ell z_0 \rangle \\ &= \sum_{j=1}^J e_j \langle \Phi(\hat{\psi}_\ell), z_j \rangle \\ &= \sum_{j=1}^J e_j \Phi(\hat{\psi}_\ell)(\Lambda_j) \\ &= \sum_{j=1}^J e_j m_j' \Phi(\psi_\ell)(\Lambda_j). \end{aligned}$$

We have $e_j m_j' = m_0' d_j$ by Lemma 5.8 and Definition 3.13, so

$$\langle \Phi(\hat{\psi}_\ell) T_\ell, z_0 \rangle = \sum_{j=1}^J m_0' d_j \Phi(\psi_\ell)(\Lambda_j).$$

Now, for a fixed j, we will analyze the term $\Phi(\psi_\ell)(\Lambda_j)$. Note that by definition,

$$\Phi(\psi_\ell)(\Lambda_j) = \psi_\ell((\Lambda_j)_\ell) \prod_{w \in W} \phi_w((\Lambda_j)_w).$$

Since t_j is an integral matrix with determinant ℓ, we know that $t_j \in \mathrm{GL}(2, \mathfrak{O}_w)$ for all $w \in W$. Then $(\Lambda_j)_w$ is the same as the lattice $(\Lambda_y)_w$. Set

$$c = \prod_{w \in W} \phi_w((\Lambda_y)_w),$$

so that

$$\Phi(\psi_\ell)(\Lambda_j) = \psi_\ell((\Lambda_j)_\ell) \cdot c.$$

Hence,

$$\langle \Phi(\hat{\psi}_\ell) T_\ell, z_0 \rangle = c m_0' \sum_{j=1}^J d_j \psi_\ell((\Lambda_j)_\ell).$$

On the other hand, since ψ_ℓ is assumed to be locally constant with respect to T_ℓ and x_0, any $s_{\beta,j}$ takes any vertex to a downhill neighbor, and one of the $s_{\beta,j}$ is equal to t_j (see Theorem 5.3(c)), we have that for each $s_{\beta,j}$,

$$\begin{aligned}\psi_\ell((s_{\beta,j}\Lambda_y)_\ell) &= \psi_\ell((t_j\Lambda_y)_\ell)\\ &= \psi_\ell((\Lambda_j)_\ell).\end{aligned}$$

Hence, using the fact that $d_j = |B_j|$, we have

$$\begin{aligned}\Phi(\Delta_\ell^n\psi_\ell)(\Lambda_y) &= (\Delta_\ell^n\psi_\ell)((\Lambda_y)_\ell)\prod_{w\in W}\phi_w((\Lambda_y)_w)\\ &= c(\Delta_\ell^n\psi_\ell)((\Lambda_y)_\ell)\\ &= c\sum_{j=1}^{J}\sum_{s_{\beta,j}\in B_j}\psi_\ell((s_{\beta,j}\Lambda_y)_\ell)\\ &= c\sum_{j=1}^{J} d_j\psi_\ell((\Lambda_j)_\ell),\end{aligned}$$

where we have used Lemma 7.22. Multiplying both sides of the last equality by m'_0 yields the assertion of the theorem. □

Corollary 8.9. *Assume that ψ_ℓ is locally constant relative to T_ℓ, that $\psi_\ell(\mathfrak{O}_\ell) = 1$, that $\Phi(\hat{\psi}_\ell)$ is q-homogeneous and $H(M,q)$-invariant, and that $\Delta_\ell^{n_\ell}\psi_\ell = \mu\psi_\ell$.*

Then $\Phi(\hat{\psi}_\ell)$, viewed as an element of $H^1(\mathrm{GL}(2,\mathbb{Z}),\mathfrak{M}(M,q)^*)$, is an eigenclass for T_ℓ with eigenvalue μ and it is an eigenclass for $T_{\ell,\ell}$ with eigenvalue $\theta(\ell)$.

Proof. First, we show that $\Phi(\hat{\psi}_\ell) \neq 0$. By definition, for Λ a lattice in K,

$$\Phi(\hat{\psi}_\ell)(\Lambda) = \hat{\psi}_\ell(\Lambda_\ell)\prod_{w\in W}\phi_w(\Lambda_w).$$

By construction, $\phi_w(\mathfrak{O}_w) = 1$ for every $w \in W$ and $\psi_\ell(\mathfrak{O}_\ell) = 1$. Since $m'_{\mathfrak{O}_\ell} = 1$, also $\hat{\psi}_\ell(\mathfrak{O}_\ell) = 1$. Therefore, $\Phi(\hat{\psi}_\ell)(\mathfrak{O}) = 1$.

Recall from Definition 4.4 that we have chosen a subset $\mathcal{A} \subset \mathfrak{A}$, such that for every $x \in \mathfrak{A}$, exactly one of x and $x\xi$ is in $\mathcal{A}$. For any $x \in \mathcal{A}$, write $\widetilde{z}_x = z_x - z_{x\xi}$. Then by Lemma 5.1, $\{\widetilde{z}_x : x \in \mathcal{A}\}$ is a basis of $H_1(\mathrm{GL}(2,\mathbb{Z}),\mathfrak{M}(M,q))$. By Theorem 8.8, linearity, and the centrality of ξ, for each $x \in \mathcal{A}$ we have

$$\begin{aligned}\langle\Phi(\hat{\psi}_\ell)T_\ell,\widetilde{z}_x\rangle &= \langle\Phi(\hat{\psi}_\ell)T_\ell,z_x\rangle - \langle\Phi(\hat{\psi}_\ell)T_\ell,z_{x\xi}\rangle\\ &= m'_x\big(\Phi(\Delta_\ell^{n_\ell}\psi_\ell)(\Lambda_y) - \Phi(\Delta_\ell^{n_\ell}\psi_\ell)(\Lambda_{y\xi})\big)\\ &= m'_x\big(\Phi(\mu\psi_\ell)(\Lambda_y) - \Phi(\mu\psi_\ell)(\Lambda_{y\xi})\big)\\ &= \mu\big(\Phi(m'_x\psi_\ell)(\Lambda_y) - \Phi(m'_x\psi_\ell)(\Lambda_{y\xi})\big)\\ &= \mu\big(\Phi(\hat{\psi}_\ell)(\Lambda_y) - \Phi(\hat{\psi}_\ell)(\Lambda_{y\xi})\big)\\ &= \mu\big(\langle\Phi(\hat{\psi}_\ell),z_x\rangle - \langle\Phi(\hat{\psi}_\ell),z_{x\xi}\rangle\big)\\ &= \langle\mu\Phi(\hat{\psi}_\ell),\widetilde{z}_x\rangle.\end{aligned}$$

Since $\Phi(\hat{\psi}_\ell)$ is in the dual space to $H_1(\mathrm{GL}(2,\mathbb{Z}),\mathfrak{M}(M,q))$, and $\{\widetilde{z}_x : x \in \mathcal{A}\}$ spans $H_1(\mathrm{GL}(2,\mathbb{Z}),\mathfrak{M}(M,q))$, we are finished with T_ℓ.

As for $T_{\ell,\ell}$, its action is given by the double coset of the central element ℓI. This is just a single coset, and its action on homology is given by $q(\ell)$, since it acts on $\mathfrak{M}(M,q)$ as multiplication by $q(\ell)$.

Since $q(\ell) = \theta(\ell)$,

$$\langle \Phi(\hat{\psi}_\ell)T_{\ell,\ell}, \widetilde{z}_x\rangle = \langle \Phi(\hat{\psi}_\ell), T_{\ell,\ell}\widetilde{z}_x\rangle = \langle \Phi(\hat{\psi}_\ell), \theta(\ell)\widetilde{z}_x\rangle = \langle \theta(\ell)\Phi(\hat{\psi}_\ell), \widetilde{z}_x\rangle.$$

Hence, $\Phi(\hat{\psi}_\ell)T_{\ell,\ell} = \theta(\ell)\Phi(\hat{\psi}_\ell)$. □

9. Constructing locally constant eigenfunctions

Fix an $\mathbb{F}$-valued character χ on the group of fractional ideals of K relatively prime to N. In this section, we will construct locally constant q-homogeneous functions ψ_ℓ^0 on $\mathcal{T}_\ell^{n_\ell}$ that are eigenfunctions of the Laplace operator with eigenvalues related to χ. We do this first for inert primes ℓ.

Theorem 9.1. *Let ℓ be a prime of $\mathbb{Q}$ that is inert in $K/\mathbb{Q}$ and does not equal the characteristic of $\mathbb{F}$. Then there is a locally constant q-homogeneous function $\psi_\ell^0 \in F(\mathcal{T}_\ell^2)$ that is an eigenvector of the Laplace operator with eigenvalue 0 and satisfies $\psi_\ell^0(\mathfrak{O}_\ell) = 1$.*

Proof. We define ψ_ℓ^0 inductively.

For vertices of tier 0, we define $\psi_\ell^0(\mathfrak{O}_\ell) = 1$ and $\psi_\ell^0(\ell\mathfrak{O}_\ell) = \theta(\ell) = -1$. Then ψ_ℓ^0 is homogeneous on the vertices of tier 0.

On vertices $t^2 \in \mathcal{T}_\ell^2$ of tier 1, we define $\psi_\ell^0(t^2) = 0$. Clearly ψ_ℓ^0 is q-homogeneous on vertices of tier 1. In addition, since all downhill neighbors of a vertex of tier 0 have tier 1, we can now compute $\Delta_\ell^2(\psi_\ell^0)$ on vertices of tier 0; we find that its value is 0, as desired. Finally, ψ_ℓ^0 is constant on all downhill neighbors of vertices of tier 0.

On each vertex $t^2 \in \mathcal{T}_\ell^2$ of tier 2, let $u^2 \in \mathcal{T}_\ell^2$ be the unique uphill neighbor of t^2 of tier 1, and we let v^2 be the unique downhill neighbor of u^2 of tier 0. We define $\psi_\ell^0(t^2) = -\psi_\ell^0(v^2)/\ell$. Because the unique uphill neighbor of ℓt^2 of tier 1 is ℓu^2, which has a unique downhill neighbor of tier 0 equal to ℓv^2, we see that with this definition, ψ_ℓ^0 is homogeneous on vertices of tier 1. In addition, for any vertex u^2 of tier 1, ψ_ℓ^0 is constant on the downhill neighbors of u^2 of higher tier, since its value on such vertices depends only on its value on the unique downhill inner neighbor of u^2. Finally, we have constructed ψ_ℓ^0 so that

$$\Delta_\ell^2(\psi_\ell^0)(u^2) = 0$$

for each vertex u^2 of tier 1.

We continue; for vertices $t^2 \in \mathcal{T}_\ell^2$ of odd tier, we define $\psi_\ell^0(t^2) = 0$. This guarantees that for vertices u^2 of even tier, $\Delta_\ell^2(\psi_\ell^0)(u^2) = 0$, and that ψ_ℓ^0 is constant on all downhill neighbors of u^2 of higher tier. Further, with this definition, $\psi_\ell^0(\ell t^2) = 0 = \theta(\ell)\psi_\ell^0(t^2)$ so that ψ_ℓ^0 is homogeneous on vertices of odd tier.

For a vertex $t^2 \in \mathcal{T}_\ell^2$ of positive even tier, let u^2 be the unique uphill inner neighbor of t^2, and let v^2 be the unique downhill inner neighbor of u^2. We define $\psi_\ell^0(t^2) = -\psi_\ell^0(v^2)/\ell$. Clearly ψ_ℓ^0 is constant on all downhill outer neighbors of u^2 (since its value on such neighbors depends only on its value on v^2). As in the case of tier 2, we see that $\psi_\ell^0(\ell t^2) = \theta(\ell)\psi_\ell^0(t^2)$, and $\Delta_\ell^2(\psi_\ell^0)(u^2) = \psi_\ell^0(v^2)+\ell(-\psi_\ell^0(v^2)/\ell) = 0$.

With this construction, we see that ψ_ℓ^0 is homogeneous, locally constant, and is an eigenfunction of Δ_ℓ^2 with eigenvalue 0. □

Lemma 9.2. *For an inert prime ℓ, the function ψ_ℓ^0 defined above is* $\mathrm{GL}(2,\mathbb{Z}_\ell)$*-invariant.*

Proof. The action of $\mathrm{GL}(2,\mathbb{Z}_\ell)$ fixes vertices of tier 0, and preserves uphill and downhill neighbors, and the tier of each vertex (Lemma 7.17). Since these relationships determine the values of ψ_ℓ^0, the function is $\mathrm{GL}(2,\mathbb{Z}_\ell)$-invariant. □

For a prime ℓ that splits in $K/\mathbb{Q}$ and does not divide N, we now prepare to construct a locally constant homogeneous function $\psi_\ell^0 \in F(\mathcal{T}_\ell^1)$ that is an eigenfunction of $\Delta_\ell = \Delta_\ell^1$. For most of the remainder of this section, we will assume that ℓ splits in K, that $(\ell) = \lambda\lambda'$, and since $\ell \nmid N$, $\chi(\lambda)$ and $\chi(\lambda')$ are defined. In this case, the function that we construct will depend on the character χ. Since we work in $\mathcal{T}_\ell^1 = \mathcal{T}_\ell$, the concepts of uphill and downhill neighbor coincide.

We begin by defining some terminology and notation for subsets of $\mathcal{T}_\ell$.

Definition 9.3. We take $\mathfrak{O}_\ell$ as the basepoint of $\mathcal{T}_\ell$ and denote it by t_0. A *descendant* of a vertex $t \in \mathcal{T}_\ell$ is a vertex $t_1 \neq t$ such that the path from t_0 to t_1 passes through t. Denote by $C(t)$ the set of all descendants t' of t such that every vertex of the path from t to t' except possibly t is non-idealistic, and let $\overline{C}(t) = C(t) \cup \{t\}$. We call $C(t)$ the *open cohort* of t, and $\overline{C}(t)$ the *closed cohort* of t.

Definition 9.4. A *simple chain* starting at a vertex $t \in \mathcal{T}_\ell$ is a collection C consisting of t and descendants of t such that for any pair $t', t'' \in C$, one of t', t'' is a descendant of the other. An *apartment* in $\mathcal{T}_\ell$ is a union of two infinite simple chains starting at a vertex t and having no other vertices in common.

For future use, we state the next lemma for all unramified primes ℓ.

Lemma 9.5. *Let t be an idealistic point in $\mathcal{T}_\ell$.*

(1) If ℓ is inert, then $t = t_0$.
(2) If $(\ell) = \lambda\lambda'$ splits and t is a distance $k > 0$ from t_0, then $t = \lambda_\ell^k$ or $t = \lambda_\ell'^k$, and both of these points are a distance k from t_0.
(3) If (ℓ) splits and $k > 0$, then λ_ℓ^k and $\lambda_\ell'^k$ define distinct points in $\mathcal{T}_\ell$.
(4) No descendant of an non-idealistic point in $\mathcal{T}_\ell$ is idealistic.
(5) The vertices of $\mathcal{T}_\ell$ are partitioned into the closed cohorts $\overline{C}(t_I)$ as $t_I = I_\ell$ runs over the idealistic points of $\mathcal{T}_\ell$ (where I is an ideal of $\mathfrak{O}$ of ℓ-power norm.)
(6) In the split case, the set of idealistic points of $\mathcal{T}_\ell$ form an apartment, namely

$$\{\lambda_\ell^k | k > 0\} \cup \{t_0\} \cup \{\lambda_\ell'^k | k > 0\}.$$

Proof. In the discussion following Definition 7.10, we proved that the set of idealistic nodes of $\mathcal{T}_\ell$ is $\{t_0\}$ if ℓ is inert and $\{\lambda_\ell^k | k > 0\} \cup \{t_0\} \cup \{\lambda_\ell'^k | k > 0\}$ if ℓ is split. Since λ^k has index ℓ in λ^{k-1}, and similarly for the powers of λ', (1) and (2) are now clear. As for (3), if $\lambda_\ell^k = \lambda_\ell'^k$ in $\mathcal{T}_\ell$, then $\lambda^k = \ell^m \lambda'^k$ as ideals for some integer m, which is absurd.

If ℓ is inert, (4) and (5) are obvious.

Assume that ℓ splits. Then the idealistic point λ_ℓ^k is at the end of a path containing the nodes $t_0, \lambda_\ell, \dots, \lambda_\ell^k$. A similar statement holds for $\lambda_\ell'^k$. Since every non-idealistic node is a descendant of t_0 and $\mathcal{T}_\ell$ is a tree, no idealistic point can be a descendant of a non-idealistic point. Hence (4) holds.

For any node $u \in \mathcal{T}_\ell$ consider the path from t_0 to u (possibly of length 0.) Let t_I be the last idealistic point in this path. Then $u \in \overline{C}(t_I)$ is in the closed cohort of this idealistic point. If u were in the closed cohort of two distinct idealistic points, there would be a nontrivial loop in $\mathcal{T}_\ell$. Hence, (5) holds.

Finally, (6) is clear, since the set of nonnegative powers of λ and of λ' each form a simple chain starting at t_0. □

Now we assume that ℓ is split with $\ell\mathfrak{O} = \lambda\lambda'$ for the rest of this section.

Definition 9.6. Let N be a positive integer and c be an $\mathbb{F}$-valued multiplicative function on the group $I_K(N)$ of nonzero fractional ideals of K relatively prime to N. Fix an unramified prime ℓ that does not divide N. Assume that c is trivial on the principal fractional ideal $\ell\mathfrak{O}$. Define $\hat{c} \in F(\mathcal{T}_\ell)$ by

$$\hat{c}(t) = \begin{cases} 0 & \text{if } t \text{ is non-idealistic,} \\ c(I) & \text{if } t = I_\ell, \text{ where } I \text{ is an ideal of } \ell\text{-power index in } \mathfrak{O}. \end{cases}$$

Lemma 9.7. *The function $\hat{c}$ is well defined.*

Proof. Suppose I and J are both ideals of $\mathfrak{O}$ of ℓ-power index, and that I_ℓ and J_ℓ are homothetic in K_ℓ by a power of ℓ.

If $(\ell) = \lambda\lambda'$, then $I = \ell^q\mu^a$ and $J = \ell^r\nu^b$ for nonnegative integers a, b, q, r, and $\mu, \nu \in \{\lambda, \lambda'\}$. The fact that I_ℓ and J_ℓ are homothetic implies that $\mu^a = \nu^b$, so I and J differ by a factor of ℓ^{q-r}. Since c is trivial on $\ell\mathfrak{O}$, $c(I) = c(J)$. □

Let $t \in \mathcal{T}_\ell$. For any s in the open cohort $C(t)$ of t, all of the neighbors of s are in the closed cohort $\overline{C}(t)$. Hence, the Laplace operator Δ_ℓ defines a linear map from functions on $\overline{C}(t)$ to functions on $C(t)$.

Lemma 9.8. *Assume that ℓ is not equal to the characteristic of $\mathbb{F}$. Let $\mu \in \mathbb{F}$, and let t be an idealistic point of $\mathcal{T}_\ell$ with closed cohort $\overline{C}(t)$. Then there is a unique $\mathbb{F}$-valued function $\theta_{t,\mu}$ on $\overline{C}(t)$ with the following properties:*

(i) *$\theta_{t,\mu}(t) = 1$,*
(ii) *$\theta_{t,\mu}(s) = 0$ for every $s \in C(t)$ that is distance 1 from t,*
(iii) *$\theta_{t,\mu}(s)$ depends only on ℓ, μ, and the distance from s to t,*
(iv) *$\Delta_\ell(\theta_{t,\mu})(s) = \mu\theta_{t,\mu}(s)$ for every $s \in C(t)$.*

Proof. Define a sequence $a_k \in \mathbb{F}$ for $k \geq 0$ by the recurrence relation $a_0 = 1$, $a_1 = 0$, and for $k \geq 2$,

$$a_k = \frac{\mu a_{k-1} - a_{k-2}}{\ell}.$$

This clearly defines a unique sequence. For s a distance k from t in $\overline{C}(t)$, set $\theta_{t,\mu}(s) = a_k$. With this definition, $\theta_{t,\mu}$ satisfies conditions (i), (ii), and (iii).

Given a point $s \in C(t)$ a distance k from t, s has one neighbor a distance $k-1$ from t, and ℓ neighbors a distance $k+1$ from t. Hence

$$\begin{aligned} \Delta_\ell(\theta_{t,\mu})(s) &= a_{k-1} + \ell a_{k+1} \\ &= a_{k-1} + \ell\left(\frac{\mu a_k - a_{k-1}}{\ell}\right) \\ &= \mu a_k \\ &= \mu\theta_{t,\mu}(s), \end{aligned}$$

so $\theta_{t,\mu}$ satisfies condition (iv).

Conversely, if $\theta_{t,\mu}$ is a function on $\overline{C}(t)$ satisfying condition (iii), then for any s a distance k from t, we may define $a_k = \theta_{t,\mu}(s)$. If in addition $\theta_{t,\mu}$ satisfies conditions (i), (ii), (iv), the a_k satisfy the recurrence relation given above. The uniqueness of $\theta_{t,\mu}$ follows from the uniqueness of the sequence $\{a_k\}$. □

Definition 9.9. Let $\mu \in \mathbb{F}$, and assume ℓ does not divide N and does not equal the characteristic of $\mathbb{F}$, and that $\chi(\ell\mathfrak{O}) = 1$. We define $\psi_\ell^0 \in F(\mathcal{T}_\ell)$ by

$$\psi_\ell^0(s) = \hat{\chi}(t)\theta_{t,\mu}(s),$$

where $t \in \mathcal{T}_\ell$ is the unique idealistic vertex with $s \in \overline{C}(t)$.

Lemma 9.10. *Let $\mu \in \mathbb{F}$ and assume that ℓ does not divide N and does not equal the characteristic of $\mathbb{F}$.*

(1) $\psi_\ell^0(\mathfrak{O}_\ell) = 1$.
(2) ψ_ℓ^0 is locally constant with respect to T_ℓ.
(3) If $\mu = \chi(\lambda) + \chi(\lambda')$, then

$$\Delta_\ell \psi_\ell^0 = \mu\psi_\ell^0.$$

Proof. The first assertion is immediate from the definitions.

Let s be any vertex in $\mathcal{T}_\ell$. We wish to show that ψ_ℓ^0 is constant on all non-idealistic outer downhill neighbors u of s. Then, by Lemma 7.24, we will obtain (2). Let $s \in \bar{C}(t)$ with t idealistic. Then any such u will be in $C(t)$. Since $\hat{\chi}(t)$ is constant for all points in $C(t)$, we need only show that $\theta_{t,\mu}(u)$ is constant for all such u. Letting the distance from t to s be $k-1$, the distance from t to u will be k. Hence, the desired constancy follows from Lemma 9.8(iii).

For (3), again assume $s \in \bar{C}(t)$ with t idealistic. Suppose that $s = t = I_\ell$ is idealistic, where I is an ideal of ℓ-power index in $\mathfrak{O}$. Then s has exactly two idealistic neighbors, namely $(\lambda I)_\ell$ and $(\lambda' I)_\ell$. The nonidealistic neighbors u of s are all in $C(t)$ and have distance 1 from t; hence $\theta_{t,\mu}$ vanishes on them all. Hence

$$(\Delta_\ell\psi_\ell^0)(s) = \chi(\lambda I) + \chi(\lambda' I) = (\chi(\lambda) + \chi(\lambda'))\chi(I) = \mu\psi_\ell^0(s).$$

Finally, suppose that s is non-idealistic. Then it belongs to the open cohort $C(t)$. Then

$$\begin{aligned}(\Delta_\ell\psi_\ell^0)(s) &= \sum_u \psi_\ell^0(u) \\ &= \sum_u \hat{\chi}(t)\theta_{t,\mu}(u) \\ &= \hat{\chi}(t)\sum_u \theta_{t,\mu}(u) \\ &= \hat{\chi}(t)(\Delta_\ell\theta_{t,\mu})(s) \\ &= \mu\hat{\chi}(t)\theta_{t,\mu}(s) \\ &= \mu\psi_\ell^0(s),\end{aligned}$$

by Lemma 9.8(iv), where the sums run over all neighbors u of s. □

10. $H(M,q)$-invariance

In this section, we construct a function on lattices as a product of the local eigenfunctions for the Laplacian constructed in the previous section, and prove that the resulting function is $H(M,q)$-invariant and q-homogeneous, which implies by Theorem 8.8 and Corollary 8.9 that it corresponds to a Hecke eigenclass in $H^1(\mathrm{GL}(2,\mathbb{Z}), \mathfrak{M}(M,q)^*)$.

Lemma 10.1. *Fix a prime ℓ that is unramified in K, and let $n = 1$ if ℓ splits in K and 2 if ℓ is inert. Let Λ be a $\mathbb{Z}$-lattice in K, and let $\alpha \in K^\times$. Let s^n be the vertex in $\mathcal{T}_\ell^n$ corresponding to Λ_ℓ, and let u^n be the vertex corresponding to $(\alpha\Lambda)_\ell$. Factor the fractional ideal $\alpha\mathfrak{O} = I_1 I_2$, where I_1 has norm a power of ℓ and I_2 is prime to ℓ.*

(1) There exists a matrix $g \in \mathrm{GL}(2, \mathbb{Q}_\ell)$ depending only on α (independent of Λ), such that $u^n = gs^n$. If ℓ is inert, then $g = \ell^k g'$ with $k \in \mathbb{Z}$ and $g' \in \mathrm{GL}(2, \mathbb{Z}_\ell)$.

(2) The vertex s^n is idealistic if and only if u^n is idealistic. If s^n corresponds to Λ_ℓ with Λ an ideal, then u^n corresponds $(I_1\Lambda)_\ell$.

(3) Suppose ℓ is split. Assume that s^n is not idealistic, but lies in the open cohort $C(t)$ of the idealistic point $t^n = M_\ell$, where M is an ideal of ℓ-power norm. Then u^n lies in the open cohort $C(t_1^n)$, where $t_1^n = (I_1 M)_\ell$ and the distance between s^n and t^n is the same as the distance between u^n and t_1^n.

Proof. (1) First, suppose that ℓ is inert. Via our identification of K_ℓ with $\mathbb{Q}_\ell^2$, multiplication by α is a $\mathbb{Q}_\ell$-linear isomorphism from $\mathbb{Q}_\ell^2$ to $\mathbb{Q}_\ell^2$; hence, it is given by a matrix $g \in \mathrm{GL}(2, \mathbb{Q}_\ell)$. We can write $\alpha \in K_\ell$ as $\alpha = \ell^k \eta$ for some $k \in \mathbb{Z}$, and some unit $\eta \in \mathfrak{O}_\ell^\times$; multiplication by η is given by a matrix in $\mathrm{GL}(2, \mathbb{Z}_\ell)$.

Now assume that ℓ is split. In this case, we identify K_ℓ with $\mathbb{Q}_\ell^2$ by mapping α to (α, α'). Then multiplication by α is defined by the matrix

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha' \end{pmatrix},$$

which is in $\mathrm{GL}(2, \mathbb{Q}_\ell)$.

(2) Λ is a fractional ideal if and only if $\alpha\Lambda$ is a fractional ideal. If $\Lambda = MP$ with M a fractional ideal of ℓ-power norm, and P a fractional ideal prime to ℓ, then

$$(\alpha\Lambda)_\ell = (I_1 M)_\ell = (I_1 \Lambda)_\ell.$$

(3) Let $g \in \mathrm{GL}(2, \mathbb{Q}_\ell)$ be the matrix from part (1) corresponding to multiplication by α. Multiplication by g is then an automorphism of $\mathcal{T}_\ell$ that takes idealistic vertices to idealistic vertices, and non-idealistic vertices to non-idealistic vertices. Let R be a simple path from t^n to s^n whose only idealistic vertex is t^n. Then gR is a simple path from gt^n to u^n of the same length as R, whose only idealistic vertex is gt^n. Moreover, u^n lies in the open cohort $C(gt^n)$ where $gt^n = (I_1 M)_\ell$. □

Theorem 10.2. *Let $\mathbb{F}$ be a field of characteristic 0 or of finite characteristic not equal to two. If $\mathbb{F}$ has characteristic 0, set $p = 1$, and otherwise let p be the characteristic of $\mathbb{F}$. Assume that χ is trivial on principal ideals generated by elements of $\mathbb{Z}_{(pdN)}^\times$. Also assume that χ is trivial on principal ideals generated by elements of $K(M, q)$. Let Φ be the function from lattices in K to $\mathbb{F}$ defined by*

$$\Phi(\Lambda) = \prod_{w \nmid pdN} \hat{\psi}_w^0(\Lambda_w)$$

where ψ_w^0 is given by Theorem 9.1 if w is inert in $K/\mathbb{Q}$, and by Definition 9.9 if w splits in $K/\mathbb{Q}$.

Then $\Phi(\alpha\Lambda) = \Phi(\Lambda)$ for all $\alpha \in H(M, q)$ and all lattices Λ in K.

Moreover, $\Phi(\alpha\Lambda) = q(\alpha)\Phi(\Lambda)$ for all $\alpha \in \mathbb{Z}_{(pdN)}^\times$.

Proof. Let Λ be a lattice in K and let $\alpha \in K(M,q)$. Note that $m'_\Lambda = m'_{\alpha\Lambda}$, since $K^\times$ is commutative. Hence, there is a single integer m' depending on Λ, such that for each prime $w \nmid pdN$, we have

$$\hat{\psi}^0_w(\Lambda_w) = m'\psi^0_w(\Lambda_w)$$

and

$$\hat{\psi}^0_w((\alpha\Lambda)_w) = m'\psi^0_w((\alpha\Lambda)_w).$$

Assume first that w is inert in K. Then we may factor $\alpha\mathfrak{O}$ as

$$\alpha\mathfrak{O} = w^j I_2,$$

with I_2 a fractional ideal that is relatively prime to w. By Lemma 10.1(1), we have

$$(\alpha\Lambda)_w = w^j g\Lambda_w$$

for some $g \in \mathrm{GL}(2,\mathbb{Z}_w)$. Since ψ^0_w is homogeneous and is $\mathrm{GL}(2,\mathbb{Z}_w)$-invariant on $\mathcal{T}^2_\ell$ (by Lemma 9.2), we have

$$\hat{\psi}^0_w((\alpha\Lambda)_w) = m'\psi^0_w(w^j g\Lambda_w) = m'q(w^j)\psi^0_w(\Lambda_w) = q(w^j)\hat{\psi}^0_w(\Lambda_w).$$

Now assume that w splits in K. Let $s \in \mathcal{T}_\ell$ be the vertex corresponding to Λ_w, and let u correspond to $(\alpha\Lambda)_w$.

If s is idealistic, so is u, and we see that

$$\psi^0_w(s) = \hat{\chi}(s)\theta_{s,\mu}(s) = \hat{\chi}(s) = \chi(\Lambda) = \chi(\alpha\Lambda) = \hat{\chi}(u) = \hat{\chi}(u)\theta_{u,\mu}(u) = \psi^0_w(u).$$

If s is nonidealistic, then so is u, and $u = gs$ for some $g \in \mathrm{GL}(2,\mathbb{Q}_w)$. Suppose s lies in the open cohort $C(t)$ of the idealistic vertex t corresponding to I_w, where I is an ideal of w-power index in $\mathfrak{O}$. By Lemma 10.1(3), u is in the open cohort $C(t_1)$ of the idealistic point t_1 corresponding to $(I_1 I)_w$, where $\alpha\mathfrak{O} = I_1 I_2$, with I_1 having norm a power of w, and I_2 having norm relatively prime to w. In addition, the distance from s to t is the same as the distance from u to t_1. Hence,

$$\hat{\chi}(t_1) = \chi(I_1 I) = \chi(I_1)\chi(I) = \chi(I_1)\hat{\chi}(t)$$

and

$$\theta_{t,\mu}(s) = \theta_{t_1,\mu}(u).$$

Therefore,

$$\begin{aligned}
\hat{\psi}^0_w((\alpha\Lambda)_w) &= m'\psi^0_w(u) \\
&= m'\hat{\chi}(t_1)\theta_{t_1,\mu}(u) \\
&= m'\chi(I_1 I)\theta_{t_1,\mu}(u) \\
&= m'\chi(I_1)\hat{\chi}(t)\theta_{t,\mu}(s) \\
&= \chi(I_1)\hat{\psi}^0_w(\Lambda_w).
\end{aligned}$$

In all of this, the fractional ideal I_1 depends on w; we will call it $I_\alpha(w)$. Then $I_\alpha(w)$ is a product of powers of primes lying over w; if w is inert, it is clear that $I_\alpha(w)$ is principal with a generator $\beta_\alpha(w)$ in $\mathbb{Z}^\times_{(pdN)}$, so that $\chi(I_\alpha(w)) = 1$.

Since $\alpha \in K(M,q)$, α is relatively prime to pdN, so that

$$\alpha\mathfrak{O} = \prod_{w \nmid pdN} I_\alpha(w) = \left(\prod_{w \text{ inert}} I_\alpha(w)\right)\left(\prod_{w \text{ split}} I_\alpha(w)\right).$$

Setting $\beta = \prod_{w \text{ inert}} \beta_\alpha(w)$, we have

$$\beta\mathfrak{O} = \left(\prod_{w \text{ inert}} I_\alpha(w)\right).$$

Since $\alpha \in K(M, q)$, $q(\alpha) = 1$. Because q depends only on inert prime factors, and the powers of inert primes dividing α and β are equal, we see that

$$1 = q(\alpha) = q(\beta).$$

In addition, we have that $\chi(\beta\mathfrak{O}) = 1$, since β is a product of powers of elements of $\mathbb{Z}_{(pdN)}^\times$, and we have assumed that χ is trivial on ideals generated by elements of $\mathbb{Z}_{(pdN)}^\times$. Hence, we see that

$$\prod_{w \text{ split}} I_\alpha(w)$$

is principal, with generator α/β, so

$$\prod_{w \text{ split}} \chi(I_\alpha(w)) = \frac{\chi(\alpha\mathfrak{O})}{\chi(\beta\mathfrak{O})} = \chi(\alpha\mathfrak{O}) = 1,$$

since we have assumed that χ is trivial on principal ideals generated by elements of $K(M, q)$.

Hence, we obtain

$$\begin{aligned}
\Phi(\alpha\Lambda) &= \prod_{w \nmid pdN} \hat{\psi}_w^0((\alpha\Lambda)_w) \\
&= \left(\prod_{w \text{ inert}} \hat{\psi}_w^0((\alpha\Lambda)_w)\right)\left(\prod_{w \text{ split}} \hat{\psi}_w^0((\alpha\Lambda)_w)\right) \\
&= \left(\prod_{w \text{ inert}} q(\beta_\alpha(w))\hat{\psi}_w^0(\Lambda_w)\right)\left(\prod_{w \text{ split}} \chi(I_\alpha(w))\hat{\psi}_w^0(\Lambda_w)\right) \\
&= q(\beta)\left(\prod_{w \text{ split}} \chi(I_\alpha(w))\right)\prod_{w \nmid pdN} \hat{\psi}_w^0(\Lambda_w) \\
&= \Phi(\Lambda),
\end{aligned}$$

so Φ is $K(M, q)$-invariant.

Next, if $\alpha \in \mathbb{Z}_{(pdN)}^\times$, it is a product of powers of primes not dividing pdN. We may thus assume that α is such a prime. The q-homogeneity of Φ then follows by Lemma 8.3 from the homogeneity of the individual ψ_w^0 functions (see Theorem 9.1 for inert primes, and note that homogeneity is trivial for split primes).

Finally, $K(M, q)$-invariance and q-homogeneity imply that Φ is $H(M, q)$-invariant. □

11. Galois representations

We now define the Galois representations to which our main theorem below applies.

As before, we let K be a real quadratic field of discriminant d, cut out by the Dirichlet character θ. Let $\mathbb{F}$ be a field of characteristic 0 (in which case we set

$p = 1$) or a field of odd characteristic p, let G_K be the absolute Galois group of K (i.e. $\mathrm{Gal}(\bar{\mathbb{Q}}/K)$), and let $\chi : G_K \to \mathbb{F}^\times$ be a character of G_K with finite image. By class field theory, we can think of χ as a character on the group of the nonzero fractional ideals of K relatively prime to N for some positive $N \in \mathbb{Z}$. Let L be the fixed field of the kernel of χ. Then L/K is Galois. We fix a positive integer M that divides pdN and define $K^\times_{(pdN)}$ and $K(M)$ as in Definition 3.3, and $K(M, q)$ as in Definition 3.7.

We place the following conditions on the character χ.

(1) χ is trivial on the principal fractional ideals of K generated by elements of $K(M, q)$.
(2) χ is trivial on the principal fractional ideals of K generated by elements of $\mathbb{Q}$ that are prime to pdN.
(3) $[L : K]$ is odd.
(4) $L/\mathbb{Q}$ is Galois.

As mentioned in the introduction, any ring class character of an order of K that cuts out a Galois extension of $\mathbb{Q}$ of odd degree over K will satisfy these conditions, for appropriate choices of M and N. As a special case of this, any unramified character of odd order will satisfy these conditions.

Let $\rho : G_\mathbb{Q} \to \mathrm{GL}(2, \mathbb{F})$ be the induced representation

$$\rho = \mathrm{Ind}_{G_K}^{G_\mathbb{Q}} \chi.$$

Note that this representation will factor through $\mathrm{Gal}(L/\mathbb{Q})$. We have an exact sequence

$$1 \to \mathrm{Gal}(L/K) \to \mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{Gal}(K/\mathbb{Q}) \to 1;$$

since $[L : K]$ is odd, this sequence splits, so there is an element τ of order 2 in $\mathrm{Gal}(L/\mathbb{Q})$ mapping to the nonidentity element of $\mathrm{Gal}(K/\mathbb{Q})$; we can lift it to an element $\tau \in G_\mathbb{Q}$, and we have that τ^2 is the identity modulo G_L.

With respect to a suitable basis, it is easy to see that for $g \in G_\mathbb{Q}$, we have the following:

(a) If $g \in G_K$, then

$$\rho(g) = \begin{pmatrix} \chi(g) & 0 \\ 0 & \chi(g') \end{pmatrix},$$

where $g' = \tau^{-1} g \tau$.

(b) If $g \notin G_K$, then $g = h\tau$ for some $h \in G_K$, and

$$\rho(g) = \begin{pmatrix} 0 & \chi(h') \\ \chi(h\tau^2) & 0 \end{pmatrix},$$

where $h' = \tau^{-1} h \tau$.

If we now let g be a Frobenius element in $G_\mathbb{Q}$ for some prime ℓ of $\mathbb{Q}$ not dividing pdN (so that ℓ is unramified in $L/\mathbb{Q}$), then we have the following two cases.

If ℓ splits in K and $\ell \nmid N$, then $g \in G_K$. If we write $\ell\mathfrak{O} = \lambda\lambda'$ with λ, λ' primes in K, then we may take g to be a Frobenius element in G_K of λ; a Frobenius element of λ' will be g'. Hence, we have

$$\mathrm{Tr}(\rho(g)) = \chi(g) + \chi(g') = \chi(\lambda) + \chi(\lambda'),$$

and

$$\det(\rho(g)) = \chi(g)\chi(g') = \chi(\lambda)\chi(\lambda') = \chi(\lambda\lambda') = \chi(\ell\mathfrak{O}) = 1$$

by condition (2) on the character χ.

On the other hand, if ℓ is inert in K and $\ell \nmid N$, write $g = h\tau$ as above. Then

$$\operatorname{Tr}(\rho(g)) = 0$$

and $\det(\rho(g)) = -\chi(h\tau^2)\chi(h')$ with $h' = \tau^{-1}h\tau$. We note that g^2 is a Frobenius element of $\ell\mathfrak{O}$ in G_K. Hence, we have

$$\det(\rho(g)) = -\chi(h\tau^2)\chi(h') = -\chi(h\tau^2 h') = -\chi((h\tau)^2) = -\chi(g^2) = -\chi(\ell\mathfrak{O}) = -1,$$

where we have again used condition (2) on χ.

Note that in each case, when g is a Frobenius element in $G_{\mathbb{Q}}$ of ℓ, we have $\det(\rho(g)) = \theta(\ell)$.

Now we check that ρ is even. Let $c \in G_{\mathbb{Q}}$ be a complex conjugation. Since c has order 2 and χ has odd order, $\chi(c) = \chi(\tau^{-1}c\tau) = 1$. From the explicit description of the matrices $\rho(g)$ above, since $c \in G_K$, $\rho(c)$ is the identity matrix.

Theorem 11.1. *Let K be a real quadratic field of discriminant d, let $\mathbb{F}$ be a field of characteristic 0 or a finite field of odd characteristic. In the first case set $p = 1$ and in the second case let p be the characteristic of $\mathbb{F}$. Let $\chi : G_K \to \mathbb{F}^\times$ be a character with finite image. Let L be the fixed field of the kernel of χ and choose $N \in \mathbb{Z}$ so that L/K is unramified outside primes of K dividing N. Let M be a positive divisor of pdN, θ the Dirichlet character cutting out K, q the extension of θ defined in Definition 3.5, and $\mathfrak{M}(M,q)$ the module defined in Definition 4.2. Assume*

(1) χ is trivial on the principal fractional ideals of K generated by elements of $K(M,q)$.
(2) χ is trivial on the principal fractional ideals of K generated by elements of $\mathbb{Q}^\times$ that are prime to pdN.
(3) $[L : K]$ is odd.
(4) $L/\mathbb{Q}$ is Galois.

Then $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(2,\mathbb{F})$ given by $\rho = \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \chi$ is an even Galois representation, and is attached to a Hecke eigenclass in $H^1(\mathrm{GL}(2,\mathbb{Z}), \mathfrak{M}(M,q)^*)$.

Proof. Recall that for w inert in K and prime to pN, we have constructed a function ψ_w^0 in the proof of Theorem 9.1, and for w split in K and prime to pN, we defined a function ψ_w^0 in Definition 9.9. Given χ satisfying the conditions of the theorem, we define an $\mathbb{F}$-valued function Φ on lattices,

$$\Phi(\Lambda) = \prod_{w \nmid pdN} \hat{\psi}_w^0(\Lambda_w)$$

where $\hat{\psi}_w^0$ is the transform (see Definition 8.4) of the function ψ_w^0.

By Theorem 10.2, Φ is $H(M,q)$-invariant and q-homogeneous. Hence, by Lemma 6.3 we may consider it as an element of $H^1(\mathrm{GL}(2,\mathbb{Q}), \mathfrak{M}(M,q)^*)$. By Corollary 8.9, combined with Lemma 9.10 and Theorem 9.1 we see that for all ℓ prime to pdN, Φ is an eigenvector for T_ℓ and $T_{\ell,\ell}$, and that the eigenvalues of T_ℓ match the trace of $\rho(\mathrm{Frob}_\ell)$. The q-homogeneity of Φ shows that the eigenvalues of $T_{\ell,\ell}$ match the determinant of $\rho(\mathrm{Frob}_\ell)$ for all ℓ prime to pdN. Hence, Φ is attached to ρ. □

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