Key for Sample Exam 3

(1) False	(2) True	(3) True	(4) False	(5) True	(6) False	(7) True	(8) False	(9) False	(10) True
(11) B	(12) C	(13) D	(14) B	(15) B	(16) D	(17) D	(18) D	(19) C	(20) A

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(21) (a) The function $f : \mathbb{N} \to \mathbb{N}$ given by $f(n) = n + 1$ is injective but not surjective.

(b) Disproof: Let $A = \{1\}$, $B = \{2, 3\}$ and $C = \{4\}$. Set $f = \{(1, 2)\}$, and $g = \{(2, 4), (3, 4)\}$. Then $g \circ f = \{(1, 4)\}$ is injective, but g is not injective (since $g(2) = g(3)$).

(c) Proof: Suppose $g(f(a_1)) = g(f(a_2))$. Then, since g is injective, $f(a_1) = f(a_2)$. Since f is injective, $a_1 = a_2$. Hence, $g \circ f$ is injective.

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(22) First we not that since $5x + 1$ can never equal $5(x - 2)$, the function f always maps elements of $\mathbb{R} - \{2\}$ to $\mathbb{R} - \{5\}$. Hence, it is a function with the given domain and codomain.

For $a, b \in \mathbb{R} - \{2\}$, suppose that $f(a) = f(b)$. Then

$$\frac{5a + 1}{a - 1} = \frac{5b + 1}{b - 2}.$$

Hence, crossmultiplying (which we can do since neither $a - 2$ nor $b - 2$ is zero),

$$(5a + 1)(b - 2) = (5b + 1)(a - 2),$$

and then simplifying, we find that $5ab - 10a + b - 2 = 5ab - 10b + a - 2$. Cancelling terms, we find that $9a = 9b$ so that $a = b$. Hence, f is injective.

Let $a \in \mathbb{R} - \{5\}$. Set $b = \frac{2a+1}{a-5}$. Note that since $2a + 1 \neq 2(a - 5)$, we see that $b \neq 2$, so $b \in \mathbb{R} - \{2\}$.

Now

$$\begin{aligned} f(b) &= \frac{5b + 1}{b - 2} \\ &= \frac{5\left(\frac{2a+1}{a-5}\right) + 1}{\left(\frac{2a+1}{a-5}\right) - 5} \\ &= \frac{5(2a + 1) + (a - 5)}{(2a + 1) - 2(a - 5)} \\ &= \frac{11a}{11} \\ &= a \end{aligned}$$

Hence, $f(b) = a$. Since a was arbitrary, we see that f is surjective.

In addition, we see that

$$f^{-1}(x) = \frac{2x + 1}{x - 5}.$$

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(23) Suppose that $\bar{a} = \bar{b}$. Then $a \equiv b \pmod 5$. Hence, $a = b + 5k$. Then

$$a^5 = (b + 5k)^5 = b^5 + 5b^4(5k) + 10b^3(5k)^2 + 10b^2(5k)^3 + 5b(5k)^4 + (5k)^5 \equiv b^5 \pmod 5.$$

Hence, $\overline{a^5} = \overline{b^5}$. Therefore, f is well defined.

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(24) See the proof of Theorem 30.4, or the proof of Theorem 31.5 (applied to $S = \mathbb{N}$ as in Corollary 31.7).

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(25) For $x \in A$ we note that $x + 6 \in (6, 11) \subseteq B$, so there is a function $f : A \to B$ given by $f(x) = x + 6$. This function is injective since $f(a) = f(b)$ implies that $a + 6 = b + 6$, so that $a = b$. Hence $|A| \leq |B|$.

For $x \in B$, we note that $0 < x < 50$, so $x/10 \in A$. We may thus define a function $g : B \to A$ by $g(x) = x/10$. This function is injective since $g(a) = g(b)$ implies that $a/10 = b/10$, so that $a = b$. Hence, $|B| \leq |A|$.

By the Schröder-Bernstein theorem, we thus see that $|A| = |B|$.