SUMS OF GALOIS REPRESENTATIONS AND ARITHMETIC HOMOLOGY

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Abstract. Let $\Gamma_0(n, N)$ denote the usual congruence subgroup of type Γ_0 and level N in $\mathrm{SL}(n, \mathbb{Z})$. Suppose for $i = 1, 2$ we have an irreducible odd n-dimensional Galois representation ρ_i attached to a homology Hecke eigenclass in $H_*(\Gamma_0(n, N_i), M_i)$, where the level N_i and the weight and nebentype making up M_i are as predicted by the Serre-style conjecture of Ash-Doud-Pollack-Sinnott. We assume that n is odd, N_1N_2 is squarefree and that $\rho_1 \oplus \rho_2$ is odd. We prove two theorems that assert that $\rho_1 \oplus \rho_2$ is attached to a homology Hecke eigenclass in $H_*(\Gamma_0(2n, N), M)$, where N and M are as predicted by the Serre-style conjecture. The first theorem requires the hypothesis that the highest weights of M_1 and M_2 are small in a certain sense. The second theorem requires the truth of a conjecture as to what degrees of homology can support Hecke eigenclasses with irreducible Galois representations attached, but no hypothesis on the highest weights of M_1 and M_2. This conjecture is known to be true for $n = 3$ so we obtain unconditional results for GL(6). A similar result for GL(4) appeared in an earlier paper.

1. Introduction

Scholze has proved [20] that any system of Hecke eigenvalues appearing in the mod p homology of a congruence subgroup of $\mathrm{GL}(n, \mathbb{Z})$ has an n-dimensional Galois representation ρ attached. (See Definition 2.3 for "attached.") Attention therefore turns to the question: given ρ, does there exist a congruence subgroup Γ of $\mathrm{GL}(n, \mathbb{Z})$, a coefficient module M and a Hecke eigenclass in $H_*(\Gamma, M)$ with ρ attached? Work of Caraiani and Le Hung [13] shows that for this to happen ρ applied to complex conjugation must be similar to a matrix with alternating 1's and -1's on the diagonal (in other words, the multiplicities of 1 and -1 as eigenvalues must differ by at most 1) or $p = 2$. We will say that a Galois representation satisfying this condition on the eigenvalues of the image of complex conjugation is odd.

The main conjecture of [10], as refined in [8, Conjecture 3.1], asserts that for odd ρ, the question in the preceding paragraph is to be answered in the affirmative. (For $n = 2$, this conjecture was made by Serre [22] and proven by Khare, Wintenberger [16, 17] and Kisin [18].) Here (and throughout this paper) we are using [6, Lemma 2.4], which shows that a system of Hecke eigenvalues appears in $H_i(\Gamma, M)$ if and only if it appears in $H^i(\Gamma, M)$, as long as the coefficients are finite-dimensional. When we use the Borel-Serre duality theorem in its usual form we need to work with the homology of Γ with coefficients in the Steinberg module tensored with M. For this reason, we find it suitable also to state our main results, namely

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Theorem 7.1, Theorem 8.2, and Corollary 8.3, in terms of homology, rather than cohomology.

In [8], given ρ we give precise recipes for the smallest possible level N of the group Γ and for a small finite set of possible irreducible modules M such that ρ should be attached to a class in $H_i(\Gamma, M)$. Such an M is determined by giving its highest weight and its nebentype character.

Definition 1.1. A level, weight, or nebentype is said to be *predicted by* ρ if it coincides with the level, one of the weights, or the nebentype occurring in [8, Conjecture 3.1] applied to ρ.

Note that any predicted level is prime to p, any predicted weight is an irreducible representation, and any predicted nebentype has conductor dividing the level.

The proof of the full conjecture of [8] appears to be a long way off. In a series of papers [3, 4, 5, 6, 7] we have proven the conjecture in various cases when ρ is reducible. We approach the problem inductively, assuming that the conjecture is true for the irreducible constituents of ρ. Even in the reducible case, the proofs are by no means easy, especially as we need to keep track of the exact level N and module M.

In this paper we extend results of [7] regarding sums of two odd Galois representations. In [7], we showed that for $p \geq 5$, given two odd two-dimensional Galois representations $\rho_1, \rho_2 : G_{\mathbb{Q}} \to \mathrm{GL}(2, \overline{\mathbb{F}}_p)$ with relatively prime squarefree Serre conductors N_1 and N_2, there is a Hecke eigenclass in $H_*(\Gamma_0(4, N_1N_2), M)$ that has $\rho_1 \oplus \rho_2$ attached. In addition, the coefficient module M conforms to the main conjecture of [8].

One hindrance in [7] to extending this result to Galois representations $\rho_i : G_{\mathbb{Q}} \to \mathrm{GL}(n, \overline{\mathbb{F}}_p)$ with $n > 2$ was that for $n > 2$, a certain Hochschild-Serre spectral sequence could not be shown to degenerate; hence, although we could construct suitable Hecke eigenvectors in the 2-page of the spectral sequence, we could not prove that they survived to the infinity-page to give rise to the eigenvectors that we needed.

In this paper, we partially solve this problem in two different ways. First, we restrict the weights that we allow to a certain set of "very small" weights, and demonstrate that for such a weight, the eigenvector that we can construct in the 2-page of the Hochschild-Serre spectral sequence does survive to the infinity-page.

Alternatively, we can make the assumption that irreducible Galois representations are attached to homology eigenvectors only in degrees in the "cuspidal range." Using this assumption instead of very small weights, we can once again demonstrate that the eigenvector that we construct in the 2-page of the Hochschild-Serre spectral sequence survives to the infinity-page.

We note that the assumption about the cuspidal range is known to be true for $n = 3$. Hence, with no conditions on the weight or unproven assumptions, we obtain, for example, the following theorem (see Corollary 8.3 with $\eta = \eta' = 0$). This theorem gives the flavor of our main results.

Theorem 1.2. *Let* $p > 7$. *Let* $\rho_1, \rho_2 : G_{\mathbb{Q}} \to \mathrm{GL}(3, \overline{\mathbb{F}}_p)$ *be odd irreducible Galois representations with predicted levels* N_1 *and* N_2*, predicted nebentypes* ϵ_1 *and* ϵ_2*, and predicted weights* $F(a+3, b+3, c+3)$ *and* $F(d, e, f)$*. Assume that* $\rho_1 \oplus \rho_2$ *is odd, that* N_1N_2 *is squarefree, that* ρ_1 *is attached to a Hecke eigenclass in* $H_2(\Gamma_0(3, N_1), F(a+3, b+3, c+3)_{\epsilon_1})$ *and that* ρ_2 *is attached to a Hecke eigenclass* *in* $H_3(\Gamma_0(3,N_2),F(d,e,f)_{\epsilon_2})$*. Then* $\rho_1\oplus\rho_2$ *is attached to a Hecke eigenclass in at least one of*

$$H_{15}(\Gamma_0(6,N_1N_2),F(a,b,c,d,e,f)_{\epsilon_1\epsilon_2})$$

or

$$H_4(\Gamma_0(6,N_1N_2),F(a,b,c,d,e,f)_{\epsilon_1\epsilon_2}),$$

and the level N_1N_2*, the nebentype* $\epsilon_1\epsilon_2$*, and the weight* $F(a,b,c,d,e,f)$ *are predicted for* $\rho_1\oplus\rho_2$ *by* [8].

The two main general theorems we prove, under the alternative assumptions discussed above, are Theorem 7.1 and Theorem 8.2. Both of these theorems require n to be odd.

Remark 1.3. The only part of our proofs that requires n to be odd is the construction in Theorem 3.5 of an eigenclass in

$$H_k(\Gamma_0^{\pm}(n,N),M)$$

having ρ attached, given the existence of an eigenclass in $H_k(\Gamma_0(n,N),M)$ with ρ attached.

By analogy with cuspidal automorphic forms, we believe that as long as p is odd, for any even n and any odd irreducible Galois representation $\rho : G_{\mathbb{Q}}\to \mathrm{GL}(n,\overline{\mathbb{F}}_p)$ attached to an eigenclass in $H_k(\Gamma_0(n,N),M)$, this construction is possible, but we have been unable to prove it. With this as additional input, we could extend Theorem 7.1 and Theorem 8.2 to arbitrary $n>1$ (with an adjustment to the degrees of the homology groups for even n).

What prevents us from extending our results to the sum of two Galois representations of unequal degrees, or to a sum of more than two Galois representations, is the breakdown of our argument concerning the spectral sequence $\mathcal{E}^*_{q,r}$ of Theorem 2.6. In the proof of Theorem 3.6 we show that when the degrees of the two Galois representations are equal, a corresponding class in the 1-page survives to the infinity page. (Note that this is different from the problem mentioned above of showing that an eigenclass in the 2-page of the Hochschild-Serre spectral sequence, which we denote by E^*_{ij}, survives to the infinity-page. The Hochschild-Serre spectral sequence is used to construct the eigenvector in the 1-page of the spectral sequence $\mathcal{E}^*_{q,r}$.) In future work we hope to extend Theorem 3.6 to the case of two Galois representations of unequal degrees.

The main new tool in this paper is a result on the admissibility of the cohomology of the unipotent radical of a parabolic subgroup with coefficients in an admissible module (see section 5). This result is crucial to allow the use of Scholze's theorem in showing sufficient degeneration of the Hochschild-Serre spectral sequence.

The contents of the other sections of the paper are as follows. In section 2 we review definitions and results from [7], in particular a new resolution of the Steinberg module of a vector space in terms of Steinberg modules of smaller dimensional spaces. In section 3 we give the construction mentioned in Remark 1.3 and we prove the result about the spectral sequence $\mathcal{E}^*_{q,r}$ mentioned two paragraphs above.

Sections 4 and 5 present new results on the homology of a parabolic subgroup in Γ viewed as a Hecke-module. In sections 6 and 7 we prove our main theorem for the case of M with very small highest weight. Here we use an explicit study of the weights for diagonal matrices acting on the homology of the unipotent radical in order to obtain the necessary amount of degeneration in the Hochschild-Serre spectral sequence. In section 8 we do the same thing for general M under the hypothesis mentioned above about Galois representations attached to homology outside the cuspidal range.

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2. Results of [7]

In this section, we review some of the main definitions and results of [7] that we will use.

2.1. $\Gamma_0(n,N)$-orbits of subspaces of $\mathbb{Q}^n$.

Let $\mathrm{GL}(n,\mathbb{Q})$ act on the right on $\mathbb{Q}^n$ via matrix multiplication (considering the elements of $\mathbb{Q}^n$ as row vectors). We define the following subsets of $\mathrm{GL}(n,\mathbb{Q})$. For the remainder of the paper, we fix a prime number p.

Definition 2.1. Let n, N be positive integers.

1. $S_0^{\pm}(n,N)$ consists of the integer matrices with determinant prime to pN whose top row is congruent modulo N to $(*,0,\dots,0)$.
2. $S_0(n,N)$ consists of the elements of $S_0^{\pm}(n,N)$ with positive determinant.
3. $\Gamma_0^{\pm}(n,N) = S_0(n,N)^{\pm} \cap \mathrm{GL}(n,\mathbb{Z})$.
4. $\Gamma_0(n,N) = S_0(n,N) \cap \mathrm{SL}(n,\mathbb{Z})$.

We note that $\Gamma_0(n,N)$ and $\Gamma_0^{\pm}(n,N)$ are subgroups of $\mathrm{GL}(n,\mathbb{Q})$, while $S_0(n,N)$ and $S_0^{\pm}(n,N)$ are subsemigroups.

In [7, Theorem 5.1], we prove the following theorem.

Theorem 2.2. *Let $0 < k < n$ and assume that N is squarefree. Then the $\Gamma_0(n,N)$-orbits of k-dimensional subspaces of $\mathbb{Q}^n$ are in one-to-one correspondence with the set of positive divisors of N, where the orbit corresponding to the divisor d contains the k-dimensional subspace spanned by*

$$e_1 + de_{k+1}, e_2, \dots, e_k,$$

where e_i denotes the standard basis element of $\mathbb{Q}^n$ with a 1 in the ith column and zeroes elsewhere.

The $\Gamma_0(n,N)$-orbits are stable under the action of $S_0^{\pm}(n,N)$.

Let M_0^k be the $k \times n$ matrix

$$\left(\begin{array}{c|c} I_k & 0 \end{array}\right),$$

and let W_0^k be the row space of M_0^k. Let $\mathrm{GL}(n,\mathbb{Q})$ act on $\mathbb{Q}^n$ via right multiplication, and set P_0^k to be the stabilizer of W_0^k.

For d a positive integer, let g_d be the $n \times n$ identity matrix with the $(1, k+1)$ entry replaced by d. We define $P_d^k = g_d^{-1} P_0^k g_d$, and note that P_d^k is the stabilizer of the row space W_d^k of $M_0^k g_d$. We see that for $d|N$, P_d^k is the stabilizer of the canonical representative of the $\Gamma_0(n,N)$-orbit of k-dimensional subspaces of $\mathbb{Q}^n$ corresponding to d.

Typically, when k is understood from the context, we omit it. We call P_d a *representative maximal parabolic subgroup*, and denote its unipotent radical by U_d and its Levi quotient by $L_d = P_d/U_d$.

For a subgroup Γ of $\mathrm{GL}(n,\mathbb{Z})$ and a maximal parabolic subgroup P, we write $\Gamma_P = \Gamma \cap P$ and $\Gamma_U = \Gamma \cap U$, where U is the unipotent radical of P. We denote by Γ_L the quotient Γ_P/Γ_U. Similarly, for a subsemigroup $S \subset \mathrm{GL}(n,\mathbb{Q})$, we write $S_P = S \cap P$, $S_U = S \cap U$ and $S_L = S_P/S_U$.

For a matrix $s \in P_0^k$, we may write s as a block lower triangular matrix

$$s = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

with A a $k \times k$ invertible matrix and C an $(n-k)\times(n-k)$ invertible matrix. We define maps $\psi_0^1 : P_0^k \to \mathrm{GL}(k,\mathbb{Q})$ and $\psi_0^2 : P_0^k \to \mathrm{GL}(n-k,\mathbb{Q})$ by $\psi_0^1(s) = A$ and $\psi_0^2(s) = C$. For $s \in P_d^k$, we then define $\psi_d^i(s) = \psi_0^i(g_d s g_d^{-1})$. The maps ψ_0^i and ψ_d^i are homomorphisms. We recall, from [7, Theorem 5.2], the exact sequence

$$1 \to U_d \cap \Gamma_0^{\pm}(n,N) \to P_d \cap \Gamma_0^{\pm}(n,N) \xrightarrow{\psi_d^1 \times \psi_d^2} \Gamma_0^{\pm}(k,d) \times \Gamma_0^{\pm}(n-k,N/d) \to 1.$$

2.2. **Hecke operators and Galois representations.** For a positive integer N prime to p, $(\Gamma_0(n,N), S_0(n,N))$ is a Hecke pair (see [1]), and we denote the $\mathbb{F}_p$-algebra of its double cosets by $\mathcal{H}_{n,N}$. We note that $\mathcal{H}_{n,N}$ is commutative, and is generated by the double cosets

$$\Gamma_0(n,N)s(\ell,n,k)\Gamma_0(n,N),$$

where $s(\ell,n,k) = \mathrm{diag}(1,\ldots,1,\ell,\ldots,\ell)$ is a diagonal matrix with k copies of ℓ on the diagonal, ℓ runs over all primes not dividing pN, and $0 \le k \le n$. The algebra $\mathcal{H}_{n,N}$ acts on the homology and cohomology of $\Gamma_0(n,N)$ with coefficients in any $\mathbb{F}_p[S_0(n,N)]$-module M. When the double coset of $s(\ell,n,k)$ acts on homology or cohomology, we denote it by $T_n(\ell,k)$.

Definition 2.3. Let V be any $\mathcal{H}_{n,N}$-module, and suppose that $v \in V$ is a simultaneous eigenvector of all the $T_n(\ell,k)$ for $\ell \nmid pN$, with eigenvalues $a(\ell,k) \in \overline{\mathbb{F}}_p$. Suppose that $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(n,\overline{\mathbb{F}}_p)$ is a Galois representation unramified outside pN. We say that ρ is *attached* to v if, for all $\ell \nmid pN$,

$$\det(I - \rho(\mathrm{Fr}_\ell)X) = \sum_{k=0}^{n} (-1)^k \ell^{k(k-1)/2} a(\ell,k) X^k.$$

2.3. **A Steinberg module exact sequence.** In [7], the Steinberg module $\mathrm{St}(W)$ of a vector space W over a field K is implicitly defined as the cokernel of the map d_0 in the sharbly complex (see the discussion after [7, Definition 4.1]). For the convenience of the reader, we include here an explicit definition:

Definition 2.4. Let K be a field, and let W be a K-vector space of dimension $n \ge 1$. The Steinberg module $\mathrm{St}(W)$ is the $\mathrm{GL}(W)$-module defined by the quotient S/R, where S is the free $\mathbb{Z}$-module on the symbols $[w_1,\ldots,w_n]$ where $w_1,\ldots,w_n$ are nonzero vectors in W and R is the $\mathbb{Z}$-submodule generated by the following elements of S:

(i) $[w_{\sigma 1},\ldots,w_{\sigma n}] - \mathrm{sgn}(\sigma)[w_1,\ldots,w_n]$ for all permutations σ of the set $\{1,\ldots,n\}$;
(ii) $[w_1,\ldots,w_n]$ whenever $w_1,\ldots,w_n$ are linearly dependent;
(iii) $\sum_{i=0}^{n+1}(-1)^n[w_1,\ldots,\widehat{w}_i,\ldots w_{n+1}]$ for all nonzero vectors $w_1,\ldots,w_{n+1} \in W$,

where as usual the notation $\widehat{w}_i$ means to omit w_i (compare [2, Section 1]). The action of $g \in GL(W)$ is given by $[w_1,\ldots,w_n]g = [w_1 g,\ldots,w_n g]$.

The module $\mathrm{St}(W)$ defined here is isomorphic to the module $\mathrm{St}(\dim(W))$ defined in [2] when K is taken to be a number field whose ring of integers is a PID, and when the coefficient ring R in [2] is taken equal to $\mathbb{Z}$.

In [7] the following exact sequence of modules is derived ([7, Theorem 4.2]).

Theorem 2.5. *Let V be an n-dimensional vector space over a field K with $n > 0$. Then there is an exact sequence of* $\mathrm{GL}(V)$*-modules*

$$0 \to \bigoplus_{W^n} \mathrm{St}(W^n) \to \bigoplus_{W^{n-1}} \mathrm{St}(W^{n-1}) \to \cdots \to \bigoplus_{W^1} \mathrm{St}(W^1) \to \mathbb{Z} \to 0$$

where W^i runs through all subspaces of V of dimension i.

Applying the spectral sequence of [12] to this exact sequence, as in [4] and [7, Section 6], we obtain the following theorem.

Theorem 2.6. *Let $(\Gamma, S) = (\Gamma_0(n, N), S_0(n, N))$ for some positive squarefree integer N, and let M be an admissible S-module. Then there is a convergent Hecke equivariant spectral sequence*

$$\mathcal{E}^1_{q,r} \implies H_{q+r}(\Gamma, M)$$

where the terms $\mathcal{E}^1_{q,r}$ are given by

$$\mathcal{E}^1_{q,r} = \begin{cases} H_r(\Gamma, \mathrm{St}(V) \otimes M) & \text{if } q = n-1 \\ \bigoplus_{d|N} H_r(\Gamma_{P_d^{q+1}}, \mathrm{St}(W_d^{q+1}) \otimes M) & \text{if } q < n-1, \end{cases}$$

where $V = \mathbb{Q}^n$.

2.4. **Admissibility of homology with trivial coefficients.** From now on, assume that $\mathbb{F}$ is either a finite field of characteristic p or $\bar{\mathbb{F}}_p$. We recall the following definition of an admissible module.

Definition 2.7. Let S be a subsemigroup of the matrices in $\mathrm{GL}(n, \mathbb{Q})$ with integer entries whose determinants are prime to pN. A (p, N)-*admissible* S-module M is an $\mathbb{F}S$-module of the form $M' \otimes \mathbb{F}_\epsilon$, where M' is an $\mathbb{F}S$-module, finite-dimensional over $\mathbb{F}$, on which $S \cap \mathrm{GL}(n, \mathbb{Q})^+$ acts via its reduction modulo p, and ϵ is a character $\epsilon : S \to \mathbb{F}^\times$ which factors through the reduction of S modulo N. An admissible module is one which is (p, N)-admissible for some choice of N.

Let P be a maximal parabolic subgroup of $\mathrm{GL}(n, \mathbb{Q})$, and let U be its unipotent radical. Let (Γ, S) be a congruence Hecke pair, such that Γ_U and S_U have the same reduction modulo p. Note that if we take (Γ, S) to be $(\Gamma_0(n, N), S_0(n, N))$, then the natural map of Hecke algebras $\mathcal{H}(\Gamma_P, S_P) \to \mathcal{H}(\Gamma, S)$ is an isomorphism (see [7, Remark 6.5]).

Theorem 2.8. [7, Theorem 11.3] *With the natural action of S_P on $H_t(\Gamma_U, \mathbb{F})$ described in* [7]*, $H_t(\Gamma_U, \mathbb{F})$ is an admissible S_P-module.*

We note that the natural action of S_P on $H_t(\Gamma_U, \mathbb{F})$ is described in the paragraph preceding [7, Theorem 11.3].

2.5. **Cohomology of Γ_U.** A p-restricted n-tuple $(a_1,\dots,a_n)$ is an n-tuple of integers satisfying $0 \le a_i - a_{i+1} \le p-1$ for $1 \le i < n$, and $0 \le a_n < p-1$. It is well known [14] that the set of isomorphism classes of irreducible $\bar{\mathbb{F}}_p[\mathrm{GL}(n,\mathbb{F}_p)]$-modules is parametrized by the set of p-restricted n-tuples, with the n-tuple $(a_1,\dots,a_n)$ corresponding to the restriction to $\mathrm{GL}(n,\mathbb{F}_p)$ of the irreducible $\bar{\mathbb{F}}_p[\mathrm{GL}(n,\bar{\mathbb{F}}_p)]$-module with highest weight $(a_1,\dots,a_n)$ with respect to (T,B), where T is the group of diagonal matrices and B is the group of upper-triangular matrices. We denote the irreducible module with highest weight $(a_1,\dots,a_n)$ by $F(a_1,\dots,a_n)$.

As in [7], we will relax the condition $0 \le a_n < p-1$ to allow a_n to be any integer, while keeping the other conditions. Any such n-tuple still corresponds to a unique $\mathrm{GL}(n,\mathbb{F}_p)$-module, which we denote by $F(a_1,\dots,a_n)$, but there are now infinitely many n-tuples that correspond to a given module (two n-tuples correspond to the same module if they differ by a multiple of the constant n-tuple $(p-1,\dots,p-1)$).

Let M be a right $\mathrm{GL}(n,\mathbb{F}_p)$-module that is finite-dimensional over $\mathbb{F}$, let N be a positive integer prime to p, and let d be a positive divisor of N. Set $(\Gamma,S) = (\Gamma_0(n,N),S_0(n,N))$, and let S act on M via reduction modulo p. Then M is an admissible $\mathbb{F}[S]$-module. There is a homomorphism $\theta : S \to (\mathbb{Z}/N\mathbb{Z})^\times$ taking each element of S to the mod N reduction of its $(1,1)$ entry. For a character $\epsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{F}$, we define a nebentype character $S \to \mathbb{F}$ (which we will also call ϵ) to be $\epsilon \circ \theta$.

Let $P_0 = P_0^k$ be a standard maximal parabolic subgroup, as defined in subsection 2.1. Set $P = P_d = g_d^{-1} P_0 g_d$, let U be the unipotent radical of P, and define Γ_U, S_P, and S_L as in subsection 2.1. We write M_ϵ^d for the S_P-module consisting of the elements of M, with S_P acting via

$$m|_\epsilon^d s = \epsilon(s) m \cdot (g_d s g_d^{-1}).$$

When using this notation, if $d = 0$, we omit it; similarly if $\epsilon = 1$, we omit it. As in [4, Section 5], we note that if M is an irreducible $\mathrm{GL}(n,\mathbb{F}_p)$-module, M and M^d are isomorphic as S_P-modules (with g_d as an intertwining operator).

We prove the following theorem in [7].

Theorem 2.9. [7, Theorem 9.1] *Let N be squarefree and prime to p, let $\epsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \overline{\mathbb{F}}_p$, let $d|N$, and let $1 \le k \le n-1$. Let $P = P_d^k$, and let $U = U_d^k$ be the unipotent radical of P. Set $(\Gamma,S) = (\Gamma_0(n,N),S_0(n,N))$. Let $r = k(n-k)$ be the $\mathbb{Q}$-dimension of U. Then*

$$H_r(\Gamma_U, F(a_1,\dots,a_n)_\epsilon) \cong \big(F(a_1+(n-k),\dots,a_k+(n-k)) \otimes F(a_{k+1}-k,\dots,a_n-k)\big)_\epsilon^d$$

as S_L-modules.

2.6. **Galois representations.** In [7], we prove the following theorem, which tells us that a system of Hecke eigenvalues in the cohomology of a congruence subgroup of a parabolic subgroup of type (n_1,n_2) has an attached Galois representation that is reducible as a sum of an n_1-dimensional and an n_2-dimensional Galois representation.

Theorem 2.10. [7, Theorem 6.3 and Theorem 11.5] *Let P be a maximal $\mathbb{Q}$-parabolic subgroup of $\mathrm{GL}(n,\mathbb{Q})$ with unipotent radical U and Levi quotient $L = P/U$ and let $(\Gamma,S) = (\Gamma_0(n,N),S_0(n,N))$. Let W be the maximal proper P-stable subspace of $\mathbb{Q}^n$. Set $n_1 = \dim W$ and $n_2 = n - n_1$. Assume that $p > n+1$. Let M be an irreducible (p,N)-admissible $\mathbb{F}[S]$-module. Then for any $t \ge 0$, $H_t(\Gamma_P, St(W) \otimes M)$*

is a finite-dimensional $\mathbb{F}$*-vector space. Let* Φ *be a system of* $\mathcal{H}(\Gamma, S)$*-eigenvalues occurring in* $H_t(\Gamma_P, \mathrm{St}(W) \otimes M)$*. Then there is some reducible Galois representation* $\rho = \sigma_1 \oplus \sigma_2$ *with* $\sigma_i : G_{\mathbb{Q}} \to \mathrm{GL}(n_i, \mathbb{F})$ *that is attached to* Φ*.*

We also prove the following in [7]. Here, $\Gamma_L^{\pm} = \Gamma_P^{\pm}/\Gamma_U^{\pm}$ is isomorphic to $\Gamma_0^{\pm}(d, k) \times \Gamma_0^{\pm}(N/d, n-k)$. The component $\Gamma_{L^1}^{\pm}$ is the portion of $\Gamma_L^{\pm}$ corresponding to the first factor, and the component $\Gamma_{L^2}^{\pm}$ is the portion corresponding to the second factor.

Theorem 2.11. [7, Corollary 10.2] *Let* $(\Gamma^{\pm}, S^{\pm}) = (\Gamma_0^{\pm}(n, N), S_0^{\pm}(n, N))$*. Let* P *be a maximal parabolic subgroup of* $\mathrm{GL}(n, \mathbb{Q})$ *of type* $(k_1, k_2) = (k, n-k)$*, with unipotent radical* U *and Levi quotient* L*, and denote the two components of the Levi quotient by* L^1 *and* L^2*. For* $i = 1, 2$*, let* M_i *be an* L^i*-module and set* $M = M_1 \otimes M_2$*. Let* $f_i \in H_{s_i}(\Gamma_{L^i}^{\pm}, M_i)$ *be an eigenclass of all the Hecke operators* $T_{k_i}(\ell, j)$*. Then* $f_1 \otimes f_2$ *may be considered as an element of* $H_{s_1+s_2}(\Gamma_L^{\pm}, M)$*, and if each* f_i *is attached to a Galois representation* ρ_i*, then* $f_1 \otimes f_2$ *is attached to* $\rho_1 \oplus \omega^{k_1}\rho_2$*.*

2.7. **Hecke equivariance of a Hochschild-Serre spectral sequence.** Let P be a maximal parabolic subgroup of $\mathrm{GL}(n, \mathbb{Q})$, U its unipotent radical, and W the maximal proper subspace of $\mathbb{Q}^n$ stabilized by P. Then there is an exact sequence

$$1 \to \Gamma_U \to \Gamma_P \to \Gamma_L \to 1$$

where $\Gamma_L = \Gamma_P/\Gamma_U$. The Hochschild-Serre spectral sequence for this exact sequence takes the following form:

$$E^2_{ij} = H_i(\Gamma_L, H_j(\Gamma_U, \mathrm{St}(W) \otimes M)) \implies H_{i+j}(\Gamma_P, \mathrm{St}(W) \otimes M).$$

In [7, Theorem 7.11], we prove

Theorem 2.12. *Let* $(\Gamma, S) = (\Gamma_0(n, N), S_0(n, N))$*. The Hochschild-Serre spectral sequence described above is Hecke equivariant for* $\mathcal{H}(\Gamma_P, S_P)$*, and a given system of Hecke eigenvalues occurs in* $H_t(\Gamma_P, \mathrm{St}(W) \otimes M)$ *if and only if it appears in*

$$\bigoplus_{i+j=t} E^{\infty}_{ij}.$$

Remark 2.13. In [7] the assertions of Theorems 2.11 and 2.12 are made only for representative maximal parabolic subgroups, but they hold for any rational maximal parabolic subgroup P, since any such P is conjugate by an element of $\Gamma_0(n, N)$ to a representative maximal parabolic subgroup.

3. Preliminary results on Galois representations and Homology

We begin our analysis of Galois representations with some results about attachment that are related to the parity of their determinant characters. Let $\mathfrak{c} \in G_{\mathbb{Q}}$ be a complex conjugation.

Definition 3.1. For a Galois representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(n, \mathbb{F})$ with $p > 2$, say that $\det(\rho)$ is odd if $\det(\rho(\mathfrak{c})) = -1$, and $\det(\rho)$ is even if $\det(\rho(\mathfrak{c})) = 1$.

Remark 3.2. This terminology matches the standard terminology for Dirichlet characters, although it differs from the terminology for Galois representations (where any one-dimensional Galois representation would be odd).

We note that if n is odd and $p > 2$, two odd representations $\rho_1, \rho_2 : G_{\mathbb{Q}} \to \mathrm{GL}(n, \mathbb{F})$ have determinants of opposite parity if and only if $\rho_1 \oplus \rho_2$ is odd.

Lemma 3.3. *Let* $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(n, \mathbb{F})$ *be a Galois representation, and assume that* ρ *is attached to an eigenclass in* $H_k(\Gamma_0(n,N), F(a_1,\dots,a_n)_\epsilon)$. *Then* $\det(\rho) = \omega^{a_1+\dots+a_n+n(n-1)/2}\epsilon$.

Proof. Since ρ is attached, for $\ell \nmid pN$, we have $\det(\rho(\mathrm{Fr}_\ell)) = \ell^{n(n-1)/2}a_n(\ell,n)$, where $a_n(\ell,n)$ is the eigenvalue of $T_n(\ell,n)$. However, $T_n(\ell,n)$ is given by the action of the scalar matrix ℓI_n, which acts via the central character of $F(a_1,\dots,a_n)_\epsilon$. Hence, for $\ell \nmid pN$, we see that

$$\det(\rho(\mathrm{Fr}_\ell)) = \ell^{a_1+\dots+a_n+n(n-1)/2}\epsilon(\ell) = \omega(\mathrm{Fr}_\ell)^{a_1+\dots+a_n+n(n-1)/2}\epsilon(\mathrm{Fr}_\ell).$$

Hence, since a Galois representation is determined by its values on Frobenius elements, we see that $\det(\rho) = \omega^{a_1+\dots+a_n+n(n-1)/2}\epsilon$. ☐

Corollary 3.4. *Let* $p > 2$, *let* n *be odd and assume that* $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(n,\mathbb{F})$ *is attached to* $H_k(\Gamma_0(n,N), F(a_1,\dots,a_n)_\epsilon)$.

1. *If* $n \equiv 1 \pmod 4$, *then* $\det(\rho)$ *is even if and only if* $\mathrm{diag}(-1,\dots,-1)$ *acts trivially on* $F(a_1,\dots,a_n)_\epsilon$.
2. *If* $n \equiv 3 \pmod 4$, *then* $\det(\rho)$ *is odd if and only if* $\mathrm{diag}(-1,\dots,-1)$ *acts trivially on* $F(a_1,\dots,a_n)_\epsilon$.

Proof. This follows since the images of complex conjugation under ω and ϵ are -1 and $\epsilon(-1)$, respectively. ☐

Theorem 3.5. *Assume* $p > 2$. *Let* n, N *be positive integers with* n *odd, and let* M *be a* $\mathrm{GL}(n,\mathbb{F}_p)$*-module on which* $-I \in \mathrm{GL}(n,\mathbb{Z})$ *acts trivially. Then for any* r,

$$H_r(\Gamma_0(n,N), M) \cong H_r(\Gamma_0^\pm(n,N), M)$$

as Hecke modules. Hence, if a Galois representation ρ *is attached to a Hecke eigenclass in* $H_r(\Gamma_0(n,N), M)$, *then it is attached to an eigenclass in* $H_r(\Gamma_0^\pm(n,N), M)$. *A similar theorem holds for cohomology.*

Proof. Since $-I$ acts trivially on M, it acts trivially on $H_r(\Gamma_0(n,N), M)$. Because n is odd, $-I \in \Gamma_0^\pm(n,N) - \Gamma_0(n,N)$. Hence,

$$H_r(\Gamma_0(n,N), M) = H_r(\Gamma_0(n,N), M)_{\Gamma_0^\pm(n,N)/\Gamma_0(n,N)}$$

Then [12, Proposition III.10.4] (adapted to homology) shows that the corestriction map induces an isomorphism

$$H_r(\Gamma_0(n,N), M)_{\Gamma_0^\pm(n,N)/\Gamma_0(n,N)} \cong H_r(\Gamma_0^\pm(n,N), M)$$

Since corestriction is Hecke invariant ([11, Lemma 1.1.3]), we see that corestriction induces an isomorphism of Hecke modules.

The proof for cohomology is similar. ☐

We next use Theorem 2.10 to prove the following theorem concerning the spectral sequence of Theorem 2.6. By [12, pg. 229], since Γ is commensurate with $\mathrm{SL}(2n,\mathbb{Z})$, the virtual cohomological dimension (VCD) of Γ is $2n(2n-1)/2 = 2n^2 - n$. We need this when we apply Borel-Serre duality [9].

Theorem 3.6. *Let* $(\Gamma, S) = (\Gamma_0(2n,N), S_0(2n,N))$, *let* $\rho = \rho_1 \oplus \rho_2$ *be a sum of two irreducible* n*-dimensional Galois representations, and assume that* ρ *is attached to a Hecke eigenclass in*

$$\mathcal{E}^1_{n-1,r} = \bigoplus_{d|N} H_r(\Gamma_{P^n_d}, \mathrm{St}(W^n_d) \otimes M).$$

Then ρ is attached to a Hecke eigenclass in at least one of

$$H_{r+n-1}(\Gamma, M) \qquad \text{or} \qquad H_{2n^2-r-1}(\Gamma, M).$$

Remark 3.7. Note that although we make no assumption that ρ is odd in the theorem, the conclusion makes it clear that any ρ attached to a Hecke eigenclass in $\mathcal{E}^1_{n-1,r}$ must be odd.

Proof. Assume that ρ is attached to an eigenclass ξ in $\mathcal{E}^1_{n-1,r}$. By Theorem 2.10, all the terms in $\mathcal{E}^1_{n-1,r}$ are finite-dimensional over $\mathbb{F}$. Because the spectral sequence is Hecke equivariant, the only way that ξ can fail to survive to the infinity-page of the spectral sequence would be for ρ to be attached to an eigenclass in some $\mathcal{E}^1_{n+k,r-k}$ with $0 \leq k \leq n-1$, or an eigenclass in $\mathcal{E}^1_{n-2-k,r+k}$ with $0 \leq k \leq n-2$.

Now, by Theorem 2.10, and the fact that the Hecke operators preserve the summands of the terms in the spectral sequence, any eigenclass in $\mathcal{E}^1_{n-2-k,r+k}$ corresponds to a Galois representation with an $(n-k-1)$-dimensional summand. Since $0 < n-k-1 < n$, this eigenclass cannot have ρ attached, so it cannot kill ξ.

Similarly, for $0 \leq k < n-1$, since $2n-(n+k+1) = n-k-1$, any eigenclass in $\mathcal{E}^1_{n+k,r-k}$ corresponds to a Galois representation with an $(n-k-1)$-dimensional summand, and so cannot have ρ attached, and cannot kill ξ.

So, either ξ survives to the infinity-page of the spectral sequence, giving rise to an eigenclass in $H_{r+n-1}(\Gamma, M)$ having ρ attached, or it is killed by an eigenclass where $k = n-1$, i.e. in

$$\mathcal{E}^1_{2n-1,r-(n-1)} = H_{r-n+1}(\Gamma, \mathrm{St}(V) \otimes M) \cong H^{2n^2-r-1}(\Gamma, M)$$

that has ρ attached, where the last isomorphism comes from Borel-Serre duality. By [6, Lemma 2.4], we see that in the latter case, ρ is attached to an eigenclass in $H_{2n^2-r-1}(\Gamma, M)$. □

4. A KÜNNETH THEOREM FOR HECKE ACTIONS ON HOMOLOGY

In the present paper, we need to extend the results of [7, Section 10] slightly, to apply to a different group. Using a proof essentially identical to that of Theorem 10.1 and Corollaries 10.2 and 10.3 of [7], we obtain the following theorem, which is similar to Theorem 2.11, but for a different group.

Theorem 4.1. *Let $(\Gamma, S) = (\Gamma_0(n,N), S_0(n,N))$, let $P = P^k_d$ be a representative maximal parabolic subgroup of $\mathrm{GL}(n,\mathbb{Q})$ for some $d|n$, let W be the maximal stable subspace associated to P, and let W_0 be the maximal stable subspace associated to P_0. Let M_1 be a $\mathrm{GL}(k,\mathbb{F}_p)$-module, let M_2 be a $\mathrm{GL}(n-k,\mathbb{F}_p)$-module, and let $(M_1 \otimes M_2)^d$ be the module $M_1 \otimes M_2$, with S_P acting via*

$$(m_1 \otimes m_2)s = m_1\psi^1_d(s) \otimes m_2\psi^2_d(s).$$

Let Γ'_L be the subgroup of Γ_L represented by elements of Γ_P in the kernel of $(\det \circ \psi^1_d, \det \circ \psi^2_d)$; it is a subgroup of index 2 in Γ_L, and is isomorphic to $\Gamma_0(k,d) \otimes \Gamma_0(n-k, N/d)$. Then the natural action of $\mathcal{H}(\Gamma_P, S_P)$ on

$$H_t(\Gamma'_L, \mathrm{St}(W) \otimes (M_1 \otimes M_2)^d)$$

is given on the component

$$H_i(\Gamma_0(k,d), \mathrm{St}(W_0) \otimes M_1) \otimes H_j(\Gamma_0(n-k, N/d), M_2)$$

with $i+j=t$ by

$$(f\otimes g)|T_n(\ell,r)=\sum_{m=\max(0,r-(n-k))}^{\min(r,k)}\ell^{(k-m)(r-m)}f|T_k(\ell,m)\otimes g|T_{n-k}(\ell,r-m).$$

Any system of Hecke eigenvalues in $H_t(\Gamma'_L,\mathrm{St}(W)\otimes(M_1\otimes M_2)^d)$ appears as the system of eigenvalues of such a product, and if f and g are attached to Galois representations ρ_1 and ρ_2, respectively, then $f\otimes g$ is attached to $\rho_1\otimes\omega^k\rho_2$.

5. Admissible modules

Throughout the remainder of the paper, we will make the following hypothesis about the Hecke pair (S,Γ), the maximal parabolic subgroup P and its unipotent radical U.

Hypothesis 5.1. Assume that the Hecke pair (S,Γ), the parabolic subgroup P and its unipotent radical U have the property that every element of $S_P^{-1}S_P\cap U(\mathbb{Q})$ is congruent to an element of $\Gamma\cap U(\mathbb{Z})$ modulo p.

We note that this assumption is easily seen to be true for

$$(S,\Gamma)=(S_0(n,N),\Gamma_0(n,N)),$$

and $P=P_d^k$ for any $d|N$ (see the proof of [7, Theorem 7.10]). The reason that we need this assumption is that it implies that for an admissible S-module M, the fixed points M^{Γ_U} are stable under the action of S_P, and hence are an admissible S_P-module.

Definition 5.2. An S-module M is *f-admissible* if there is a finite S-stable filtration of M whose associated graded module is admissible.

Remark 5.3. Note that an f-admissible module must be finite-dimensional. In addition, the contragredient of an f-admissible module is f-admissible.

Theorem 5.4. *Let M be an admissible $\mathbb{F}[S_P]$-module. Then $H_*(\Gamma_U,M)$ is an f-admissible S_P-module.*

Proof. We may assume that $M\neq 0$.

Now Γ_U acts on M via reduction module p, and Γ_U modulo p is isomorphic to an elementary abelian p-group of rank $r=\dim U$. Since any p-group acting linearly on a finite-dimensional $\mathbb{F}$-vector space has a nontrivial space of invariants, M^{Γ_U} is nontrivial.

We now argue by induction on the dimension of M. If $\dim M=1$, then M is trivial as a Γ_U-module, so by Theorem 2.8, we see that $H_*(\Gamma_U,M)=H_*(\Gamma_U,\mathbb{F})\otimes M$ is an admissible S_P-module, since it is a tensor product of admissible S_P-modules.

If $\dim M>1$, let $A=M^{\Gamma_U}$ and let $B=M/A$. Then A is stable under the action of S_P, so that

$$0\to A\to M\to B\to 0$$

is an exact sequence of S_P-modules. Since M is an admissible S_P-module, so is its submodule A and its quotient B. Then the long exact homology sequence for this short exact sequence includes

$$\cdots\to H_j(\Gamma_U,A)\xrightarrow{\alpha}H_j(\Gamma_U,M)\xrightarrow{\beta}H_j(\Gamma_U,B)\to\cdots.$$

Since Γ_U acts trivially on A, Theorem 2.8 shows that $H_j(\Gamma_U, A) \cong H_j(\Gamma_U, \mathbb{F}) \otimes A$ is a tensor product of admissible S_P-modules, and is therefore admissible. Since $H_j(\Gamma_U, A)$ is admissible, so is its image under α. By induction, $H_j(\Gamma_U, B)$ has a finite filtration by S_P-modules whose associated graded module is admissible. Pulling this filtration back to $H_j(\Gamma_U, M)$ via β, and inserting the image of α below its lowest term yields a finite filtration of $H_j(\Gamma_U, M)$ by S_P-modules whose associated graded module is admissible. □

Corollary 5.5. *Let M be an admissible $\mathbb{F}S$-module.*

1. *Any irreducible S_P-subquotient C of $H_*(\Gamma_U, M)$ is admissible.*
2. *Any system of $\mathcal{H}(\Gamma_P, S_P)$-eigenvalues appearing in $H_*(\Gamma_L, H_*(\Gamma_U, M))$ also appears in $H_*(\Gamma_L, C)$ for some admissible subquotient C of $H_*(\Gamma_U, M)$.*

Proof. (1) Let C be an irreducible S_P-subquotient of $H_*(\Gamma_U, M)$. Choose a filtration of $H_*(\Gamma_U, M)$ by S_P-submodules, such that the associated graded module of the filtration is admissible, and refine it to a filtration $0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k = H_*(\Gamma_U, M)$ such that each quotient in the filtration is irreducible and admissible. Then, C is isomorphic to some quotient from the filtration, and is hence admissible.

(2) The usual argument ([11, Lemma 2.1]) shows that the given system of $\mathcal{H}(\Gamma_P, S_P)$-eigenvalues appears in $H_*(\Gamma_L, C)$ for some irreducible S_P-subquotient C of $H_*(\Gamma_U, M)$. By (1), C is admissible. □

6. VERY SMALL WEIGHTS AND THE HOCHSCHILD-SERRE SPECTRAL SEQUENCE

6.1. **Weight spaces in** $\mathrm{GL}(n, \mathbb{F}_p)$**-modules.** Since every irreducible $\mathrm{GL}(n, \bar{\mathbb{F}}_p)$-module is a direct sum of weight spaces for the maximal torus T of diagonal matrices, it follows that so too is every irreducible $\mathrm{GL}(n, \mathbb{F}_p)$-module. As in subsection 2.5 above, $(a_1, \cdots, a_n)$ stands for the character that sends $\mathrm{diag}(t_1, \cdots, t_n)$ to $t_1^{a_1} \cdots t_n^{a_n}$. The term "dominant root" is taken with respect to (T, B), where B is the group of upper triangular matrices. Note however that the unipotent radicals U_d defined in Section 2 are conjugates by g_d of groups of lower triangular matrices.

Let e_i be the n-tuple of all zeroes except for a one in the i-th position. The weight-lowering operators adjust a weight by subtracting $e_i - e_j$ for $i < j$. A weight $\vec{b} = (b_1, \ldots, b_n)$ is lower than a weight $\vec{a} = (a_1, \ldots, a_n)$ if $\vec{b}$ can be obtained by applying a sequence of weight lowering operators to $\vec{a}$. In this case we write $\vec{b} \le \vec{a}$.

Suppose, now, that $V = F(a_1, \ldots, a_n)$ is an irreducible $\mathrm{GL}(n, \mathbb{F}_p)$-module. The long element w_0 of the Weyl group takes the highest weight $(a_1, \ldots, a_n)$ to the weight $(a_n, \ldots, a_1)$. This weight $(a_n, \ldots, a_1)$ will be the lowest weight appearing in V (since any lower weight appearing in V would be taken by w_0 to a weight higher than $(a_1, \ldots, a_n)$).

Lemma 6.1. *Let $0 \le m \le n$, and let $V = F(a_1, \ldots, a_n)$, where $\vec{a} = (a_1, \ldots, a_n)$ is a p-restricted dominant weight. Set $\vec{a}' = w_0(\vec{a}) = (a_n, \ldots, a_1)$. Then any weight $\vec{b} = (b_1, \ldots, b_n)$ appearing in V satisfies*

1. $\vec{a}' \le \vec{b} \le \vec{a}$,
2. $a_n + \cdots + a_{n-m+1} \le b_1 + \cdots + b_m \le a_1 + \ldots + a_m$.

Proof. (1) Follows since $(a_1, \ldots, a_n)$ is the highest weight in V and $(a_n, \ldots, a_1)$ is the lowest.

(2) For the weight $(c_1,\dots,c_n)=\vec{a}-(e_i-e_j)$ with $i<j$ it is clear that $c_1+\dots+c_m\le a_1+\dots+a_m$, with equality if and only if $j\le m$ or $i>m$. Since $\vec{b}$ is obtained from $\vec{a}$ by a sequence of such subtractions, we see that $b_1+\cdots+b_m\le a_1+\dots+a_m$. A similar argument, using the fact that $\vec{a}'\le\vec{b}$ completes the proof. □

6.2. **Actions of half-scalar matrices on homology.** Set $\Gamma=\Gamma_0(2n,N)$ and $S=S_0(2n,N)$. Let $P=P^n_d$ be a representative maximal parabolic subgroup of $\mathrm{GL}(2n)$ for some $d|N$, and let $U=U^n_d$ be its unipotent radical. Let

$$G=\left\{\sigma_\alpha=\mathrm{diag}(\underbrace{\alpha,\dots,\alpha}_{n},\underbrace{1,\dots,1}_{n}):\alpha\in\mathbb{Z},\gcd(\alpha,p)=1,\text{ and }\alpha\equiv 1 \pmod N\right\}.$$

Clearly, G is a subsemigroup of $P^n_0(\mathbb{Q})$ that normalizes $U^n_0(\mathbb{Q})$; conjugating by g_d, we see that $G'=g_d^{-1}Gg_d$ is a subsemigroup of S_P that normalizes $U(\mathbb{Q})$, and for all $\tau_\alpha=g_d^{-1}\sigma_\alpha g_d\in G'$, right conjugation by τ_α takes $U(\mathbb{Z})$ into $U(\mathbb{Z})$. Hence, by [7, Theorem 7.10], for an admissible $\mathbb{F}[S_p]$-module M, G' acts on $H_*(\Gamma_U,M)$ by right conjugation. By Theorem 5.4, this action is f-admissible; since each τ_α modulo p has order prime to p the action is semisimple. Hence, $H_*(\Gamma_U,M)$ is a direct sum of eigenspaces of G'. For an eigenspace Λ of G', we call the character taking each element τ_α to its eigenvalue on Λ a weight of G'. In studying these eigenspaces, we note that there is an isomorphism of abelian groups $\phi: H_*(\Gamma_U,M)\to H_*(g_d\Gamma_U g_d^{-1},M)$ such that the action of τ_α on $H_*(\Gamma_U,M)$ corresponds to the action of σ_α on $H_*(g_d\Gamma_U g_d^{-1},M)$, in the sense that the diagram

$$\begin{tikzcd}
H_*(\Gamma_U,M) \arrow[r, "\phi"] \arrow[d, "\tau_\alpha"] & H_*(g_d\Gamma_U g_d^{-1},M) \arrow[d, "\sigma_\alpha"] \\
H_*(\Gamma_U,M) \arrow[r, "\phi"] & H_*(g_d\Gamma_U g_d^{-1},M)
\end{tikzcd}$$

commutes. Under this isomorphism, there is a bijection between weights of G' occurring on $H_*(\Gamma_U,M)$ and weights of G occurring on $H_*(g_d\Gamma_U g_d^{-1},M)$. Note that these latter weights arise from restrictions of weights on the maximal torus $T\subset\mathrm{GL}(2n,\mathbb{F}_p)$.

As described in section 5, since $\Gamma=\Gamma_0(2n,N)$ and $S=S_0(2n,N)$, Hypothesis 5.1 is true for (Γ,S), so M^{Γ_U} is stable under the action of S_P.

Lemma 6.2. *Let Γ, S, P, and U be defined as above, and let $V=F(a_1,\dots,a_{2n})$. Then*

(1) G' acts on $H_{n^2}(\Gamma_U,V)$ by right conjugation with weight $\sigma_\alpha\mapsto\alpha^e$, where $e=a_1+\dots+a_n+n^2$.

(2) If $0\le j<n^2$, then any weight for G' acting on $H_j(\Gamma_U,V)$ by right conjugation is of the form $\sigma_\alpha\mapsto\alpha^f$, with $f=r+j$, where $a_{n+1}+\dots+a_{2n}\le r\le a_1+\dots+a_n$.

Proof. (1) This follows from Theorem 2.9.

(2) We prove the following stronger statement:

Let M be a $P(\mathbb{F}_p)$-module such that every T-weight λ in M satisfies

$$(*)\qquad (a_{2n},\dots,a_1)\le\lambda\le(a_1,\dots,a_{2n}).$$

Then any weight for G' acting on $H_j(\Gamma_U, M)$ is of the form $\tau_\alpha \mapsto \alpha^f$ with $f = r+j$, where $a_{n+1} + \ldots + a_{2n} \leq r \leq a_1 + \ldots + a_n$.

The proof is by induction on the dimension of M. First, note that condition (*) is inherited by any T-subquotient of M.

Before beginning the induction, we prove the assertion whenever M is a trivial Γ_U-module. In this case, [7, Theorem 11.3] shows that

$$H_j(\Gamma_U, M) \cong \left(\bigwedge^j U(\mathbb{Z}) \right) \otimes M$$

is an admissible S_P-module. The element τ_α acts as multiplication by α^j on the wedge product, so each weight vector for G' will be of the form $v \otimes m$ for v in the wedge product and m a weight vector in M. Hence, on a weight vector in the homology, the element τ_α acts as the scalar α^f, where $f = j + b_1 + \ldots + b_n$ and $(b_1, \ldots, b_{2n})$ is some T-weight in M. Setting $r = b_1 + \ldots + b_n$, the result follows from Lemma 6.1.

If $\dim(M) = 1$, then M is the trivial module for Γ_U, and we are done.

Suppose $\dim(M) > 1$. Set $A = M^{\Gamma_U}$, and $B = M/A$. We know that A is nontrivial, since Γ_U acts via a p-group on a finite-dimensional $\mathbb{F}$-vector space. In addition, A is stable under S_P.

Hence,

$$0 \to A \to M \to B \to 0$$

is an exact sequence of S_P-modules. Since A and B are subquotients of M, both satisfy (*). Then the long exact sequence for homology includes

$$\cdots \to H_j(\Gamma_U, A) \to H_j(\Gamma_U, M) \to H_j(\Gamma_U, B) \to \cdots .$$

If q is a weight vector in $H_j(\Gamma_U, M)$ with weight λ, then it either maps to a nonzero weight vector in $H_j(\Gamma_U, B)$ with weight λ, or it is the image of a vector in $H_j(\Gamma_U, A)$ with weight λ. In either case, the desired result follows by induction. □

Corollary 6.3. *Let $M = F(a_1, \ldots, a_{2n})$. Let P be a maximal parabolic subgroup of type (n, n), and let W be the stable subspace associated to P. Suppose $i_0 > n^2 - n - 2$, or $n^2 + (a_1 + \cdots + a_n) - (a_{n+1} + \cdots + a_{2n}) - n - i_0 - 1 < p - 1$ with $i_0 \leq n^2 - n - 2$. Then any class $z \in E^2_{i_0, n^2} = H_{i_0}(\Gamma_L, \mathrm{St}(W) \otimes H_{n^2}(\Gamma_U, M))$ survives to E^∞ in the Hochschild-Serre spectral sequence for $1 \to \Gamma_U \to \Gamma_P \to \Gamma_L \to 1$.*

Proof. Note that the VCD of Γ_L is at most $n^2 - n$ (by [12, Prop. VIII.2.4(b)] and [12, pg. 185], since Γ_L contains a torsion free subgroup of finite index that is isomorphic to a direct product of two subgroups of finite index in $\mathrm{SL}(n, \mathbb{Z})$) and the VCD of Γ_U is n^2 [12, pg. 185, Example 5].

In the case that $i_0 > n^2 - n - 2$, we see that all terms of the Hochschild-Serre spectral sequence that could kill off z are 0. Hence, z survives to the infinity-page.

Suppose, then, that $i_0 \leq n^2 - n - 2$ and $n^2 + (a_1 + \cdots + a_n) - (a_{n+1} + \cdots + a_{2n}) - n - i_0 - 1 < p - 1$. Note that G' centralizes Γ_L and acts trivially on $\mathrm{St}(W)$, so that the action of G' on any $H_i(\Gamma_L, \mathrm{St}(W) \otimes H_j(\Gamma_U, M))$ is via its action on $H_j(\Gamma_U, M)$, which we know from Lemma 6.2(2) has all its weights of the form $\tau_\alpha \mapsto \alpha^f$ with $f = j + r$ and $a_{n+1} + \cdots + a_{2n} \leq r \leq a_1 + \ldots + a_n$.

By Lemma 6.2(1), z must be a weight vector for G', with weight $\tau_\alpha \mapsto \alpha^e$ for $e = n^2 + a_1 + \cdots + a_n$. Since the spectral sequence is G'-equivariant, we will be finished if we can prove that no possible f is congruent modulo $p - 1$ to e. We will do this by showing that $0 < e - f < p - 1$ for all possible f arising from pairs (i, j) of degrees of homology that could kill off z.

The terms in the Hochschild-Serre spectral sequence that could kill off z lie in $E^2_{ij} = H_i(\Gamma_L, \mathrm{St}(W) \otimes H_j(\Gamma_U, M))$ with $i = i_0 + 1 + k \leq n^2 - n$ and $j = n^2 - k$ for $k \geq 1$. Then $1 \leq k \leq n^2 - n - i_0 - 1$. It follows that

$$0 < e - f < n^2 - n - i_0 - 1 + (a_1 + \cdots + a_n) - (a_{n+1} + \cdots + a_{2n}) < p - 1$$

where the final inequality is by our hypothesis. $\square$

Definition 6.4. We say that a weight $(a_1, \dots, a_{2n})$ is *very small* for i_0 if $i_0 > n^2 - n - 2$ or if $i_0 \leq n^2 - n - 2$ and $n^2 + (a_1 + \cdots + a_n) - (a_{n+1} + \cdots + a_{2n}) - n - i_0 - 1 < p - 1$.

Remark 6.5. If $i_0 > n^2 - n - 2$, every weight is very small for i_0, so the terminology is somewhat forced in this case, but it is convenient to group both kinds of i_0 together in one definition.

7. Theorem for very small weights

Theorem 7.1. *Let $n > 1$ be odd, Assume that $p > 2n + 1$, and let $\rho_1, \rho_2 : G_{\mathbb{Q}} \to \mathrm{GL}(n, \overline{\mathbb{F}}_p)$ be odd irreducible Galois representations such that $\rho_1 \oplus \rho_2$ is odd. Assume that ρ_1 has predicted level N_1, predicted nebentype ϵ_1, and predicted weight $M_1 = F(a_1 + n, \dots, a_n + n)$, and that ρ_2 has predicted level N_2, predicted nebentype ϵ_2, and predicted weight $M_2 = F(a_{n+1}, \dots, a_{2n})$. Assume also that the n-tuple $(a_1 + n, \dots, a_n + n)$ is chosen so that $0 \leq a_n - a_{n+1} \leq p - 1$, that $N_1 N_2$ is squarefree, that ρ_1 is attached to a Hecke eigenclass in $H_{s_1}(\Gamma_0(n, N_1), (M_1)_{\epsilon_1})$ and that ρ_2 is attached to a Hecke eigenclass in $H_{s_2}(\Gamma_0(n, N_2), (M_2)_{\epsilon_2})$. Finally, assume that the weight $(a_1, \dots, a_{2n})$ is very small for $i_0 = \frac{n^2 - n}{2} - s_1 + s_2$. Then $\rho_1 \oplus \rho_2$ is attached to a Hecke eigenclass in at least one of*

$$H_{\frac{3n^2+n}{2} - s_1 + s_2 - 1}(\Gamma_0(2n, N_1 N_2), F(a_1, \dots, a_{2n})_{\epsilon_1 \epsilon_2})$$

or

$$H_{\frac{n^2+n}{2} + s_1 - s_2 - 1}(\Gamma_0(2n, N_1 N_2), F(a_1, \dots, a_{2n})_{\epsilon_1 \epsilon_2}).$$

Moreover, $N_1 N_2$, $\epsilon_1 \epsilon_2$, and $F(a_1, \dots, a_{2n})$ are the predicted level, the predicted nebentype, and a predicted weight for $\rho_1 \oplus \rho_2$.

Proof. Because $\rho_1 \oplus \rho_2$ is odd, $\det(\rho_1)$ and $\det(\rho_2)$ have opposite parity. Since attachment is stable under twisting by ω, we can, if needed, twist ρ_1 and ρ_2 by ω, so that $\det(\rho_1)$ is even if $n \equiv 1 \pmod 4$ and odd if $n \equiv 3 \pmod 4$. Since $\det(\rho_1)$ and $\det(\omega^{-n}\rho_2)$ have the same parity, we know, by [6, Lemma 2.4] and Theorem 3.5 that ρ_1 is attached to an eigenclass in $H^{s_1}(\Gamma_0^{\pm}(n, N_1), F(a_1 + n, \dots, a_n + n)_{\epsilon_1})$ and $\omega^{-n}\rho_2$ is attached to an eigenclass in $H_{s_2}(\Gamma_0^{\pm}(n, N_2), F(a_{n+1} - n, \dots, a_{2n} - n)_{\epsilon_2})$. By Borel-Serre duality, ρ_1 is attached to an eigenclass in

$$H_{\frac{n^2-n}{2} - s_1}(\Gamma_0(n, N_1)^{\pm}, \mathrm{St}(W_0) \otimes F(a_1 + n, \dots, a_n + n)_{\epsilon_1}),$$

where W_0 is the maximal stable subspace of P^n_0.

Hence, by [7, Corollary 10.3], we see that $\rho_1 \oplus \rho_2$ is attached to an eigenclass in

$$H_{\frac{n^2-n}{2} - s_1 + s_2}(\Gamma_L^{\pm}, \mathrm{St}(W) \otimes (F(a_1 + n, \dots, a_n + n)_{\epsilon_1} \otimes F(a_{n+1} - n, \dots, a_{2n} - n)_{\epsilon_2})^{N_1}),$$

where $\Gamma^{\pm} = \Gamma_0^{\pm}(2n, N_1 N_2)$, $P = P^n_{N_1} \in \mathrm{GL}(2n, \mathbb{Q})$, $W = W_{N_1}$ is the maximal stable subspace of P, U is the unipotent radical of P, and $\Gamma_L^{\pm} = \Gamma_P^{\pm}/\Gamma_U^{\pm} \cong$

$\Gamma_0^{\pm}(n, N_1) \times \Gamma_0^{\pm}(n, N_2)$ (see [7, Theorem 5.2]). By Theorem 2.9 this homology group is isomorphic to

$$H_{\frac{n^2-n}{2}-s_1+s_2}(\Gamma_L^{\pm}, H_{n^2}(\Gamma_U, \mathrm{St}(W) \otimes F(a_1, \dots, a_n, a_{n+1}, \dots, a_{2n})_{\epsilon_1\epsilon_2})).$$

By [7, Theorem 10.4], we see that $\rho_1 \oplus \rho_2$ is attached to an eigenclass in

$$H_{\frac{n^2-n}{2}-s_1+s_2}(\Gamma_L, H_{n^2}(\Gamma_U, \mathrm{St}(W) \otimes F(a_1, \dots, a_n, a_{n+1}, \dots, a_{2n})_{\epsilon_1\epsilon_2})).$$

Since the weight $F(a_1, \dots, a_n, a_{n+1}, \dots, a_{2n})$ is very small for

$$i_0 = \frac{n^2-n}{2} - s_1 + s_2,$$

we see that by Corollary 6.3 this eigenclass survives to the infinity-page of the Hochschild-Serre spectral sequence for

$$1 \to \Gamma_U \to \Gamma_P \to \Gamma_L \to 1,$$

giving rise to an eigenclass in

$$H_{\frac{3n^2-n}{2}-s_1+s_2}(\Gamma_P, \mathrm{St}(W) \otimes F(a_1, \dots, a_n, a_{n+1}, \dots, a_{2n})_{\epsilon_1\epsilon_2}) \in \mathcal{E}^1_{n-1, \frac{3n^2-n}{2}-s_1+s_2}.$$

By Theorem 3.6, the desired attachment follows.

An easy computation shows that N_1N_2, $\epsilon_1\epsilon_2$ and $F(a_1, \dots, a_{2n})$ are predicted for $\rho_1 \oplus \rho_2$. □

8. GL(6) AND LARGER REPRESENTATIONS

A cohomological cuspidal automorphic representation of $\mathrm{GL}(n)/\mathbb{Q}$ can occur in the cohomology of Γ with coefficients in a finite-dimensional complex representation only in a certain range of degrees, called the "cuspidal range" (see [21, Proposition 3.5] or [15, Table 1]). For n odd, the top degree of the cuspidal range is

$$t(n) = (n+1)^2/4 - 1$$

and the bottom degree is

$$b(n) = (n^2-1)/4$$

(see [19, Theorem 2.15]). In the proof of Theorem 8.2, we will assume the following hypothesis, which we conjecture by analogy to be true, but which seems to be out of reach of current techniques in this generality.

Hypothesis 8.1. An irreducible n-dimensional Galois representation in characteristic p cannot be attached to mod p cohomology with admissible coefficient module in any degree outside the cuspidal range for n.

By [6, Lemma 2.4], any system of Hecke eigenvalues occurring in cohomology occurs also in homology with the same coefficients, so Hypothesis 8.1 implies the analogous statement for homology also.

Theorem 8.2. *Let $n > 1$ be odd, $p > 2n+1$, and assume Hypothesis 8.1 for n. Choose $(\eta, \eta') = (0,0), (0,1)$, or $(1,0)$. Let $\rho_1, \rho_2 : G_{\mathbb{Q}} \to \mathrm{GL}(n, \overline{\mathbb{F}}_p)$ be irreducible odd Galois representations with predicted levels N_1 and N_2, predicted nebentypes ϵ_1 and ϵ_2, and predicted weights $M_1 = F(a_1+n, \dots, a_n+n)$ and $M_2 = F(a_{n+1}, \dots, a_{2n})$ with the n-tuple $(a_1+n, \dots a_n+n)$ representing the predicted weight of ρ_1 chosen so that $0 \le a_n - a_{n+1} \le p-1$. Let $b = b(n) + \eta'$ and $t = t(n) - \eta$. Assume that $\rho_1 \oplus \rho_2$ is odd, that N_1N_2 is squarefree, that ρ_1 is attached* *to an eigenclass in* $H_b(\Gamma_0(n, N_1), (M_1)_{\epsilon_1})$ *and that* ρ_2 *is attached to an eigenclass in* $H_t(\Gamma_0(n, N_2, (M_2)_{\epsilon_2})$*. Then* $\rho_1 \oplus \rho_2$ *is attached to a Galois representation in*

$$H_m(\Gamma_0(2n, N_1N_2), M_\epsilon)$$

where $M = F(a_1, \ldots, a_{2n})$*,* $\epsilon = \epsilon_1\epsilon_2$*, and*

$$m = 3\left(\frac{n^2-1}{2}\right) + n - (\eta + \eta') \qquad \text{or} \qquad m = \frac{n^2-1}{2} + (\eta + \eta').$$

Moreover, the weight M*, level* N_1N_2*, and nebentype* ϵ *are predicted for* $\rho_1 \oplus \rho_2$ *by* [8].

Proof. Because $\rho_1 \oplus \rho_2$ is odd, $\det(\rho_1)$ and $\det(\rho_2)$ have opposite parity. Since the main conjecture of [8] is stable under twisting, we may twist ρ_1 and ρ_2 by ω (if necessary), so that without loss of generality, we assume that $\det(\rho_1)$ is even if $n \equiv 1 \pmod 4$ and odd if $n \equiv 3 \pmod 4$.

Letting $\mathbb{Q}^n$ be an n-dimensional space on which $\mathrm{GL}(n, \mathbb{Q})$ acts via right multiplication, we use [6, Lemma 2.4] and Borel-Serre duality to find that ρ_1 is attached to an eigenclass in

$$H_{\nu-b}(\Gamma_0(n, N_1), \mathrm{St}(\mathbb{Q}^n) \otimes (M_1)_{\epsilon_1}),$$

where

$$\nu = n(n-1)/2.$$

By Corollary 3.4, the matrix $-I_n$ acts trivially on $(M_1)_{\epsilon_1}$, hence on $\mathrm{St}(\mathbb{Q}^n) \otimes (M_1)_{\epsilon_1}$. Similarly, letting $M_2' = F(a_{n+1} - n, \ldots, a_{2n} - n)$, the matrix $-I_n$ also acts trivially on $(M_2')_{\epsilon_2}$, since $\det(\omega^{-n}\rho_2)$ has the same parity as $\det(\rho_1)$.

Hence, by Theorem 3.5, ρ_1 is attached to an eigenclass in

$$H_{\nu-b}(\Gamma_0^{\pm}(n, N_1), \mathrm{St}(\mathbb{Q}^n) \otimes (M_1)_{\epsilon_1})$$

and $\omega^{-n}\rho_2$ is attached to an eigenclass in

$$H_t(\Gamma_0^{\pm}(n, N_2), (M_2')_{\epsilon_2}).$$

Taking $\Gamma = \Gamma_0(n, N_1, N_2)$, $\Gamma^{\pm} = \Gamma_0^{\pm}(2n, N_1N_2)$, $P = P_{N_1}^n$, U the unipotent radical of P, and W the maximal stable subspace under the action of P, we see from [7, Theorem 5.2] that $\Gamma_L^{\pm} \cong \Gamma_0^{\pm}(n, N_1) \times \Gamma_0^{\pm}(n, N_2)$ via the isomorphism $s \mapsto (\psi_{N_1}^1(s), \psi_{N_1}^2(s))$. Defining $\Gamma_{L_i}^{\pm}$ for $i = 1, 2$ to be the preimage under this map of $\Gamma_0^{\pm}(n, N_i)$, each $\Gamma_0(n, N_i)$-module becomes a $\Gamma_{L_i}^{\pm}$-module with multiplication by $s \in \Gamma_{L_i}$ given by $m \cdot \psi_{N_1}^i(s)$. Note that $\Gamma_{L_2}^{\pm}$ acts trivially on $\mathrm{St}(W)$. The maps $\psi_{N_1}^i : \Gamma_{L_i}^{\pm} \to \Gamma_0^{\pm}(N_i)$, together with conjugation by g_{N_1} on the coefficient modules, induce isomorphisms

$$H_*(\Gamma_{L_1}^{\pm}, (\mathrm{St}(W) \otimes (M_1)_{\epsilon_1})^{N_1}) \to H_*(\Gamma_0^{\pm}(n, N_1), \mathrm{St}(\mathbb{Q}^n) \otimes (M_1)_{\epsilon_1})$$

and

$$H_*(\Gamma_{L_2}^{\pm}, (M_2')_{\epsilon_2}^{N_1}) \to H_*(\Gamma_0^{\pm}(n, N_2), (M_2')_{\epsilon_2}).$$

Hence, ρ_1 is attached to an eigenclass in

$$H_*(\Gamma_{L_1}^{\pm}, (\mathrm{St}(\mathbb{Q}^n) \otimes (M_1)_{\epsilon_1})^{N_1})$$

and $\omega^{-n}\rho_2$ is attached to an eigenclass in

$$H_*(\Gamma_{L_2}^{\pm}, (M_2')_{\epsilon_2}^{N_1}).$$

Now, applying Theorem 2.11, and taking this twist by N_1 into account (which converts $\mathrm{St}(\mathbb{Q}^n)^{N_1}$ into $\mathrm{St}(W)$), we see that $\rho_1 \oplus \rho_2$ is attached to an eigenclass in $H_{\nu-b+t}(\Gamma_L^{\pm}, \mathrm{St}(W) \otimes ((M_1)_{\epsilon_1} \otimes (M_2')_{\epsilon_2})^{N_1}) = H_{\nu-b+t}(\Gamma_L^{\pm}, \mathrm{St}(W) \otimes ((M_1) \otimes (M_2'))_{\epsilon}^{N_1})$.

By [7, Theorem 10.4], $\rho_1 \oplus \rho_2$ is attached to an eigenclass in

$$H_{\nu-b+t}(\Gamma_L, \mathrm{St}(W) \otimes ((M_1) \otimes (M_2'))_{\epsilon}^{N_1}).$$

By Theorem 2.9, this is isomorphic to

$$H_{\nu-b+t}(\Gamma_L, \mathrm{St}(W) \otimes H_{n^2}(\Gamma_U, M_\epsilon)).$$

Since Γ_U acts trivially on $\mathrm{St}(W)$, this is the $E^2_{\nu-b+t,n^2}$ term in the Hochschild-Serre spectral sequence for the exact sequence

$$1 \to \Gamma_U \to \Gamma_P \to \Gamma_L \to 1.$$

So, we have that $\rho_1 \oplus \rho_2$ is attached to an eigenclass z in $E^2_{\nu-b+t,n^2}$.

Suppose z does not survive to $E^{\infty}_{\nu-b+t,n^2}$. Since the VCD of Γ_U is n^2, the only terms of the spectral sequence that can kill this eigenclass are the terms $E^2_{\nu-b+t+k,n^2-(k-1)}$ with $2 \le k$. Because Γ_U acts trivially on W, such a term has the form

$$H_{\nu-b+t+k}(\Gamma_L, \mathrm{St}(W) \otimes H_{n^2-(k-1)}(\Gamma_U, M_\epsilon)).$$

Consider the admissible S_P-module $V_k = H_{n^2-(k-1)}(\Gamma_U, M_\epsilon)$. Then $\rho_1 \oplus \rho_2$ is attached to an eigenclass in

$$H_{\nu-b+t+k}(\Gamma_L, \mathrm{St}(W) \otimes V_k).$$

As in the proof of [7, Theorem 11.5] (see also [11, Lemma 2.1]), $\rho_1 \oplus \rho_2$ is also attached to an eigenclass in

$$H_{\nu-b+t+k}(\Gamma_L, \mathrm{St}(W) \otimes V_k'))$$

for some irreducible S_L-subquotient V_k' of V_k. By Corollary 5.5, V_k' is an admissible module.

Let Γ_L' be the subgroup of Γ_L represented by elements of Γ_P in the kernel of $(\det \circ \psi^1_{N_1}, \det \circ \psi^2_{N_1})$. Then Γ_L' is a subgroup of index 2 in Γ_L; since 2 is relatively prime to p, the corestriction map from Γ_L' to Γ_L is surjective (see the proof of [7, Theorem 10.4]), and we see that any system of eigenvalues appearing in

$$H_{\nu-b+t+k}(\Gamma_L, \mathrm{St}(W) \otimes V_k')$$

appears in

$$H_{\nu-b+t+k}(\Gamma_L', \mathrm{St}(W) \otimes V_k').$$

Now $\Gamma_L' \cong \Gamma_0(n, N_1) \times \Gamma_0(n, N_2)$, via the isomorphism $\psi^1_{N_1} \times \psi^2_{N_1}$. Since V_k' is an irreducible admissible S_L-module (and hence an irreducible $L(\mathbb{Z}/p\mathbb{Z})$-module), it can be written as $(V_1'' \otimes V_2'')^{N_1}$ with each V_i'' an irreducible $\mathrm{GL}(n, \mathbb{Z}/p\mathbb{Z})$-module; using Theorem 4.1, we see that the system of eigenvalues of interest must appear in some

$$H_i(\Gamma_0(n, N_1), \mathrm{St}(W_0) \otimes V_1'') \otimes H_j(\Gamma_0(n, N_2), V_2'').$$

with $i + j = \nu - b + t + k$. By Scholze's theorem [20, Theorem 1.0.3], Hypothesis 8.1 and the definition of $t(n)$, we see that any system of eigenvalues appearing in $H_j(\Gamma_0(n, N_2), V_2'')$ with $j > t(n)$ is attached to a reducible Galois representation. Similarly, if $i > \nu - b(n)$, by Borel-Serre duality and the definition of $b(n)$, any system of eigenvalues appearing in $H_i(\Gamma_0(n, N_1), \mathrm{St}(W_0) \otimes V_1'')$ is attached to a reducible Galois representation. Since $i+j=\nu-b+t+k>(\nu-b(n))+t(n)$, and neither ρ_1 nor ρ_2 is reducible, we see that no values of i and j can yield a system of eigenvalues with $\rho_1\oplus\rho_2$ attached.

This shows that the eigenclass in $H_{\nu-b+t}(\Gamma_L, H_{n^2}(\Gamma_U, \mathrm{St}(W)\otimes M_\epsilon))$ that has $\rho_1\oplus\rho_2$ attached survives to the infinity-page of the Hochschild-Serre spectral sequence, so that there is a system of eigenvalues with $\rho_1\oplus\rho_2$ attached appearing in

$$H_{\nu-b+t+n^2}(\Gamma_P, \mathrm{St}(W)\otimes M_\epsilon)\subseteq \mathcal{E}^1_{n-1,\nu-b+t+n^2}.$$

Hence, by Theorem 3.6, we see that $\rho_1\oplus\rho_2$ is attached to a Hecke eigenclass in at least one of

$$H_{\nu-b+t+n^2+n-1}(\Gamma, M_\epsilon)$$

or

$$H_{2n^2-\nu+b-t-n^2-1}(\Gamma, M_\epsilon).$$

Simplifying, we find that

$$\nu-b+t+n^2+n-1=3\frac{n^2-1}{2}+n-(\eta+\eta')$$

and

$$2n^2-\nu+b-t-n^2-1=\frac{n^2-1}{2}+(\eta+\eta').$$

The fact that M is a predicted weight, N_1N_2 the predicted level and ϵ the predicted nebentype for $\rho_1\oplus\rho_2$ is easily verified. □

We now specialize to the case $n=3$, for which Hypothesis 8.1 is known to be true.

Corollary 8.3. *Let $p>7$. Let $\rho_1,\rho_2: G_{\mathbb{Q}}\to \mathrm{GL}(3,\overline{\mathbb{F}}_p)$ be irreducible odd Galois representations with predicted levels N_1 and N_2, predicted nebentypes ϵ_1 and ϵ_2, and predicted weights $F(a+3,b+3,c+3)$ and $F(d,e,f)$. Choose the triple $(a+3,b+3,c+3)$ representing the weight $F(a+3,b+3,c+3)$ so that $0\le c-d\le p-1$. Let (η,η') be one of $(0,0)$, $(0,1)$ or $(1,0)$. Assume that $\rho_1\oplus\rho_2$ is odd, that N_1N_2 is squarefree, that ρ_1 is attached to an eigenclass in $H_{2+\eta'}(\Gamma_0(3,N_1), F(a+3,b+3,c+3)_{\epsilon_1})$ and that ρ_2 is attached to an eigenclass in $H_{3-\eta}(\Gamma_0(3,N_2), F(d,e,f)_{\epsilon_2})$. Then $\rho_1\oplus\rho_2$ is attached to a Galois representation in at least one of*

$$H_{15-(\eta+\eta')}(\Gamma_0(6,N_1N_2), F(a,b,c,d,e,f)_{\epsilon_1\epsilon_2})$$

or

$$H_{4+(\eta+\eta')}(\Gamma_0(6,N_1N_2), F(a,b,c,d,e,f)_{\epsilon_1\epsilon_2}),$$

where the level N_1N_2, the nebentype $\epsilon_1\epsilon_2$ and the weight $F(a,b,c,d,e,f)$ are predicted for $\rho_1\oplus\rho_2$ by [8].

Proof. We note that by [1, Theorems 4.1.4 and 4.1.5], Hypothesis 8.1 is true for $n=3$, since three-dimensional Galois representations corresponding to cohomology in degrees 0 and 1 must be sums of characters (and hence reducible), the cuspidal range for $\mathrm{GL}(3)/\mathbb{Q}$ is from 2 to 3, and the VCD of $\mathrm{GL}(3,\mathbb{Z})$ equals 3 (by [12, pg. 229], since $\mathrm{GL}(3,\mathbb{Z})$ is commensurable with $\mathrm{SL}(3,\mathbb{Z})$).

The corollary is now a special case of Theorem 8.2. □

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