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Math 290
Sample Exam 3

Note that the first 10 questions are true-false. Mark A for true, B for false. Questions 11 through 20 are multiple choice; mark the correct answer on your bubble sheet. Answers to the last five questions should be written directly on the exam, and should be written neatly and with full justification.

True-false questions

(1) Any infinite subset of an uncountable set is uncountable.

(2) The set $\mathbb{N} \times \mathbb{Q} \times \mathbb{N} \times \mathbb{Q}$ is countably infinite.

(3) Let X be a set. The set of ordered pairs $\{(x, x) : x \in X\}$ is a function from X to X.

(4) The image of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin(x)$ is the interval $[0, 1]$.

(5) For any non-empty set S, there is a surjective function $f : \mathcal{P}(S) \to S$ where $\mathcal{P}(S)$ is the power set of S.

(6) There exists a set S such that the cardinality of any other set is less than or equal to $|S|$.

(7) It holds that $|\mathcal{P}(\mathbb{Z})| = |\mathbb{R}|$ where $\mathcal{P}$ is the power set.

(8) If $f : A \to B$ and $g : B \to A$ are functions and $g \circ f = 1_A$, then f must be bijective.

(9) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 + 3$. Then $f^{-1}(\{4, 7\}) = \{1, 2\}$.

(10) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Then $f|_{[0,1]}$ is injective.

Multiple choice section

(11) Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 5, 9, 14, 20\}$. Let $R : A \to B$ be the relation given by

$$R = \{(1, 9), (2, 2), (3, 20), (4, 5)\}.$$

Which of the following statements is **most complete** about the relation $R : A \to B$ and the inverse relation $R^{-1} : B \to A$?

(a) R is a function and R^{-1} is a function.
(b) R is a function but R^{-1} is not a function.
(c) R^{-1} is a function but R is not a function.
(d) R is not a function and R^{-1} is not a function.

(12) Which of the following functions is/are bijective? Choose the **most complete** answer.

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.
(b) $g : \mathbb{N} \to \mathbb{N}$ defined by $g(n) = n + 1$.
(c) $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = 2x + 1$.
(d) $\psi : \mathbb{Z} \to \mathbb{Z}$ defined by $\psi(n) = 2n + 1$.
(e) More than one of the above.

(13) Let $X = \{1, 2, 3\}$. Which of the following is *not* a function from X to $\mathbb{Z}$?

(a) $\{(1, 1), (2, 3), (3, 5)\}$.
(b) $f(n) = n^2$ for $n \in X$.
(c) $\{(1, 1), (2, 1), (3, 1)\}$.
(d) $\{(1, 1), (1, 2), (1, 3)\}$.
(e) All of the above are functions from X to $\mathbb{Z}$.

(14) Which of the following is *not* a well-defined function?

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 - x$.
(b) $g : \mathbb{Z}_4 \to \mathbb{Z}_3$ defined by $f(\bar{x}) = [x]$ (where $\bar{x}$ is the congruence class mod 4 and $[x]$ is the congruence class mod 3).
(c) $h : \mathbb{Q} - \{0\} \to \mathbb{R}$ defined by $h(q) = \frac{1}{q}$.
(d) $\{(1, 1), (2, 1), (3, 1)\}$ from $\{1, 2, 3\}$ to $\mathbb{N}$.
(e) All of the above are well-defined functions.

(15) Which of the following statements is true, concerning the four sets $\mathcal{P}(\mathbb{N})$, $(0, 1)$, $\mathcal{P}(\mathbb{R})$, $\mathbb{R} \times \mathbb{R}$?

(a) All four have the same cardinality.
(b) Exactly three have the same cardinality.
(c) Exactly two have the same cardinality.
(d) None of them have the same cardinality.
(e) None of the above.

(16) If $S = \{1,2,3,4,5,6,7\}$ and $f : S \to \mathcal{P}(S)$ is the function given by the rule

$$f = \{(1,\{2,5\}),(2,\{1,2,3\}),(3,S),(4,\emptyset),(5,\{6,7\}),(6,S),(7,\{1,7\}\},$$

then the barber set is:
(a) $\{1,2,3,4,5,6,7\}$.
(b) $\{1,3,5,7\}$.
(c) $\{2,4,6\}$.
(d) $\{1,4,5\}$.
(e) $\{2,3,6,7\}$.
(f) None of the above.

(17) If S is a countably infinite set, and T is a set such that $|T| > |S|$, then which of the following statements gives the **most complete** information about the cardinality of T?
(a) T is finite.
(b) T is countable.
(c) T is countably infinite.
(d) T is uncountable.
(e) T has continuum cardinality.

(18) Let $f : \mathbb{Z}_4 \to \mathbb{Z}_4$ be given by $f(\overline{a}) = \overline{a}^3$. Which of the following statements is **most complete**?
(a) f is injective.
(b) f is surjective.
(c) f is bijective.
(d) f is neither injective nor bijective.

(19) Let $f : A \to B$ and $g : B \to C$ be functions. Which statement is always true?
(a) If f is an injection and g is a surjection, then $g \circ f$ is a surjection.
(b) If f is a surjection and g is an injection, then $g \circ f$ is an injection.
(c) If f is a bijection and g is a surjection, then $g \circ f$ is a surjection
(d) If f is a bijection and g is a surjection, then $g \circ f$ is an injection.

(20) Suppose $f : A \to B$ is a function. Assume $X, Y \subseteq A$ and $W, Z \subseteq B$. Which of the following statements is not necessarily true?
(a) $f^{-1}(f(X)) = X$.
(b) $f(X \cap Y) \subseteq f(X) \cap f(Y)$.
(c) $f(X \cup Y) \subseteq f(X) \cup f(Y)$.
(d) $f^{-1}(Z \cap W) = f^{-1}(Z) \cap f^{-1}(W)$.
(e) $f^{-1}(B) \subseteq A$.

Free Response Section

(21) (a) Give an example of a function with domain $\mathbb{N}$ and codomain $\mathbb{N}$ which is injective but not surjective.

(b) Prove or disprove: If $f : A \to B$ and $g : B \to C$ are functions and $g \circ f$ is injective, then g is injective.

(c) Prove or disprove: If $f : A \to B$ and $g : B \to C$ are injective functions, then $g \circ f$ is an injective function.

(22) Prove that $f : \mathbb{R} - \{2\} \to \mathbb{R} - \{5\}$ given by $f(x) = \frac{5x+1}{x-2}$ is bijective and find its inverse.

(23) Determine whether or not the rule $f(\overline{a}) = \overline{a^5}$ describes a well-defined function f from $\mathbb{Z}_5$ to $\mathbb{Z}_5$. You must provide an example that shows f is not a well-defined function, or you must prove that f is a well-defined function.

(24) Prove that there exists an uncountable set. (There are theorems in the book that say certain sets are uncountable. You may **not** cite these theorems because they say what you need to prove. However, you can reproduce one of the proofs, if you wish.)

(25) Let $A = (0, 5) \subseteq \mathbb{R}$ and $B = [1, 3] \cup (6, 20) \cup (23, 24] \cup \{31\} \subseteq \mathbb{R}$. Prove or disprove that $|A| = |B|$.