On the Stability of Viscous-Dispersive Fronts

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Abstract. We consider the class of viscous-dispersive and higher-order conservation laws. We generalize the work of Kawashima & Shizuta, and others, by extending to higher-order the notions of symmetrizability, strict dissipativity, and genuine coupling. We prove, for symmetrizable systems, that strict dissipativity is equivalent to both (i) genuine coupling and (ii) the existence of a skew-symmetric compensating function.

1. Introduction

In recent years, there have been numerous advances in the stability theory of front propagation for systems of viscous conservation laws

$$u_t + f(u)_x = (B(u)u_x)_x. \tag{1}$$

In particular, for a very general class of hyperbolic-parabolic systems with degenerate viscosities, Zumbrun and collaborators [18, 9], generalizing the earlier work of [4, 27, 15, 26, 16, 20, 13] and others, have recently proven nonlinear stability for small-amplitude fronts and have also made considerable progress in the large-amplitude case [19]. Their work relies heavily on the use of a skew-symmetric compensating function developed by Kawashima in his doctoral thesis and subsequent work with Shizuta [11, 12, 23, 14] (see also [17]).

Specifically, Kawashima and Shizuta showed that if systems of (1) are:

(i). symmetrizable, that is, there exists a symmetric positive-definite $A_0(u)$ (denoted $A_0(u) > 0$) so that $A_0(u)df(u)$ symmetric and $A_0(u)B(u)$ is symmetric and positive semi-definite (denoted $A_0(u)B(u) \geq 0$), and

(ii). genuinely coupled, that is, no eigenvector of $df(u)$ is in the kernel of $B(u)$,

then there exists a skew-symmetric compensating function $K(u)$ satisfying

$$A_0(u)B(u) + [K(u), A_0(u)df(u)] \geq \theta(u)I > 0, \tag{2}$$

for some scalar function $\theta(u) > 0$, or in other words, the left-hand side of (2) is positive-definite. Note that $A_0(u)B(u)$ is only positive semi-definite, and it is precisely the addition of the commutator term in (2) that is the key to strict definiteness, thus yielding the coercive expression needed in the energy estimates used to prove stability (see for example [9, 18]).

Kawashima's theory develops by considering the second-order constant coefficient systems

$$v_t = Lv := -Av_x + Bv_{xx}, \quad x \in \mathbb{R},\, t > 0,\, v \in \mathbb{R}^m, \tag{3}$$

where A and B are symmetric and $B \geq 0$. By taking the Fourier transform, the evolution of (3) reduces to solving the eigenvalue problem

$$(\lambda + i\xi A + \xi^2 B)\widehat{v} = 0. \tag{4}$$

We have the theorem:

Theorem 1.1 (Shizuta–Kawashima [23]). *The following statements are equivalent:*

(i). *L is strictly dissipative, that is, $\Re e(\lambda(\xi)) < 0$ for all $\xi \neq 0$.*
(ii). *L is genuinely coupled, that is, no eigenvalue of A is in $\mathcal{N}(B)$.*
(iii). *There exists a skew-Hermitian K such that $[K, A] + B > 0$.*

In this paper, we generalize this theorem by considering the general linear system

$$v_t = \mathcal{L}v := -\sum_{k=0}^{n} D_k \partial_x^k v, \quad x \in \mathbb{R},\, t > 0,\, v \in \mathbb{R}^m, \tag{5}$$

where each $m \times m$ matrix D_k is constant. Likewise, by taking the Fourier transform, the evolution of (5) reduces to solving the eigenvalue problem

$$\lambda\widehat{v} + \sum_{k=0}^{n} (i\xi)^k D_k \widehat{v} = 0. \tag{6}$$

We simplify by separating out odd- and even-ordered terms in (6) to get

$$(\lambda + i\xi A(\xi) + B(\xi))\,\widehat{v} = 0 \tag{7}$$

where

$$A(\xi) := \sum_{k \text{ odd}} D_k (i\xi)^{k-1} \quad \text{and} \quad B(\xi) = \sum_{k \text{ even}} D_k \xi^k. \tag{8}$$

We refer to the matrix-valued terms $A(\xi)$ and $B(\xi)$, respectively, as the generalized flux and generalized viscosity. We have the following definitions:

Definition 1.2.

(i). *$\mathcal{L}$ is called strictly dissipative if for each $\xi \neq 0$, we have that $\Re e(\lambda(\xi)) < 0$.*
(ii). *$\mathcal{L}$ is said to be genuinely coupled if no eigenvector of $A(\xi)$ is in $\mathcal{N}(B(\xi))$, for all fixed $\xi \neq 0$.*

We remark that genuine coupling has physical relevance. In the case of viscous or relaxed conservation laws, it has been shown in many cases that genuine coupling implies time-asymptotic smoothing (see [17, 11, 12, 13, 20]). We remark that a loss of coupling in these instances means that a purely hyperbolic direction exists whereby discontinuous "shock wave" solutions can persist. Mathematically, it is easy to see that genuine coupling is a necessary condition for strict dissipativity. The main result of this paper tells us that for either symmetric or symmetrizable systems, it is also a sufficient condition.

2. Main Result

In this section, we state the main result of the paper. Indeed, we show that for symmetric systems, the properties of strict dissipativity, genuine coupling, and the existence of a skew-symmetric compensating function K are all equivalent. We make the following assumptions:

(H1) $A(\xi)$ is symmetric and of constant multiplicity in ξ.
(H2) $B(\xi) \geq 0$ (symmetric and positive semi-definite).

Lemma 2.1. *Assuming (H1), the distinct eigenvalues $\{\mu_j(\xi)\}_{j=1}^r$ of $A(\xi)$ and their corresponding orthogonal eigen-projections $\{\pi_j(\xi)\}_{j=1}^r$ are real-analytic in ξ.*

Proof. This is a direct consequence of [10, Thm II.6.1, pg. 120]. □

Lemma 2.2. *Assuming $(H2)$, we have that $\mathcal{L}$ is genuinely coupled iff*

$$\theta(\xi) = \inf_{\|x\|=1} \sum_{j=1}^{r} \langle \pi_j(\xi)x, B(\xi)\pi_j(\xi)x\rangle. \tag{9}$$

is positive for all $\xi \neq 0$.

Proof. Omitted. See [8] for the proof. □

Theorem 2.3 (Symmetric Case)**.** *Given $(H1)$ and $(H2)$ above, the following statements are equivalent:*

(i). *$\mathcal{L}$ is strictly dissipative.*
(ii). *$\mathcal{L}$ is genuinely coupled.*
(iii). *There exists a real-analytic skew-Hermitian matrix-valued $K(\xi)$ such that $[K(\xi), A(\xi)] + B(\xi) > 0$ for all $\xi \neq 0$.*

The proof is given in Section 4.

3. Lemmata from Linear Algebra

In this section we state four key lemmata used in our analysis (see [8] for the proofs). We generalize the work of Ellis and Pinsky [2] and Shizuta and Kawashima [23] (see also [6, 17]) by developing, in the language of eigen-projections, the spectral decomposition of the commutator operator $[A, \cdot]$. This allows for the compensating function K, developed by Kawashima, to be expressed in closed form as a Drazin inverse [1] of the commutator operator. By considering symmetric real-analytically varying matrices $A(\xi)$, we likewise obtain the corresponding real-analytically varying compensating function $K(\xi)$, which is the key to extending Kawashima's program to arbitrarily higher-order symmetrizable systems.

Let M_n denote the set of $n \times n$ matrices over $\mathbb{C}$ with the Frobenius inner product

$$\langle X, Y\rangle = \mathrm{tr}\ (X^*Y).$$

Given $A \in M_n$, we define the commutator or Ad operator on M_n as

$$\mathrm{Ad}_A(X) = [A, X].$$

Lemma 3.1.

(i). *Linearity:*

$$Ad_A(aX + bY) = a(Ad_A X) + b(Ad_A Y).$$

(ii). *Leibniz Rule:*

$$Ad_A(BC) = (Ad_A B)C + B(Ad_A C).$$

(iii). *Jacobi Identity:*

$$Ad_{[A,B]} = Ad_A Ad_B - Ad_B Ad_A.$$

(iv). *Adjoint preservation:*

$$(Ad_A)^* = Ad_{A^*}.$$

Lemma 3.2. *Let A be semi-simple. Denote the distinct eigenvalues of A as $\{\mu_j\}_{j=1}^r$ with corresponding eigen-projections $\{\pi_j\}_{j=1}^r$. Then Ad_A is also semi-simple, with eigenvalues of the form $\mu_i - \mu_j$ and corresponding eigen-projections*

$$\pi_{ij}(X) := \pi_i X \pi_j. \tag{10}$$

Moreover, the following hold

(i). *Idempotence and independence:*

$$\pi_{ij}(\pi_{mn}(X)) = \delta_{im}\delta_{jn}\pi_{ij}(X). \tag{11}$$

(ii). *A-invariance:*

$$Ad_A(\pi_{ij}(X)) = \pi_{ij}(Ad_A(X)) = (\mu_i - \mu_j)\pi_{ij}(X). \tag{12}$$

(iii). *Completeness:*

$$\sum_{i,j} \pi_{ij}(X) = X. \tag{13}$$

Lemma 3.3. *Let A be semi-simple. Define following linear operator on M_n:*

$$\Pi_A(X) := \sum_{i=j} \pi_{ij}(X) = \sum_{j=1}^{r} \pi_j X \pi_j. \tag{14}$$

The following hold:

(i). Π_A *is the projection onto $\mathcal{N}(Ad_A)$ along $\mathcal{R}(Ad_A)$.*

(ii). *For each $B \in M_n$ there exists $K \in M_n$ such that*

$$B = \Pi_A(B) + Ad_A(K). \tag{15}$$

Moreover the canonical solution K, which we call the compensating matrix, is of the form

$$K = \sum_{i \neq j} \frac{\pi_{ij}(B)}{\mu_i - \mu_j}. \tag{16}$$

This is the Drazin inverse [1] *(or reduced resolvent) of the Ad_A operator.*

Lemma 3.4. *Assume A and B are Hermitian and $B \geq 0$. Then:*

(i). $\Pi_A(B)$ *is Hermitian.*
(ii). $\Pi_A(B) \geq 0$.
(iii). *The canonical solution K given in* (16) *is skew-Hermitian.*

4. Proof of Theorem 2.3

The proof goes as follows:

(i) $\Rightarrow$ *(ii)*: Suppose for some fixed $\xi \neq 0$ that $A(\xi)v = \mu(\xi)v$ and $v \in \mathcal{N}(B(\xi))$. Then

$$(i\xi A(\xi) + B(\xi))v = i\xi\mu(\xi)v.$$

Hence, $\Re e\lambda(\xi) = \Re e(-i\xi\mu(\xi)) = 0$, which contradicts *(i)*.

(ii) $\Rightarrow$ *(iii)*: Let $\xi \neq 0$ be fixed. By Lemmas 3.3 and 3.4, there exists a skew-Hermitian $K(\xi)$ satisfying

$$B(\xi) = \Pi_{A(\xi)}(B(\xi)) + \mathrm{Ad}_{A(\xi)}(K(\xi)).$$

In addition, we know that $\Pi_{A(\xi)}(B(\xi))$ is Hermitian and positive semi-definite. Note that $\Pi_{A(\xi)}(B(\xi)) \geq \theta(\xi)I$. Since genuine coupling implies that $\theta(\xi) > 0$, strict positive-definiteness directly follows. To show that $K(\xi)$ is real-analytic, we need only apply Lemma 2.1 to the canonical choice of $K(\xi)$ given in (16). Note that

$$K(\xi) = \sum_{i \neq j} \frac{\pi_i(\xi)B(\xi)\pi_j(\xi)}{\mu_i(\xi) - \mu_j(\xi)}$$

is well-defined for all $\xi \neq 0$ and real-analytic in ξ. Note that constant multiplicity is used here to keep the denominator of $K(\xi)$ bounded away from zero.

(iii) $\Rightarrow$ *(i)*: We suppose that for some fixed ξ there is a λ and v satisfying (7). We combine two spectral energy estimates. First, by taking the inner product of (7) with v and taking the real part, we get the following standard Friedrichs-type estimate [3]:

$$\Re e(\lambda)\|v\|^2 + \langle v, B(\xi)v\rangle = 0. \tag{17}$$

We also have

$$\Re e(\lambda)\|v\|^2 + \frac{1}{\|B(\xi)\|}\|B(\xi)v\|^2 \leq 0. \tag{18}$$

Note that both (17) and (18) imply that $\Re e(\lambda) \leq 0$. For the second estimate, we multiply (7) by the Hermitian matrix $2i\xi K(\xi)$ and take the inner product with v to get

$$2\xi^2\langle v, K(\xi)A(\xi)v\rangle = 2i\lambda\xi\langle v, K(\xi)v\rangle + 2i\xi\langle v, K(\xi)B(\xi)v\rangle \tag{19}$$

Taking the real part and using Young's inequality yields

$$\begin{aligned}
&2\xi^2\langle v, [K(\xi), A(\xi)]v\rangle \\
&\quad = 4i\xi\Re e(\lambda)\langle v, K(\xi)v\rangle + 2i\xi\langle v, (K(\xi)B(\xi) + B(\xi)K(\xi))v\rangle \\
&\quad \leq 4|\Re e(\lambda)||\xi|\|K(\xi)\|\|v\|^2 + 4|\xi|\|v\|\|K(\xi)\|\|B(\xi)v\| \\
&\quad \leq 4|\Re e(\lambda)||\xi|\|K(\xi)\|\|v\|^2 + \theta(\xi)\xi^2\|v\|^2 + 4\frac{\|K(\xi)\|^2}{\theta(\xi)}\|B(\xi)v\|^2,
\end{aligned}$$

where $[K(\xi), A(\xi)] + B(\xi) \geq \theta(\xi)I > 0$. Multiplying $2\xi^2$ by (17) and adding gives

$$\xi^2\theta(\xi)\|v\|^2 + 4\Re e(\lambda)|\xi|\|K(\xi)\|\|v\|^2 \leq 4\frac{\|K(\xi)\|^2}{\theta(\xi)}\|B(\xi)v\|^2.$$

Using (18) to cancel the right-hand side yields

$$\xi^2\theta(\xi)\|v\|^2 + 4\Re e(\lambda)|\xi|\|K(\xi)\|\|v\|^2 + 4\Re e(\lambda)\frac{\|K(\xi)\|^2}{\theta(\xi)}\|B(\xi)\|\|v\|^2 \leq 0.$$

Hence

$$\Re e(\lambda) \leq \frac{-\xi^2\theta(\xi)^2}{4|\xi|\|K(\xi)\|\theta(\xi) + 4\|K(\xi)\|^2\|B(\xi)\|}. \tag{20}$$

Thus $\Re e(\lambda(\xi)) \leq 0$ for all $\xi \neq 0$. This completes the proof.

5. Symmetrizability

In both viscous and relaxed conservation laws, symmetrizability has been proven, repeatedly, to be important (see for example [11, 17]). Pego [21, 22] was the first to consider symmetrizability in the general higher-order case. Indeed he found necessary conditions for the admissibility of term-wise symmetrizable systems. We remark, however, that Slemrod's model (given below) is not term-wise symmetrizable, but by extending our notion of a symmetrizer from a positive-definite matrix to a positive-definite differential operator, we are able to symmetrize it.

Slemrod's model [24, 25, 5] for a compressible isentropic gas with capillarity is defined as

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p(v)_x &= (b(v)u_x)_x + dv_{xxx}, \end{aligned} \tag{21}$$

where physically, v is the specific volume, u is the velocity in Lagrangian coordinates, $p(v)$ is the pressure law for an ideal gas, that is, $p'(v) < 0$ and $p''(v) > 0$,

$b(v)$ is the viscosity, satisfying $b(v) > 0$, $b'(v) \leq 0$, and the capillarity term $d < 0$ is constant. The Fourier-transformed constant coefficient case is given as

$$\begin{aligned} v_t - i\xi u &= 0, \\ u_t - i\xi(c^2 - d\xi^2)v + \xi^2 bu &= 0, \end{aligned} \tag{22}$$

where $p'(v) = -c^2$ and $b > 0$. Note that (22) can only be symmetrized by permitting a ξ-valued symmetrizer, namely,

$$A_0(\xi) = \begin{pmatrix} c^2 - \xi^2 d & 0 \\ 0 & 1 \end{pmatrix}, \tag{23}$$

which is symmetric and positive-definite, as required. Left multiplying (22) by $A_0(\xi)$ yields

$$\lambda A_0(\xi) \begin{pmatrix} v \\ u \end{pmatrix} + i\xi \begin{pmatrix} 0 & -c^2 + \xi^2 d \\ -c^2 + \xi^2 d & 0 \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix} = \xi^2 \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix},$$

which is symmetrized, and thus satisfies $(H1)$ and $(H2)$ above. Hence, we propose the definition:

Definition 5.1. *$\mathcal{L}$ is called symmetrizable if there exists a symmetric, real-analytic matrix-valued function $A_0(\xi) > 0$ such that both $A_0(\xi)A(\xi)$ and $A_0(\xi)B(\xi)$ are symmetric, and $A_0(\xi)B(\xi) \geq 0$.*

With this definition of symmetrizability, its easy to see that Theorem 2.3 extends from the symmetric to the fully symmetrizable case. We remark also that extending the idea of a symmetrizer to a differential operator has profound consequences as to what it means to have a convex entropy for higher-order systems. We further explore this topic and others in [8].

6. Acknowledgments

The author would like to thank Jeffrey Rauch and Kevin Zumbrun for useful and insightful conversations during this work.

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