ODD PERFECT NUMBERS HAVE A PRIME FACTOR EXCEEDING 10^7

by

Paul M. Jenkins

Submitted to Brigham Young University in partial fulfillment
of graduation requirements for University Honors

Department of Mathematics

September 2000

Advisor: David A. Clark

Signature: ______________________

Defense Chair: C. Wilfred Griggs

Signature: ______________________

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Abstract

Every odd perfect number is divisible by a prime greater than 10^7.

1. Introduction

A perfect number is an integer N which satisfies $\sigma(N) = \sum_{d|N} d = 2N$. All known perfect numbers are even, with a one-to-one correspondence between even perfect numbers and Mersenne primes (prime numbers of the form $2^p - 1$, where p is also prime). Even perfect numbers have the form $N = 2^{p-1}(2^p - 1)$, where p is prime and $2^p - 1$ is a Mersenne prime. It is hypothesized that no odd perfect numbers exist, but this has yet to be proven. However, certain conditions that a hypothetical odd perfect number must satisfy have been found. Brent, Cohen, and teRiele [2] proved that such a number must be greater than 10^{300}. Iannucci showed that its second largest prime divisor must exceed 10^4 [10] and that the third largest prime divisor must be greater than 100 [11]. Chein [3] and Hagis [6] each showed that an odd perfect number must have at least 8 distinct prime factors.

The bound for the largest prime divisor of an odd perfect number has been raised from 100110 in 1975 by Hagis and McDaniel [9] to 300000 in 1978 by Condict [4] to 500000 in 1982 by Brandstein [1]. Most recently, Hagis and Cohen [8] proved that the largest prime divisor of an odd perfect number must be greater than 10^6.

This paper improves the lower bound for the largest prime divisor of an odd perfect number to 10^7.

2. Necessary theorems

The notation used by Hagis and Cohen [8] will be followed here. Non-negative integers will be symbolized by $a, b, c, \ldots$, and p, q and r will represent prime numbers. The dth cyclotomic polynomial will be denoted by F_d, so that $F_p(x) = 1 + x + x^2 + \cdots + x^{p-1}$. If p and m are relatively prime, $h(p, m)$ will represent the order of p modulo m.

2.1. **An expression for $\sigma(p^a)$.** The divisors of p^a are $1, p, p^2, \ldots, p^a$, so

$$\sigma(p^a) = 1 + p + p^2 + \ldots + p^a = \frac{p^{a+1} - 1}{p - 1}.$$

From section 13.6 in [5] we have the factorization

$$x^n - 1 = \prod_{d|n} F_d(x).$$

Thus,

$$\sigma(p^a) = \frac{\prod_{d|(a+1)} F_d(p)}{p - 1};$$

since $F_1(p) = p - 1$, we have

Theorem 2.1.

$$\sigma(p^a) = \prod_{\substack{d|(a+1) \\ d>1}} F_d(p).$$

2.2. **Divisors of $F_m(p)$.** Suppose $q \nmid n$.

If $q \equiv 1 \pmod{n}$, then $n|(q-1)$. By Fermat's theorem, $a^{q-1} \equiv 1 \pmod{q}$ for all $a \not\equiv 0 \pmod{q}$. Then

$$(x^{q-1} - 1) = \prod_{a \not\equiv 0 \pmod{q}} (x - a),$$

so

$$F_n(x) \mid \prod_{a\not\equiv 0 \pmod q} (x-a),$$

and all of the primitive nth roots of $1 \pmod q$ are solutions to $F_n(x) \equiv 0 \pmod q$.

If $F_n(x) \equiv 0 \pmod q$ has solutions, then these solutions will be the primitive nth roots of $1 \pmod q$. Since $a^{q-1} \equiv 1 \pmod q$ for all $a \not\equiv 0 \pmod q$ by Fermat's theorem, $n|(q-1)$ and $q \equiv 1 \pmod n$.

Thus, $q|F_n(p)$ if and only if $n = h(p;q)$.

Nagell [15] proves in his Theorem 94 that if $q|F_n(x)$, then $F_n(x)$ is divisible by exactly the same power of q as $x^n - 1$.

Suppose $q|n$, with $n = q^a n_1$, where n_1 is not divisible by q.

If $q|F_n(x)$, then $q|(x^n-1)$, and $x^n \equiv 1 \pmod q$. By Fermat's theorem, $x^q \equiv x \pmod q$, so $x^n \equiv (x^{q^a})^{n_1} \equiv x^{n_1} \equiv 1 \pmod q$. Since $x^{n_1} - 1 \equiv 0 \pmod q$, $q|F_{n_1}(x)$ and $n_1 = h(p;q)$ from the above proof.

Nagell also shows in his Section 46 that $F_n(x) = \frac{F_{n_1}(x^{q^a})}{F_{n_1}(x^{q^{a-1}})}$. But $F_{n_1}(x^{q^a})$ is divisible by the same power of q as $x^{q^a n_1} - 1$, and

$$F_{n_1}(x^{q^a}) = (x^{q^{a-1}n_1} - 1)(x^{(q-1)q^{a-1}n_1} + x^{(q-2)q^{a-1}n_1} + \ldots + x^{q^{a-1}n_1} + 1).$$

Looking modulo q, $F_n(x) \equiv x^{(q-1)q^{a-1}n_1} + x^{(q-2)q^{a-1}n_1} + \ldots + x^{q^{a-1}n_1} + 1$. But if $n_1 = h(p;q)$, each of the q terms of this expression is equivalent to $1 \pmod q$, and $F_n(x) \equiv (1+1+\ldots+1) = q \pmod q$, so $q|F_n(x)$.

Thus $q|F_n(x)$ if and only if $n = q^b h(p;q)$.

Nagell proves in his Theorem 95 that if $q|n$ and $q|F_n(x)$, then $q\|F_n(x)$ for $n > 2$.

Combining these results, it follows that

Theorem 2.2. *It is true that* $q|F_m(p)$ *if and only if* $m = q^b h(p;q)$. *If* $b > 0$, *then* $q \| F_m(p)$. *If* $b = 0$, *then* $q \equiv 1 \pmod{m}$.

It follows from Theorem 2.2 that, for r prime,

Theorem 2.3. *If* $q|F_r(p)$ *then either* $r = q$ *and* $p \equiv 1 \pmod{q}$, *so that* $q \| F_r(p)$, *or* $q \equiv 1 \pmod{r}$.

Note that $h(p;3) = 2$ if $p \equiv 2 \pmod{3}$ and that $h(p;5) = 2$ when $p \equiv 4 \pmod{5}$ and 4 when $p \equiv 2$ or $3 \pmod{5}$. Therefore, if $q = 3$ or 5, $h(p;q)$ is even unless $p \equiv 1 \pmod{q}$, and it also follows from Theorem 2.2 that

Theorem 2.4. *If* $q = 3$ *or* 5 *and* $m > 1$ *is odd, then* $q|F_m(p)$ *(and* $q \| F_m(p)$*) if and only if* $m = q^b$ *and* $p \equiv 1 \pmod{q}$.

2.3. **Kanold's Theorem.** Recall that $F_m(x) = (x-\epsilon_1)(x-\epsilon_2)\cdots(x-\epsilon_{\varphi(m)})$, where ϵ_i is a primitive nth root of unity, $|\epsilon_i| = 1$, and $\varphi(m)$ is Euler's φ-function.

Kanold [12] proved the following theorem:

Theorem 2.5. *For an integer* x, *if* $r|F_m(x)$ *and* $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \geq 3, p_1 < p_2 < \ldots < p_k$, *then:*

If $r = p_k$ *then* $p_k \equiv 1 \pmod{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{k-1}^{\alpha_{k-1}}}$.

If $p_k \not\equiv 1 \pmod{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{k-1}^{\alpha_{k-1}}}$ *then* $r \equiv 1 \pmod{m}$ *and* $p_k^2 \nmid F_m(x)$.

His proof is as follows: From Theorem 2.2, if $r|F_m(x)$ then either $r|m$ or $r \equiv 1 \pmod{m}$.

Note that

$$\frac{x^m - 1}{x - 1} = x^{m-1} + x^{m-2} + \ldots + x + 1 = F_m(x) \cdot \prod_{\substack{d|m \\ d \neq 1,m}} F_d(x).$$

Set $m_l = m/p_l$ for $l = 1, 2, \ldots, k$.

Then

$$(1 + x + x^2 + \ldots + x^{m_l - 1})(1 + x^{m_l} + x^{2m_l} + \ldots + x^{(p_l - 1)m_l}) = x^{m-1} + x^{m-2} + \ldots + x + 1.$$

Let

$$a_l = (1 + x + x^2 + \ldots + x^{m_l - 1})$$

and

$$b_l = (1 + x^{m_l} + x^{2m_l} + \ldots + x^{(p_l - 1)m_l}).$$

Since

$$a_l = F_{m_l}(x) \prod_{\substack{d|m_l \\ d \neq 1, m_l}} F_d(x),$$

it must be true that

$$a_l | \prod_{\substack{d|m \\ d \neq 1, m}} F_d(x).$$

But

$$a_l b_l = F_m(x) \cdot \prod_{\substack{d|m \\ d \neq 1, m}} F_d(x),$$

so $F_m(x) | b_l$.

Suppose that $r_l | b_l$, r_l a prime. Then $x^m = x^{m_l p_l} \equiv 1 \pmod{r_l}$, since $r_l | b_l$ and $a_l b_l | (x^m - 1)$.

Suppose that $x^{m_l} \equiv 1 \pmod{r_l}$. Then

$$b_l \pmod{r_l} = 1 + x^{m_l} + x^{2m_l} + \ldots + x^{(p_l - 1)m_l} \equiv 1 + 1 + \ldots + 1 = p_l \pmod{r_l}.$$

But since $r_l | b_l$, $b_l \equiv 0 \pmod{r_l}$, so $r_l = p_l$ since both are prime.

If such is the case, then $x^{m_l} = 1 + sp_l$ for some s. Thus,

$$\begin{aligned} b_l &= 1 + x^{m_l} + x^{2m_l} + \ldots + x^{(p_l-1)m_l} \\ &= 1 + (1 + sp_l) + (1 + sp_l)^2 + \ldots + (1 + sp_l)^{p_l-1} \\ &= p_l + sp_l + 2sp_l + \ldots + (p_l - 1)sp_l + Ap_l^2 \\ &= p_l + sp_l(\frac{p_l(p_l-1)}{2}) + Ap_l^2 \equiv p_l \pmod{p_l^2}. \end{aligned}$$

Thus, $p_l^2 \not| \, b_l$.

The other case is that $x^{m_l} \not\equiv 1 \pmod{r_l}$. Knowing that $x^m = x^{m_l p_l} \equiv 1 \pmod{r_l}$ and that $x^{r_l - 1} \equiv 1 \pmod{r_l}$ by Fermat's theorem, let $t = \gcd(r_l - 1, m_l p_l)$. Thus, $x^t \equiv 1 \pmod{r_l}$.

Let $n_l = m/p_l^{\alpha_l}$. Then $(x^{n_l p_l^{\alpha_l - 1}})^{p_l} \equiv 1 \pmod{r_l}$, but $x^{n_l p_l^k} \not\equiv 1 \pmod{r_l}$ for $k < \alpha_l$. Thus, the order of $x^{n_l} \pmod{r_l}$ is $p_l^{\alpha_l}$. The order of an element divides the order of a group (in this case, the multiplicative group with $r_l - 1$ elements), so $p_l^{\alpha_l} | (r_l - 1)$. Thus, $r_l \equiv 1 \pmod{p_l^{\alpha_l}}$ for $l = 1 \ldots k$.

Thus, if r is a prime divisor of $F_m(x)$, then for all l either $r = p_l$ or $r \equiv 1 \pmod{p_l^{\alpha_l}}$. But $r = p_l$ for at most one l, so $r = p_d$ and $r \equiv 1 \pmod{p_l^{\alpha_l}}$ for $l \neq d$. If $d \neq k$ then we have $p_k^{\alpha_k} | (r-1)$ and $(r-1) < p_k^{\alpha_k}$, a contradiction that gives us $d = k$.

It is thus shown that if $r = p_k$ then $p_k \equiv 1 \pmod{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{k-1}^{\alpha_{k-1}}}$.

If $p_k \not\equiv 1 \pmod{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{k-1}^{\alpha_{k-1}}}$, then by the contrapositive it is true that $p_k \neq r$, and that $r \equiv 1 \pmod{p_l^{a_l}}$ for all l, so $r \equiv 1 \pmod{m}$. Since $F_m(x) | b_l$ and $p_l^2 \not| \, b_l$, it must be true that $p_l^2 \not| \, F_m(x)$.

Thus, if $F_m(x)$ has no prime divisors congruent to $1 \pmod{m}$, p_k is the only divisor of $F_m(x)$. Since $p_k^2 \not| F_m(x)$, then $p_k = F_m(x)$.

Suppose that $m \geq 3$ and $x \geq 3$. Since $x \geq 3$, $|x - \epsilon_i| > 2$, and $F_m(p) > 2^{\varphi(m)}$. But φ is multiplicative, so $\varphi(m) = \varphi(p_1)\cdots\varphi(p_k) \geq \varphi(p_k) = (p_k - 1)$, and $2^{\varphi(m)} \geq 2^{p_k - 1} \geq p_k$. Thus, $F_m(p) > p_k$, and p_k cannot be the only divisor of $F_m(x)$, so it is true that

Theorem 2.6. *If p is an odd prime and $m \geq 3$, then $F_m(p)$ has at least one prime factor q such that $q \equiv 1 \pmod{m}$.*

2.4. **Factorization of N.** Note that for odd p, $F_d(p) = 1 + p + p^2 + \ldots + p^{d-1}$ has d odd terms, and is thus even if d is even and odd if d is odd.

Suppose N is odd and $\sigma(N) = 2N$, with

$$N = p_0^{a_0} p_1^{a_1} \cdots p_u^{a_u}. \tag{2.1}$$

Since σ is multiplicative, $2N = \sigma(N) = \sigma(p_0^{a_0})\sigma(p_1^{a_1})\cdots\sigma(p_u^{a_u})$. From Theorem 2.1 it is obvious that

Theorem 2.7.

$$2N = \prod_{i=0}^{u} \sigma(p_i^{a_i}) = \prod_{i=0}^{u} \prod_{\substack{d|(a_i+1) \\ d>1}} F_d(p_i)$$

Since $2 \| \sigma(N)$, there must be exactly one even $\sigma(p_i^{a_i})$ and thus only one even $F_d(p)$ in the factorization of N. Since $d|(a_i + 1)$, it must be true that $2|a_i$ for all a_i except one. Assume without loss of generality that this is a_0, and call p_0 the *special* prime.

If $a_0 \equiv 3 \pmod{4}$, then $4|(a_0 + 1)$ and $F_4(p_0)|N$. But $F_4(p_0) = p_0^3 + p_0^2 + p_0 + 1 \equiv 0 \pmod{4}$, and then $4|N$, a contradiction. Thus, it must be true that $a_0 \equiv 1 \pmod{4}$.

If $p_0 \equiv 3 \pmod 4$, then $p_0^b \equiv 1 \pmod 4$ when b is even and 3 (mod 4) when b is odd, so pairs of terms of $F_{a_0+1}(p_0)$ add to $0 \pmod 4$; since there are an even number of terms, $4|F_{a_0+1}(p_0)$ and $4|N$, a contradiction. Thus, $p_0 \equiv 1 \pmod 4$.

3. Statement of theorem

This paper extends the previous result of Hagis and Cohen [7] by proving the following:

Theorem 3.1. *If N is odd and perfect, then N has a prime factor greater than 10^7.*

The proof, following the methods used by Hagis and Cohen, is by contradiction. Without further explicit mention, assume that $p_i < 10^7$ for every p_i in equation 2.1. It is obvious that the set of p_i in equation 2.1 is identical to the set of odd prime factors of the $F_d(p_i)$ in Theorem 2.7, so all prime factors of each $F_d(p_i)$ must be less than 10^7. In particular, if r is a prime divisor of $a_i + 1$, then every prime factor of $F_r(p_i)$ must be less than 10^7.

4. Acceptable values of $F_r(p)$

If r and $p > 2$ are prime, we will define $F_r(p)$ to be *acceptable* if every prime divisor of $F_r(p)$ is less than 10^7. If $r > 5000000$, then since p is odd, from equation 2.6 we know that $F_r(p)$ has at least one prime factor q congruent to $1 \pmod r$. Since 1 is not prime and $r+1$ is even, $q \geq 2r + 1 > 10^7$, and thus $F_r(p)$ is not acceptable.

Define p to be *inadmissible* if $F_r(p)$ is unacceptable for every prime r, considering $r = 2$ only if it is possible that p is the special prime for N.

Through a lengthy computer search, it was discovered that if $3 \leq p < 10^7$ and $r \geq 7$, then $F_r(p)$ is unacceptable except for the pairs of values of p and r listed in Table A, located in the appendix to this paper.

5. Inadmissible small primes

It is shown in this section that N is not divisible by any of a certain set of primes.

Lemma 5.1. *Let X be the set of primes*

$$\{3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 43, 61, 71,$$
$$113, 127, 131, 151, 197, 211, 239, 281, 1093\}$$

If $p \in X$, then $p \nmid N$.

The proof considers each prime p in X one by one, but in the order

$$1093, 151, 31, 127, 19, 11, 7, 23, 31, 37, 43, 61,$$
$$13, 3, 5, 29, 43, 17, 71, 113, 197, 211, 239, 281.$$

We assume $p \,|\, N$ and find all acceptable values of $F_r(p)$ by checking Table A to find acceptable values with $r \geq 7$, factoring $F_3(p)$ and $F_5(p)$ to check for acceptability, and factoring $F_2(p)$ if it is possible that p is the special prime—that is, if $p \equiv 1 \pmod 4$ and no other prime is already assumed to be the special prime. From Theorem 2.7 we know that $F_r(p)|2N$ for at least one acceptable $F_r(p)$, and that each odd prime divisor of this $F_r(p)$ divides N. A single odd prime divisor q is selected from each acceptable $F_r(p)$, and the acceptable values of $F_r(q)$ are found. This procedure is repeated for all acceptable values until a contradiction is found (such as an inadmissible prime), proving that $p \nmid N$.

To illustrate, we show that $1093 \nmid N$ and $151 \nmid N$. The complete proof of Lemma 5.1 is found in the appendix. Write p^* to indicate that p is the special prime; begin each line by listing the primes that are assumed to be factors of N.

A. Proof that $1093 \nmid N$.

Assume that $1093|N$. One value of $F_r(1093)$ must divide N. List values of $F_r(1093)$ from Table A and for $r = 2, 3, 5$.

A, $1093 : F_2(1093) = 2 \cdot 547; F_3(1093) = 3 \cdot 398581;\ F_5(1093) = 11 \cdot 31 \cdot 4189129561$ is unacceptable. Thus, either $547|N$ or $398581|N$.

Assume that $547|N$. Then 1093 is the special prime. One value of $F_r(547)$ must divide N. List values of $F_r(547)$ from Table A and for $r = 3, 5$.

A, $1093^*, 547 : F_3(547) = 3 \cdot 163 \cdot 613;\ F_5(547) = 431 \cdot 208097431$ is unacceptable. Thus, $613|N$.

One value of $F_r(613)$ must divide N. List values of $F_r(613)$ from Table A and for $r = 3, 5$.

A, $1093^*, 547, 613 : F_3(613) = 3 \cdot 7 \cdot 17923;\ F_5(613) = 131 \cdot 20161 \cdot 53551$. Thus, either $17923|N$ or $20161|N$.

Assume that $17923|N$. One value of $F_r(17923)$ must divide N. List values of $F_r(17923)$ from Table A and for $r = 3, 5$.

A, $1093^*, 547, 613, 17923$: $F_3(17923) = 3 \cdot 13 \cdot 31 \cdot 265717$; $F_5(17923) = 11 \cdot 27308381 \cdot 343540871$ is unacceptable. Thus, $265717|N$.

One value of $F_r(265717)$ must divide N. List values of $F_r(265717)$ from Table A and for $r = 3, 5$.

A, $1093^*, 547, 613, 17923, 265717$: $F_3(265717) = 3 \cdot 7 \cdot 3362180467$ is unacceptable; $F_5(265717) = 31 \cdot 41 \cdot 101 \cdot 38833995162778271$ is unacceptable, so 265717 is inadmissible. Thus, $17923 \not| N$, so $20161|N$.

One value of $F_r(20161)$ must divide N. List values of $F_r(20161)$ from Table A and for $r = 3, 5$.

A, $1093^*, 547, 613, 20161$: $F_3(20161) = 3 \cdot 135495361$ is unacceptable; $F_5(20161) = 5 \cdot 11 \cdot 1801 \cdot 1667989905611$ is unacceptable, so 20161 is inadmissible. Thus, $547 \not| N$, so $398581|N$.

One value of $F_r(398581)$ must divide N. List values of $F_r(398581)$ from Table A and for $r = 2, 3, 5$.

A, $1093, 398581$: $F_2(398581) = 2 \cdot 17 \cdot 19 \cdot 617$; $F_3(398581) = 3 \cdot 1621 \cdot 32668561$ is unacceptable; $F_5(398581) = 5 \cdot 1866871 \cdot 2703853428809791$ is unacceptable. Thus, $617|N$, and 398581 is the special prime.

One value of $F_r(617)$ must divide N. List values of $F_r(617)$ from Table A and for $r = 3, 5$.

A, $1093, 398581^*, 617$: $F_3(617) = 97 \cdot 3931$; $F_5(617) = 145159381141$ is unacceptable. Thus, $3931|N$.

One value of $F_r(3931)$ must divide N. List values of $F_r(3931)$ from Table A and for $r = 3, 5$.

A, $1093, 398581^*, 617, 3931$: $F_3(3931) = 3 \cdot 7 \cdot 31 \cdot 23743$; $F_5(3931) = 5 \cdot 11 \cdot 4342701505151$ is unacceptable. Thus, $23743|N$.

One value of $F_r(23743)$ must divide N. List values of $F_r(23743)$ from Table A and for $r = 3, 5$.

A, $1093, 398581^*, 617, 3931, 23743$: $F_3(23743) = 3 \cdot 37 \cdot 5078863$; $F_5(23743) = 11^2 \cdot 251 \cdot 10464092501131$ is unacceptable. Thus, $5078863|N$.

One value of $F_r(5078863)$ must divide N. List values of $F_r(5078863)$ from Table A and for $r = 3, 5$.

A, $1093, 398581^*, 617, 3931, 23743, 5078863$: $F_3(5078863) = 3 \cdot 19 \cdot 2341 \cdot 193311109$ is unacceptable. In addition, $F_5(5078863) = 11^2 \cdot 10831 \cdot 507705831495587075591$ is unacceptable, so 5078863 is inadmissible. These contradictions show that $1093 \nmid N$.

B. $151 \nmid N$.

B, 151 : $F_3(151) = 3 \cdot 7 \cdot 1093$, contradiction to A; $F_5(151) = 5 \cdot 104670301$ is unacceptable.

6. Restrictions on exponents in the prime power decomposition of N

Suppose that $p^a \| N$ and $r|(a+1)$ with $r > 5$. Then $F_r(p)$ appears in Table A, and $F_r(p)|N$, from Theorem 2.7. It follows from Table A and Lemma 5.1 that $r = 7$ and $p \in \{67, 173, 607, 619, 653, 1063, 1453, 2503, 4289, 5953, 9103, 9397, 10889, 12917, 19441, 63587, 109793, 113287, 191693, 6450307, 7144363\}$. From Table A, if $p = 67$ then $175897|N$. Since

$F_r(175897)$ is acceptable only for $r = 2$ and $r = 3$, with $F_2(175897) = 2 \cdot 37 \cdot 2377$ and $F_3(175897) = 3 \cdot 3121 \cdot 3304489$, and since $3 \nmid N$ and $37 \nmid N$ from Lemma 5.1, it follows that $p \neq 67$.

Similarly, if $p = 173$ then $3144079|N$; only $F_3(3144079)$ is acceptable, with $F_3(3144079) = 3 \cdot 13 \cdot 67 \cdot 13267 \cdot 285151$. Since $3 \nmid N$, $p \neq 173$. If $p = 607$ then $2210419|N$; 2210419 is inadmissible so $607 \nmid N$. If $p = 619$ then $4453751|N$; only $F_3(4453751) = 13 \cdot 522061 \cdot 2922721$ is acceptable and $13 \nmid N$, so $p \neq 619$. If $p = 653$ then $706763|N$; only $F_3(706763) = 31 \cdot 37 \cdot 631 \cdot 751 \cdot 919$ is acceptable and $31 \nmid N$, so $p \neq 653$. If $p = 1063$ then $1371511|N$; only $F_3(1371511) = 3 \cdot 1579 \cdot 1759 \cdot 225751$ is acceptable and $3 \nmid N$, so $p \neq 1063$. If $p = 1453$ then $2219491|N$; only $F_3(2219491) = 3 \cdot 421 \cdot 487 \cdot 8008933$ is acceptable and $3 \nmid N$, so $p \neq 1453$. If $p = 2503$ then $3591869|N$; 3591869 is inadmissible so $p \neq 2503$. If $p = 4289$ then $1538951|N$; 1538951 is inadmissible so $p \neq 4289$. If $p = 5953$ then $1591927|N$; 1591927 is inadmissible so $p \neq 4289$. If $p = 9103$ then $1338331|N$; 1338331 is inadmissible so $p \neq 9103$. If $p = 9397$ then $1551383|N$; 1551383 is inadmissible so $p \neq 9397$. If $p = 10889$ then $686057|N$; only $F_2(686057) = 2 \cdot 3 \cdot 114343$ is acceptable and $3 \nmid N$, so $p \neq 10889$. If $p = 12917$ then $5194337|N$; only $F_2(5194337) = 2 \cdot 3 \cdot 181 \cdot 4783$ is acceptable and $3 \nmid N$, so $p \neq 12917$. If $p = 19441$ then $2196979|N$; 2196979 is inadmissible so $p \neq 19441$. If $p = 63587$ then $6079823|N$; 6079823 is inadmissible so $p \neq 63587$. If $p = 109793$ then $7258639|N$; 7258639 is inadmissible so $p \neq 109793$. If $p = 113287$ then $5980619|N$; 5980619 is inadmissible so $p \neq 113287$. If

$p = 191693$ then $208279|N$; 208279 is inadmissible so $p \neq 191693$. If $p = 6450307$ then $901279|N$; 901279 is inadmissible so $p \neq 6450307$. If $p = 7144363$ then $171823|N$; 171823 is inadmissible so $p \neq 7144363$.

Thus, it is true that

Lemma 6.1. *If $p^a \| N$ and p is not the special prime p_0, then $a + 1 = 3^b \cdot 5^c$ where $(b + c) > 0$. If $p_0^{a_0} \| N$, then $a_0 + 1 = 2 \cdot 3^b \cdot 5^c$ where $(b + c) \geq 0$.*

7. Four sets

Let $S = \{47, 53, 59, \ldots\}$ be the set of all primes p such that $p \not\equiv 1 \pmod 3$, $p \not\equiv 1 \pmod 5$ and $37 < p < 10^7$.

If $p|N$ and $p|F_d(p_i)$ and $d \neq 2$, then, since $d|(a_i + 1)$, either $3|d$ or $5|d$ by Lemma 6.1. By Theorem 2.4, either $p \equiv 1 \pmod 3$ or $p \equiv 1 \pmod 5$, so $p \notin S$.

Suppose that $p_i \in S$ and $p_i^{a_i} \| N$ and $p_i | F_2(p_0)$. Then $p_i^{a_i} \| F_2(p_0)$ from the previous statement, and if two elements of S were divisors of $F_2(p_0)$, then $F_2(p_0) = p_0 + 1 \geq 2 \cdot 47^2 \cdot 53^2 = 12410162$. This is impossible since $p_0 < 10^7$. Thus, at most one element of S can divide $F_2(p_0)$. Note also that if $p_0 \in S$, then $p_0 \equiv 2 \pmod 3$ and $3|(p_0 + 1) = F_2(p_0)$, contradicting Lemma 5.1. Thus, $p_0 \notin S$.

We have proved

Proposition 7.1. *The number N is divisible by at most one element of S. If there is such an element s, then $s \neq p_0$ and $s \geq 47$.*

A computer search showed that S has 249278 elements, and that

$$S^* = \prod_{p \in S} \frac{p}{p-1} > 1.7331909144375899931. \tag{7.1}$$

Let $T = \{61, 151, 181, \ldots\}$ be the set of all primes p such that $p \equiv 1 \pmod{15}$ and $37 < p < 10^7$.

Suppose that $p_i \in T$ and $p_i \neq p_0$. If $p_i^{a_i} \| N$, then either $3|(a_i+1)$ or $5|(a_i+1)$ by Lemma 6.1. By Theorem 2.4, either $F_3(p_i)|N$, in which case $3|N$, or $F_5(p_i)|N$, in which case $5|N$. In either case Lemma 6.1 is contradicted, so $p_i \nmid N$.

Thus,

Proposition 7.2. *The number N is divisible by at most one element of T. If there is such an element it is p_0, and then $p_0 \geq 61$.*

A computer search showed that T has 83002 elements, and that

$$T^* = \prod_{p \in T} \frac{p}{p-1} > 1.1791835683407662159. \tag{7.2}$$

Let $U = \{73, 79, 103, \ldots\}$ be the set of all primes p such that $p \equiv 1 \pmod{3}, p \not\equiv 1 \pmod{5}$, $F_5(p)$ has a prime factor greater than 10^7, and $37 < p < 10^7$.

Suppose $p_i \in U$ and $p_i \neq p_0$. If $p_i^{a_i} \| N$, then by Lemma 6.1 either $3|(a_i+1)$ or $5|(a_i+1)$. If $3|(a_i+1)$, then $F_3(p_i)|N$ and $3|N$, contradicting Lemma 5.1. If $5|(a_i+1)$, then $F_5(p_i)|N$ and N has a factor greater than 10^7, a contradiction. Thus, $p_i \nmid N$.

It is, therefore, true that

Proposition 7.3. *The number N is divisible by at most one element of U. If there is such an element it is p_0, and then $p_0 \geq 73$.*

A computer search showed that U has 694 elements less than 20000, and that

$$U^* = \prod_{p \in U} \frac{p}{p-1} > \prod_{\substack{p \in U \\ p < 20000}} \frac{p}{p-1} > 1.239225225. \tag{7.3}$$

Let $V = \{3221, 3251, 3491, \ldots\}$ be the set of all primes p such that $p \equiv 1 \pmod 5, p \not\equiv 1 \pmod 3, F_3(p)$ has a prime factor greater than 10^7, and $37 < p < 10^7$.

Suppose $p_i \in V$. Since $p_i \not\equiv 1 \pmod 3$, it must be true that $p_i \equiv 2 \pmod 3$ and thus that $3|(p_i + 1) = F_2(p_i)$. But $F_2(p_0)|N$ and $3 \not| N$, so $p_i \neq p_0$. If $p_i^{a_i} \| N$, then by Lemma 6.1 either $3|(a_i + 1)$ or $5|(a_i + 1)$. If $5|(a_i+1)$, then $F_5(p_i)|N$ and $5|N$, contradicting Lemma 5.1. If $3|(a_i+1)$, then $F_3(p_i)|N$ and N has a factor greater than 10^7, a contradiction. Thus, $p_i \not| N$.

It is, therefore, true that

Proposition 7.4. *The number N is not divisible by any element of V.*

A computer search showed that V has 57 elements less than 20000, and that

$$V^* = \prod_{p \in V} \frac{p}{p-1} > \prod_{\substack{p \in V \\ p < 20000}} \frac{p}{p-1} > 1.006054597. \tag{7.4}$$

Note that S, T, U, and V are pairwise disjoint.

8. Proof of the theorem

There are 664567 primes p such that $37 < p < 10^7$, and

$$P^* = \prod_{41 \le p < 10^7} \frac{p}{p-1} < 4.269448664996309337. \tag{8.1}$$

If $p^a \| N$, then

$$1 < \sigma(p^a)/p^a = (p^{a+1}-1)/(p^a(p-1)) < p/(p-1).$$

Since σ is a multiplicative function,

$$\frac{\sigma(N)}{N} = \frac{\sigma(p_0)\sigma(p_1)\cdots}{p_0 p_1 \cdots} < \prod_{i=0}^{u} \frac{p_i}{p_i - 1}.$$

From Lemma 5.1, $p_i > 37$. Since $x/(x-1)$ is monotonic decreasing for $x > 1$, then if $p_i \in S$ then $p_i/(p_i-1) < 47/46$ and if $p_i \in T$ or U, then $p_i/(p_i-1) < 61/60$. Thus, it follows from Propositions 7.1, 7.2, 7.3, and 7.4, Theorem 2.7, and equations 7.1, 7.2, 7.3, 7.4, and 8.1 that

$$2 = \frac{\sigma(N)}{N} < \prod_{i=0}^{u} \frac{p_i}{p_i - 1} \le \frac{47}{46}\frac{61}{60}\frac{P^*}{S^* T^* U^* V^*} < 1.740567 \tag{8.2}$$

This contradiction proves the theorem.

9. Details on the search for acceptable values of $F_r(p)$

Let $Q(r)$ be the product of all primes less than 10^7 and congruent to 1 (mod r).

Suppose that $2142 < r < 5000000$. A computer search showed that if $10^2 < p < 10^7$, then $Q(r)^2 < 10^{2(r-1)}$. Searches showed that if $q < 10^7$, then $q^3 \nmid F_r(p)$ except that $60647^3 \| F_{30323}(6392117)$ and $10709^3 \| F_{2677}(6619441)$.

With these limits on p, $F_r(p) > p^{r-1} > (10^2)^{r-1}$.

But if $r \notin \{30323, 2677\}$, then from Theorem 2.2, $F_r(p) < (Q(r))^2 < 10^{2(r-1)}$. There are 22 primes less than 10^7 congruent to $1 \pmod{30323}$ and 247 primes less than 10^7 congruent to $1 \pmod{2677}$. If $r = 30323$ then $F_r(p) > p^{r-1} > (10^2)^{30322} = 10^{60644}$, but $F_r(p) < 60647^3((10^7)^2)^{21} < (10^5)^3 10^{294} = 10^{309}$. If $r = 2677$, then $F_r(p) > (10^2)^{2676} = 10^{5352}$, but

$$F_r(p) < 10709^3((10^7)^2)^{246} < 10^{15}10^{3444} = 10^{3459}.$$

These contradictions prove

Proposition 9.1. *If $r > 2142$ and $10^2 < p < 10^7$, then $F_r(p)$ has a prime factor greater than 10^7.*

Suppose that $1472 < r < 2142$ and $10^2 < p < 10^7$. A computer search showed that if $q < 10^7$, then $q^3 \nmid F_r(p)$ except that $3119^3 \| F_{1559}(146917)$ and $2999^3 \| F_{1499}(8474027)$, and $q^2 \| F_r(p)$ for at most one q for each $F_r(p)$. Searches also showed that $10^7 \cdot Q(r) < 10^{2(r-1)}$ for all r in this range.

Again, it is true that $F_r(p) > p^{r-1} > (10^2)^{r-1}$.

If $r \notin \{1559, 1499\}$, then from Theorem 2.2, $F_r(p) < 10^7 \cdot Q(r) < 10^{2(r-1)}$. There are 433 primes less than 10^7 congruent to $1 \pmod{1559}$ and 428 primes less than 10^7 congruent to 1 (mod 1499), of which 7 are less than 10^5. If $r = 1559$ then $F_r(p) > p^{r-1} > (10^2)^1 558 = 10^{3116}$, but $F_r(p) < 3119^3(10^7)^{432} < 10^{12}10^{3024} = 10^{3036}$. If $r = 1499$, then $F_r(p) > (10^2)^{1498} = 10^{2996}$, but

$$F_r(p) < 2999^3(10^7)^{421}(10^5)^6 < 10^{12}10^{2947}10^{30} = 10^{2989}.$$

These contradictions prove

Proposition 9.2. *If* $1472 < r < 2142$ *and* $10^2 < p < 10^7$*, then* $F_r(p)$ *has a prime factor greater than* 10^7.

For $7 \le r < 1472$, more computation was necessary. For each $F_r(p)$, the primes $q < 10^7$ that divide $F_r(p)$ were determined. Since $F_r(p)$ must factor into primes smaller than 10^7 to be acceptable, it follows that if

$$\prod_{\substack{q^b \| F_r(p) \\ q<10^7}} q^b < p^{r-1},$$

then $F_r(p)$ has a factor greater than 10^7, since $p^{r-1} < F_r(p)$.

These computations yielded

Proposition 9.3. *If* $7 \le r < 1472$ *and* $10^2 < p < 10^7$*, then* $F_r(p)$ *has a prime factor greater than* 10^7*, except for the values of* r *and the values of* p *exceeding* 10^2 *listed in Table A.*

During the computations for Proposition 9.3, the following fourth and fifth powers dividing values of $F_r(p)$ were found: $29^4 \| F_7(1006441)$, $29^4 \| F_7(1202363)$, $29^4 \| F_7(4321703)$, $29^4 \| F_7(5147951)$, $23^4 \| F_{11}(859891)$, $23^4 \| F_{11}(972557)$, $23^4 \| F_{11}(1739839)$, $23^4 \| F_{11}(2651603)$, $23^4 \| F_{11}(3125833)$, $67^4 \| F_{11}(3246107)$, $23^4 \| F_{11}(4328677)$, $23^5 \| F_{11}(4330649)$, $23^4 \| F_{11}(4890331)$, $23^4 \| F_{11}(5097931)$, $23^4 \| F_{11}(6007723)$, $23^4 \| F_{11}(6455557)$, $23^4 \| F_{11}(6569377)$, $23^4 \| F_{11}(6651583)$, $23^4 \| F_{11}(6783701)$, $23^4 \| F_{11}(6924787)$, $23^4 \| F_{11}(7343383)$, $23^4 \| F_{11}(7574921)$, $23^4 \| F_{11}(7603289)$, $23^4 \| F_{11}(7896341)$, $23^4 \| F_{11}(8162971)$, $23^4 \| F_{11}(8248423)$, $23^4 \| F_{11}(8462747)$, $23^4 \| F_{11}(9842017)$, $53^4 \| F_{13}(4388179)$, $53^4 \| F_{13}(4633627)$, $53^4 \| F_{13}(6760483)$, $47^4 \| F_{23}(1513367)$, $47^4 \| F_{23}(2279741)$, $59^4 \| F_{29}(9140297)$, $83^4 \| F_{41}(2864371)$, $83^4 \| F_{41}(4939939)$.

p	$R(p)$	p	$R(p)$	p	$R(p)$
3	3122	29	1748	61	1560
5	2478	31	1748	67	1532
7	2298	37	1700	71	1532
11	2054	41	1700	73	1532
13	2004	43	1638	79	1494
17	1949	47	1622	83	1494
19	1934	53	1584	89	1490
23	1788	59	1572	97	1472

TABLE 1. Values of $R(p)$ for $3 \leq p < 100$

Assume now that $p < 10^2$ and $r \geq 7$. The table in McDaniel [13] gives $48947^2 \| F_{24473}(17)$, $47^2 \| F_{23}(53)$, $59^2 \| F_{29}(53)$, $47^2 \| F_{23}(71)$, and $4871^2 \| F_{487}(83)$. However, the range is not broad enough for $p = 31, 37, 41, 43, 53, 59, 67, 71, 79, 83$. The table in Montgomery [14] gives $p = 31, r = 2806861$ as the only possible new solution to $q^2 | F_r(p)$, and direct computation shows that $F_{2806861}(31)$ has a prime divisor exceeding 10^7. Direct computation also shows that $F_{24473}(17)$, $F_{23}(53)$, $F_{29}(53)$, $F_{23}(71)$, and $F_{487}(83)$ have prime divisors greater than 10^7. These cases may therefore be disregarded, and it may be assumed that if every prime divisor of $F_r(p)$ is less than 10^7, then $F_r(p)$ is squarefree.

Again, let $Q(r)$ be the product of all primes less than 10^7 and congruent to $1 \pmod r$. In Table 9, for each prime p, $3 \leq p < 100$, $R(p)$ is an integer such that if $r > R(p)$, then $Q(r) < p^{r-1}$. Thus, if $r > R(p)$, $F_r(p) > p^{r-1}$, but since $F_r(p)$ is squarefree, $F_r(p) < Q(r) < p^{r-1}$. This contradiction proves

Proposition 9.4. *If $p < 10^2$ and $r > R(p)$, then $F_r(p)$ has a prime factor greater than 10^7.*

If $p < 10^2$ and $7 \leq r < R(p)$, direct computations similar to those performed in proving Proposition 9.3 give

Proposition 9.5. *If $p < 10^2$ and $7 \leq r < R(p)$, then $F_r(p)$ has a prime factor greater than 10^7, except for the values of $F_r(p)$ with $p < 10^2$ listed in Table A.*

Propositions 9.1 through 9.5 show the validity of Table A.

10. Concluding remarks

Let R be the largest prime factor of the odd perfect number N. It has been shown here that $R > 10^7$. It seems probable that this proof could be extended to raise the lower bound for R, using the same methods, since the inequality proving the theorem is much stronger than is necessary. Unfortunately, the time that would be required to generate Table A for a larger lower bound seems to be great enough to make this computation impractical. If $\pi(x)$ is the number of primes not exceeding x, then to generate Table A for a lower bound of R for the largest prime divisor of N, $\pi(R) \cdot \pi(R/2)$ values of $F_r(p)$ must be examined for acceptability.

Hagis and Cohen[8] spent approximately 700 hours of computing time proving that $R \geq 10^6$, using a CYBER 860 and a 486 PC. The computations in this paper were performed on a dual-processor 866 MHz computer and twenty-two 300 MHz Pentium II's, and required approximately 25800 hours of processor time. The bound was increased only by a factor of 10, but the time required, even with advances in computer technology, increased by a factor of 36.

The method used to prove Lemma 5.1 could theoretically be used for the entire proof of the theorem, provided one has sufficient patience to eliminate every prime less than R one by one. This suggests that if it can be proved that every prime can be eliminated eventually for any value of R, then no odd perfect number exists. A possible proof would assume the existence of a set X of primes that could not be so eliminated and derive a contradiction. An element p_i of X would have the property that for some r_i, $F_{r_i}(p_i) = p_1 p_2 \cdots p_k$, where $p_1, p_2, \ldots$ are elements of X as well.

Appendix A. Table A: Acceptable values of $F_r(p)$ for $3 \le p < 10^7$ and $p \ge 7$

p	r	$F_r(p)$
3	7	1093
3	11	$23 \cdot 3851$
3	13	797161
3	17	$1871 \cdot 34511$
3	19	$1597 \cdot 363889$
3	31	$683 \cdot 102673 \cdot 4404047$
5	7	19531
5	19	$191 \cdot 6271 \cdot 3981071$
7	7	$29 \cdot 4733$
7	11	$1123 \cdot 293459$
11	7	$43 \cdot 45319$
11	11	$15797 \cdot 1806113$
13	7	5229043
13	11	$23 \cdot 419 \cdot 859 \cdot 18041$
13	13	$53 \cdot 264031 \cdot 1803647$
19	7	$701 \cdot 70841$
23	7	$29 \cdot 5336717$
31	13	$42407 \cdot 2426789 \cdot 7908811$
43	7	$7 \cdot 5839 \cdot 158341$
59	7	$43 \cdot 281 \cdot 757 \cdot 4691$
67	7	$175897 \cdot 522061$
79	7	$281 \cdot 337 \cdot 1289 \cdot 2017$
113	7	$7 \cdot 44983 \cdot 6670903$
127	7	$7 \cdot 43 \cdot 86353 \cdot 162709$
131	7	$127 \cdot 189967 \cdot 211093$
167	11	$23 \cdot 89 \cdot 331 \cdot 397 \cdot 1013 \cdot 32099 \cdot 1940599$
173	7	$3144079 \cdot 8576317$
191	7	$127 \cdot 197 \cdot 10627 \cdot 183569$
197	7	$7 \cdot 29 \cdot 97847 \cdot 2957767$
223	7	$29 \cdot 491 \cdot 1709 \cdot 5076443$
239	7	$7 \cdot 29 \cdot 245561 \cdot 3754507$
269	7	$43 \cdot 211 \cdot 631 \cdot 2633 \cdot 25229$
293	7	$43 \cdot 2197609 \cdot 6718489$
359	7	$211 \cdot 449 \cdot 1303 \cdot 4019 \cdot 4327$
389	7	$127 \cdot 337 \cdot 659 \cdot 827 \cdot 148933$
397	7	$29 \cdot 127 \cdot 927137 \cdot 1149457$
401	7	$29 \cdot 337 \cdot 263047 \cdot 1621397$
431	7	$29 \cdot 953 \cdot 967 \cdot 1009 \cdot 238267$

p	r	$F_r(p)$
607	7	$54517 \cdot 415759 \cdot 2210419$
619	7	$3389 \cdot 3732919 \cdot 4453751$
653	7	$21757 \cdot 706763 \cdot 5049773$
919	7	$29 \cdot 43 \cdot 3851 \cdot 21407 \cdot 5866379$
953	7	$7 \cdot 29 \cdot 71 \cdot 113 \cdot 127 \cdot 379 \cdot 9566159$
977	7	$29 \cdot 5573 \cdot 914047 \cdot 5893273$
1063	7	$337 \cdot 2423 \cdot 1289513 \cdot 1371511$
1181	7	$71 \cdot 27791 \cdot 202021 \cdot 6812527$
1283	7	$29 \cdot 631 \cdot 3739 \cdot 24781 \cdot 2632673$
1453	7	$1051 \cdot 11117 \cdot 363119 \cdot 2219491$
1523	7	$71 \cdot 337 \cdot 449 \cdot 235537 \cdot 4935043$
1693	7	$43 \cdot 337 \cdot 7673 \cdot 37171 \cdot 5700731$
1823	7	$29 \cdot 71 \cdot 547 \cdot 5709019 \cdot 5711539$
2053	7	$29 \cdot 161869 \cdot 3482179 \cdot 4582817$
2141	7	$29 \cdot 1163 \cdot 11719 \cdot 112967 \cdot 2158157$
2381	7	$7 \cdot 43 \cdot 2689 \cdot 3613 \cdot 72997 \cdot 853903$
2473	11	$23 \cdot 463 \cdot 2927 \cdot 8647 \cdot 81071 \cdot 451793 \cdot 640531 \cdot 1353551$
2503	7	$757 \cdot 19013 \cdot 3591869 \cdot 4758517$
2633	7	$7 \cdot 29 \cdot 232919 \cdot 2103613 \cdot 3351223$
2657	7	$71 \cdot 631 \cdot 1289 \cdot 1991389 \cdot 3060667$
2713	7	$29^2 \cdot 43 \cdot 73361 \cdot 258469 \cdot 581729$
3301	7	$29^2 \cdot 911 \cdot 38669 \cdot 186733 \cdot 233941$
3313	7	$29 \cdot 211 \cdot 216259 \cdot 884857 \cdot 1129619$
3623	7	$43^3 \cdot 35911 \cdot 353263 \cdot 2242843$
3779	7	$197 \cdot 2311 \cdot 23773 \cdot 455407 \cdot 591053$
4283	7	$29 \cdot 29611 \cdot 41539 \cdot 100003 \cdot 1730891$
4289	7	$1471 \cdot 8807 \cdot 9619 \cdot 32467 \cdot 1538951$
4327	7	$7 \cdot 3221 \cdot 5503 \cdot 5657 \cdot 92401 \cdot 101221$
5953	7	$4663 \cdot 1352107 \cdot 1591927 \cdot 4434949$
7759	7	$29^2 \cdot 71 \cdot 337 \cdot 1051 \cdot 2213 \cdot 2689 \cdot 1733929$
8009	7	$7 \cdot 43 \cdot 127 \cdot 491 \cdot 127247 \cdot 305873 \cdot 361313$
8681	7	$7 \cdot 62903 \cdot 285979 \cdot 1563101 \cdot 2174593$
9103	7	$631 \cdot 911 \cdot 514417 \cdot 1338331 \cdot 1437899$
9397	7	$617 \cdot 8779 \cdot 11579 \cdot 1551383 \cdot 7077197$
9403	7	$29 \cdot 2311 \cdot 2633 \cdot 9521 \cdot 54629 \cdot 7531651$
9719	7	$281 \cdot 3067 \cdot 8219 \cdot 19937 \cdot 30773 \cdot 193957$
10889	7	$2003 \cdot 22093 \cdot 116341 \cdot 471997 \cdot 686057$
10949	7	$7 \cdot 29 \cdot 197 \cdot 547 \cdot 1009 \cdot 6917 \cdot 25523 \cdot 442177$

p	r	$F_r(p)$
12553	7	$29 \cdot 701 \cdot 1471 \cdot 49463 \cdot 594119 \cdot 4452841$
12917	7	$953 \cdot 6007 \cdot 352661 \cdot 442961 \cdot 5194337$
15307	7	$29 \cdot 883 \cdot 2381 \cdot 51437 \cdot 1001953 \cdot 4093811$
15313	7	$71 \cdot 463 \cdot 547 \cdot 44633 \cdot 3918587 \cdot 4099943$
15601	7	$43 \cdot 2423 \cdot 3335263 \cdot 5823749 \cdot 7125049$
17431	7	$7 \cdot 43 \cdot 127 \cdot 3347 \cdot 50527 \cdot 558881 \cdot 7764079$
17491	7	$239 \cdot 827 \cdot 1093 \cdot 12097 \cdot 2287951 \cdot 4789219$
19441	7	$421 \cdot 2143 \cdot 4663 \cdot 15233 \cdot 383489 \cdot 2196979$
20441	7	$7 \cdot 29 \cdot 43 \cdot 659 \cdot 2213 \cdot 27763 \cdot 68279 \cdot 3023077$
23339	7	$7 \cdot 29 \cdot 1051 \cdot 21841 \cdot 92779 \cdot 353137 \cdot 1058639$
23671	7	$29 \cdot 43 \cdot 239 \cdot 9829 \cdot 90203 \cdot 237301 \cdot 2805587$
27457	7	$29 \cdot 42463 \cdot 65171 \cdot 71261 \cdot 91813 \cdot 816047$
30557	7	$29 \cdot 449 \cdot 631 \cdot 10067 \cdot 90679 \cdot 99667 \cdot 1089047$
35983	7	$29 \cdot 43 \cdot 113 \cdot 21001 \cdot 30983 \cdot 2938307 \cdot 8057323$
41203	7	$7 \cdot 29 \cdot 211 \cdot 1163 \cdot 6329 \cdot 16339 \cdot 340397 \cdot 2790481$
45341	7	$113 \cdot 127 \cdot 13259 \cdot 2515003 \cdot 2804117 \cdot 6474833$
48473	7	$113 \cdot 30829 \cdot 42743 \cdot 241249 \cdot 352073 \cdot 1025669$
53831	7	$7 \cdot 29 \cdot 39341 \cdot 104651 \cdot 257489 \cdot 269221 \cdot 420001$
56053	7	$29 \cdot 743 \cdot 104707 \cdot 1805833 \cdot 2248681 \cdot 3385579$
59467	7	$43 \cdot 1373 \cdot 140449 \cdot 469631 \cdot 2847139 \cdot 3988783$
63587	7	$13063 \cdot 380059 \cdot 1341257 \cdot 1632751 \cdot 6079823$
93251	7	$29 \cdot 2129 \cdot 2437 \cdot 25439 \cdot 148471 \cdot 385939 \cdot 2998031$
109793	7	$2371 \cdot 6287 \cdot 46831 \cdot 314581 \cdot 950111 \cdot 7288639$
113287	7	$2339 \cdot 3319 \cdot 5419 \cdot 2085931 \cdot 4027927 \cdot 5980619$
118189	7	$7 \cdot 71 \cdot 57373 \cdot 60397 \cdot 230567 \cdot 1121051 \cdot 6122999$
123493	7	$211 \cdot 449 \cdot 827 \cdot 57667 \cdot 143263 \cdot 880909 \cdot 6220579$
128833	7	$211 \cdot 701 \cdot 1723 \cdot 160441 \cdot 161267 \cdot 205423 \cdot 3375751$
146801	7	$197 \cdot 911 \cdot 2801 \cdot 49547 \cdot 343897 \cdot 403957 \cdot 2892667$
156703	7	$7 \cdot 29 \cdot 43 \cdot 827 \cdot 5209 \cdot 4776647 \cdot 8992691 \cdot 9167047$
172489	7	$43 \cdot 71 \cdot 197 \cdot 1723 \cdot 8233 \cdot 231547 \cdot 2129107 \cdot 6261781$
174637	7	$7 \cdot 71 \cdot 491 \cdot 3571 \cdot 33587 \cdot 315281 \cdot 832987 \cdot 3690499$
181739	7	$29 \cdot 43^2 \cdot 127 \cdot 757 \cdot 4691 \cdot 44507 \cdot 3790739 \cdot 8831593$
191693	7	$7561 \cdot 11887 \cdot 14869 \cdot 16759 \cdot 89839 \cdot 118399 \cdot 208279$
218971	7	$197 \cdot 1471 \cdot 2647 \cdot 9871 \cdot 1557613 \cdot 2178877 \cdot 4289783$
219031	7	$7 \cdot 29 \cdot 197 \cdot 22541 \cdot 174721 \cdot 183317 \cdot 771653 \cdot 4955987$
244457	7	$29 \cdot 883 \cdot 1583 \cdot 3361 \cdot 6203 \cdot 32341 \cdot 2406307 \cdot 3244907$
246319	7	$71 \cdot 1303 \cdot 1709 \cdot 6217 \cdot 6553 \cdot 87683 \cdot 141121 \cdot 2802311$
285559	7	$7 \cdot 29 \cdot 449 \cdot 1667 \cdot 3389 \cdot 14771 \cdot 210071 \cdot 224267 \cdot 1513163$

p	r	$F_r(p)$
304501	7	$7 \cdot 883 \cdot 69427 \cdot 430739 \cdot 532099 \cdot 882967 \cdot 9179003$
441841	7	$7 \cdot 43 \cdot 127 \cdot 281 \cdot 449 \cdot 9059 \cdot 2587943 \cdot 7274807 \cdot 9045191$
472457	7	$43 \cdot 1460957 \cdot 2076803 \cdot 2178653 \cdot 4680733 \cdot 8359331$
493397	7	$29^2 \cdot 127 \cdot 1163 \cdot 2129 \cdot 4229 \cdot 26041 \cdot 50177 \cdot 71359 \cdot 138349$
592919	7	$43 \cdot 6833 \cdot 60607 \cdot 237707 \cdot 704243 \cdot 2362571 \cdot 6169087$
692353	7	$29 \cdot 71 \cdot 449 \cdot 911 \cdot 1499 \cdot 128969 \cdot 204331 \cdot 1443989 \cdot 2292767$
704731	7	$43 \cdot 127 \cdot 337 \cdot 150011 \cdot 304739 \cdot 323009 \cdot 626011 \cdot 7200971$
737129	7	$7 \cdot 29 \cdot 113 \cdot 491 \cdot 883 \cdot 911 \cdot 16073 \cdot 109201 \cdot 2807239 \cdot 3593549$
844957	7	$7 \cdot 37493 \cdot 150431 \cdot 662551 \cdot 889687 \cdot 1603267 \cdot 9753493$
851297	7	$127 \cdot 211 \cdot 4271 \cdot 10151 \cdot 52963 \cdot 804329 \cdot 1531181 \cdot 5022613$
923599	7	$29 \cdot 757 \cdot 3823 \cdot 16493 \cdot 223217 \cdot 422689 \cdot 1165711 \cdot 4077221$
971039	7	$113 \cdot 491 \cdot 71527 \cdot 2404823 \cdot 2447551 \cdot 4459687 \cdot 8047691$
1135451	7	$239 \cdot 827 \cdot 4663 \cdot 42967 \cdot 89041 \cdot 223007 \cdot 309583 \cdot 8802809$
1242781	7	$7 \cdot 547 \cdot 13903 \cdot 48413 \cdot 265511 \cdot 1095403 \cdot 2171261 \cdot 2263829$
1310251	7	$43 \cdot 113 \cdot 3347 \cdot 25733 \cdot 855191 \cdot 1249669 \cdot 1599809 \cdot 7071443$
1630021	7	$7 \cdot 113 \cdot 13679 \cdot 477863 \cdot 513899 \cdot 984859 \cdot 1458619 \cdot 4913959$
1709483	7	$29^2 \cdot 197 \cdot 953 \cdot 24809 \cdot 305621 \cdot 358667 \cdot 6433841 \cdot 9033991$
1740199	7	$29 \cdot 449 \cdot 6917 \cdot 7001 \cdot 11383 \cdot 39901 \cdot 61223 \cdot 337751 \cdot 4689427$
1899047	7	$127 \cdot 197 \cdot 337 \cdot 164837 \cdot 908573 \cdot 1938161 \cdot 2651111 \cdot 7229069$
2090681	7	$43 \cdot 1583 \cdot 2521 \cdot 12391 \cdot 25229 \cdot 33349 \cdot 97259 \cdot 122921 \cdot 3904447$
2522543	7	$29 \cdot 4957 \cdot 7351 \cdot 168491 \cdot 168869 \cdot 178613 \cdot 6155129 \cdot 7794557$
4024961	7	$239 \cdot 10949 \cdot 25229 \cdot 160343 \cdot 200467 \cdot 462239 \cdot 637729 \cdot 6796763$
4538603	7	$29 \cdot 127 \cdot 2129 \cdot 9199 \cdot 220529 \cdot 336211 \cdot 700127 \cdot 775349 \cdot 3010673$
6450307	7	$7253 \cdot 32789 \cdot 33629 \cdot 46327 \cdot 251063 \cdot 331339 \cdot 901279 \cdot 2592829$
6602399	7	$239 \cdot 1303 \cdot 2423 \cdot 33013 \cdot 53117 \cdot 86171 \cdot 312509 \cdot 472249 \cdot 4922681$
7144363	7	$449 \cdot 659 \cdot 1429 \cdot 4943 \cdot 6917 \cdot 10739 \cdot 146819 \cdot 169667 \cdot 171823 \cdot 200117$
8233787	7	$29 \cdot 449 \cdot 1163 \cdot 6091 \cdot 6581 \cdot 1598633 \cdot 6115061 \cdot 7036499 \cdot 7462547$
8693947	7	$43 \cdot 2591 \cdot 18257 \cdot 20903 \cdot 25747 \cdot 159167 \cdot 355517 \cdot 1188601 \cdot 5864783$
8837557	7	$7 \cdot 71 \cdot 127 \cdot 43093 \cdot 785303 \cdot 857669 \cdot 5669441 \cdot 6260297 \cdot 7327139$

Appendix B. Proof of Lemma 5.1

Here, p^* means that p is the special prime. Two different primes cannot both be special at the same time. Each line begins by listing the numbers assumed to divide N.

A. $1093 \nmid N$.

A, 1093 : $F_2(1093) = 2 \cdot 547$; $F_3(1093) = 3 \cdot 398581$; $F_5(1093) = 11 \cdot 31 \cdot 4189129561$ is unacceptable.

A, $1093^*, 547$: $F_3(547) = 3 \cdot 163 \cdot 613$; $F_5(547) = 431 \cdot 208097431$ is unacceptable.

A, $1093^*, 547, 613$: $F_3(613) = 3 \cdot 7 \cdot 17923$; $F_5(613) = 131 \cdot 20161 \cdot 53551$.

A, $1093^*, 547, 613, 17923$: $F_3(17923) = 3 \cdot 13 \cdot 31 \cdot 265717$; $F_5(17923) = 11 \cdot 27308381 \cdot 343540871$ is unacceptable.

A, $1093^*, 547, 613, 17923, 265717$: $F_3(265717) = 3 \cdot 7 \cdot 3362180467$ is unacceptable; $F_5(265717) = 31 \cdot 41 \cdot 101 \cdot 38833995162778271$ is unacceptable, so 265717 is inadmissible.

A, $1093^*, 547, 613, 20161$: $F_3(20161) = 3 \cdot 135495361$ is unacceptable; $F_5(20161) = 5 \cdot 11 \cdot 1801 \cdot 1667989905611$ is unacceptable, so 20161 is inadmissible.

A, $1093, 398581$: $F_2(398581) = 2 \cdot 17 \cdot 19 \cdot 617$; $F_3(398581) = 3 \cdot 1621 \cdot 32668561$ is unacceptable; $F_5(398581) = 5 \cdot 1866871 \cdot 2703853428809791$ is unacceptable.

A, $1093, 398581^*, 617$: $F_3(617) = 97 \cdot 3931$; $F_5(617) = 145159381141$ is unacceptable.

A, $1093, 398581^*, 617, 3931$: $F_3(3931) = 3 \cdot 7 \cdot 31 \cdot 23743$; $F_5(3931) = 5 \cdot 11 \cdot 4342701505151$ is unacceptable.

A, $1093, 398581^*, 617, 3931, 23743$: $F_3(23743) = 3 \cdot 37 \cdot 5078863$; $F_5(23743) = 11^2 \cdot 251 \cdot 10464092501131$ is unacceptable.

A, $1093, 398581^*, 617, 3931, 23743, 5078863$: $F_3(5078863) = 3 \cdot 19 \cdot 2341 \cdot 193311109$ is unacceptable; $F_5(5078863) = 11^2 \cdot 10831 \cdot 50770583149558707 5591$ is unacceptable, so 5078863 is inadmissible.

B. $151 \nmid N$.

B, 151 : $F_3(151) = 3 \cdot 7 \cdot 1093$, contradiction to A; $F_5(151) = 5 \cdot 104670301$ is unacceptable.

C. $31 \nmid N$.

C, 31 : $F_3(31) = 3 \cdot 331$; $F_5(31) = 5 \cdot 11 \cdot 17351$; $F_{13}(31) = 42407 \cdot 2426789 \cdot 7908811$.

C, $31, 331$: $F_3(331) = 3 \cdot 7 \cdot 5233$; $F_5(331) = 5 \cdot 37861 \cdot 63601$.

C, $31, 331, 5233$: $F_2(5233) = 2 \cdot 2617$; $F_3(5233) = 3 \cdot 7 \cdot 31 \cdot 42073$; $F_5(5233) = 2351 \cdot 7741 \cdot 41213191$ is unacceptable.

C, $31, 331, 5233^*, 2617$: $F_3(2617) = 3 \cdot 193 \cdot 11833$; $F_5(2617) = 46922470889141$ is unacceptable.

C, $31, 331, 5233^*, 2617, 11833$: $F_3(11833) = 3 \cdot 13 \cdot 199 \cdot 18043$; $F_5(11833) = 22651 \cdot 865622988431$ is unacceptable.

C, $31, 331, 5233^*, 2617, 11833, 18043$: $F_3(18043) = 3 \cdot 7 \cdot 15503233$ is unacceptable; $F_5(18043) = 11 \cdot 1155061 \cdot 8341832431$ is unacceptable, so 18043 is inadmissible.

C, $31, 331, 5233, 42073$: $F_2(42073) = 2 \cdot 109 \cdot 193$; $F_3(42073) = 3 \cdot 19 \cdot 409 \cdot 75931$; $F_5(42073) = 11 \cdot 284860058207025151$ is unacceptable.

C, $31, 331, 5233, 42073^*, 193$: $F_3(193) = 3 \cdot 7 \cdot 1783$; $F_5(193) = 1394714501$ is unacceptable.

C, $31, 331, 5233, 42073^*, 193, 1783$: $F_3(1783) = 3{\cdot}829{\cdot}1279$; $F_5(1783) = 31 \cdot 67271 \cdot 4849081$.

C, $31, 331, 5233, 42073^*, 193, 1783, 1279$: $F_3(1279) = 3 \cdot 229 \cdot 2383$; $F_5(1279) = 11 \cdot 31 \cdot 7853576701$ is unacceptable.

C, $31, 331, 5233, 42073^*, 193, 1783, 1279, 2383$: $F_3(2383) = 3 \cdot 151 \cdot 12541$, contradiction to B; $F_5(2383) = 32261046755681$ is unacceptable.

C, $31, 331, 5233, 42073^*, 193, 1783, 4849081$: $F_3(4849081) = 3 \cdot 13 \cdot 602912599837$ is unacceptable; $F_5(4849081) = 5{\cdot}2327441802281{\cdot}47510435334281$ is unacceptable, so 4849081 is inadmissible.

C, $31, 331, 5233, 42073, 75931$: $F_3(75931) = 3 \cdot 7^2 \cdot 19 \cdot 43 \cdot 61 \cdot 787$; $F_5(75931) = 5 \cdot 11 \cdot 751 \cdot 1171 \cdot 687262545131$ is unacceptable.

C, $31, 331, 5233, 42073, 75931, 787$: $F_3(787) = 3 \cdot 37^2 \cdot 151$, contradiction to B; $F_5(787) = 570821 \cdot 672901$.

C, $31, 331, 5233, 42073, 75931, 787, 570821$: $F_2(570821) = 2 \cdot 3 \cdot 7 \cdot 13591$; $F_3(570821) = 325837184863$ is unacceptable; $F_5(570821) = 5 \cdot 11 \cdot 3729471281 \cdot 517595595851$ is unacceptable.

C, $31, 331, 5233, 42073, 75931, 787, 570821^*, 13591$: $F_3(13591) = 3 \cdot 7^2 \cdot 1256659$; $F_5(13591) = 5 \cdot 6824449137004381$ is unacceptable.

C, $31, 331, 5233, 42073, 75931, 787, 570821^*, 13591, 1256659$: $F_3(1256659) = 3{\cdot}8599{\cdot}61216153$ is unacceptable; $F_5(1256659) = 391331{\cdot}6372735252852647251$ is unacceptable, so 13591 is inadmissible.

C, $31, 331, 63601$: $F_2(63601) = 2 \cdot 7^2 \cdot 11 \cdot 59$; $F_3(63601) = 3 \cdot 2203 \cdot 612067$; $F_5(63601) = 5 \cdot 41 \cdot 271 \cdot 1381 \cdot 4231 \cdot 50408381$ is unacceptable.

C, $31, 331, 63601^*, 37861$: $F_3(37861) = 3{\cdot}37{\cdot}1201{\cdot}10753$; $F_5(37861) = 5{\cdot}41{\cdot}101{\cdot}99244068137581$ is unacceptable. (Note: 37861 appears along with 63601 as a factor of $F_5(331)$ in the third line assuming C. This variation from the normal procedure is clearly valid, and will be used occasionally below.)

C, $31, 331, 63601^*, 37861, 10753 : F_3(10753) = 3 \cdot 151 \cdot 397 \cdot 643$, contradiction to B; $F_5(10753) = 13370848663151621$ is unacceptable.

C, $31, 331, 63601, 612067 : F_3(612067) = 3 \cdot 4801 \cdot 26010319$ is unacceptable; $F_5(612067) = 11 \cdot 151 \cdot 421 \cdot 4861 \cdot 6701 \cdot 6161402801$ is unacceptable, so 612067 is inadmissible.

C, $31, 17351 : F_3(17351) = 13 \cdot 1063 \cdot 21787$; $F_5(17351) = 5 \cdot 11 \cdot 1648012040336791$ is unacceptable.

C, $31, 17351, 21787 : F_3(21787) = 3 \cdot 31 \cdot 5104249$; $F_5(21787) = 41 \cdot 491 \cdot 67231 \cdot 166484861$ is unacceptable.

C, $31, 17351, 21787, 5104249 : F_2(5104249) = 2 \cdot 5^3 \cdot 17 \cdot 1201$; $F_3(5104249) = 3 \cdot 61 \cdot 216781 \cdot 656737$; $F_5(5104249) = 9654791 \cdot 307208711 \cdot 228850092101$ is unacceptable.

C, $31, 17351, 21787, 5104249^*, 1201 : F_3(1201) = 3 \cdot 7 \cdot 68743$; $F_5(1201) = 5 \cdot 416450882401$ is unacceptable.

C, $31, 17351, 21787, 5104249^*, 1201, 68743 : F_3(68743) = 3 \cdot 1575222931$ is unacceptable; $F_5(68743) = 11 \cdot 31 \cdot 65488623694306861$ is unacceptable, so 68743 is inadmissible.

C, $31, 17351, 21787, 5104249, 656737 : F_2(656737) = 2 \cdot 41 \cdot 8009$; $F_3(656737) = 3 \cdot 7 \cdot 13 \cdot 661 \cdot 787 \cdot 3037$; $F_5(656737) = 11 \cdot 31^2 \cdot 311 \cdot 971 \cdot 58273473869371$ is unacceptable.

C, $31, 17351, 21787, 5104249, 656737^*, 8009 : F_3(8009) = 6661 \cdot 9631$; $F_5(8009) = 71 \cdot 96451 \cdot 600900161$ is unacceptable; $F_7(8009) = 7 \cdot 43 \cdot 127 \cdot 491 \cdot 127247 \cdot 305873 \cdot 361313$.

C, $31, 17351, 21787, 5104249, 656737^*, 8009, 9631 : F_3(9631) = 3 \cdot 151 \cdot 204781$, contradiction to B; $F_5(9631) = 5 \cdot 41 \cdot 101 \cdot 2411 \cdot 172368611$ is unacceptable.

C, $31, 17351, 21787, 5104249, 656737^*, 8009, 361313 : F_3(361313) = 130547445283$ is unacceptable; $F_5(361313) = 31 \cdot 101 \cdot 5443177355892122111$ is unacceptable, so 361313 is inadmissible.

C, $31, 17351, 21787, 5104249, 656737, 787 : F_3(787) = 3 \cdot 37^2 \cdot 151$, contradiction to B; $F_5(787) = 570821 \cdot 672901$.

C, $31, 17351, 21787, 5104249, 656737, 787, 570821 : F_2(570821) = 2 \cdot 3 \cdot 7 \cdot 13591$; $F_3(570821) = 325837184863$ is unacceptable; $F_5(570821) = 5 \cdot 11 \cdot 3729471281 \cdot 517595595851$ is unacceptable.

C, $31, 17351, 21787, 5104249, 656737, 787, 570821^*, 13591$: $F_3(13591) = 3 \cdot 7^2 \cdot 1256659$; $F_5(13591) = 5 \cdot 6824449137004381$ is unacceptable.

C, $31, 17351, 21787, 5104249, 656737, 787, 570821^*, 13591, 1256659$: $F_3(1256659) = 3 \cdot 8599 \cdot 61216153$ is unacceptable; $F_5(1256659) = 391331 \cdot 6372735252852647251$ is unacceptable, so 1256659 is inadmissible.

D. $127 \nmid N$.

D, 127 : $F_3(127) = 3 \cdot 5419$; $F_5(127) = 262209281$ is unacceptable; $F_7(127) = 7 \cdot 43 \cdot 86353 \cdot 162709$.

D, $127, 5419$: $F_3(5419) = 3 \cdot 31 \cdot 313 \cdot 1009$, contradiction to C; $F_5(5419) = 1031 \cdot 836561914831$ is unacceptable.

D, $127, 162709$: $F_2(162709) = 2 \cdot 5 \cdot 53 \cdot 307$; $F_3(162709) = 3 \cdot 8824793797$ is unacceptable; $F_5(162709) = 700888562389531127981$ is unacceptable.

D, $127, 162709^*, 307$: $F_3(307) = 3 \cdot 43 \cdot 733$; $F_5(307) = 1051 \cdot 1621 \cdot 5231$.

D, $127, 162709^*, 307, 733$: $F_3(733) = 3 \cdot 19 \cdot 9439$; $F_5(733) = 5641 \cdot 51245141$ is unacceptable.

D, $127, 162709^*, 307, 733, 9349$: $F_3(9349) = 3 \cdot 7 \cdot 163 \cdot 25537$; $F_5(9349) = 7640241654796301$ is unacceptable.

D, $127, 162709^*, 307, 733, 9349, 25537$: $F_3(25537) = 3 \cdot 217387969$ is unacceptable; $F_5(25537) = 92068621 \cdot 4619392601$ is unacceptable, so 25537 is inadmissible.

D, $127, 162709^*, 307, 5231$: $F_3(5231) = 7 \cdot 3909799$; $F_5(5231) = 5 \cdot 601 \cdot 249216868661$ is unacceptable.

D, $127, 162708^*, 307, 5231, 3909799$: $F_3(3909799) = 3 \cdot 2671 \cdot 1907716477$ is unacceptable; $F_5(3909799) = 11 \cdot 11551 \cdot 5296621 \cdot 12881171 \cdot 26456251$ is unacceptable, so 3909799 is inadmissible.

E. $19 \nmid N$.

E, 19 : $F_3(19) = 3 \cdot 127$, contradiction to D; $F_5(19) = 151 \cdot 911$, contradiction to B; $F_7(19) = 701 \cdot 70841$.

E, $19, 70841$: $F_2(70841) = 2 \cdot 3 \cdot 11807$; $F_3(70841) = 39103 \cdot 128341$; $F_5(70841) = 5 \cdot 61 \cdot 71 \cdot 1163018639068051$ is unacceptable.

E, $19, 70841^*, 701$: $F_3(701) = 492103$; $F_5(701) = 5 \cdot 101 \cdot 478851301$ is unacceptable.

E, $19, 70841^*, 701, 492103$: $F_3(492103) = 3 \cdot 307 \cdot 1609 \cdot 163417$; $F_5(492103) = 4356227321 \cdot 13462149171001$ is unacceptable.

E, $19, 70841^*, 701, 492103, 163417$: $F_3(163417) = 3 \cdot 7 \cdot 463 \cdot 2746609$; $F_5(163417) = 31 \cdot 230054057658564120 11$ is unacceptable.

E, $19, 70841^*, 701, 492103, 163417, 2746609$: $F_3(2746609) = 3 \cdot 19 \cdot 127 \cdot 1879 \cdot 554611$, contradiction to D; $F_5(2746609) = 17471 \cdot 1854991 \cdot 16585171 \cdot 105878551$ is unacceptable.

E, $19, 70841, 39103$: $F_3(39103) = 3 \cdot 7561 \cdot 67411$; $F_5(39103) = 11 \cdot 57344741 \cdot 3706509671$ is unacceptable.

E, $19, 70841, 39103, 67411$: $F_3(67411) = 3 \cdot 1514770111$ is unacceptable; $F_5(67411) = 5 \cdot 11 \cdot 79631 \cdot 4715032190881$ is unacceptable, so 67411 is inadmissible.

F. $11 \nmid N$.

F, 11 : $F_3(11) = 7 \cdot 19$, contradiction to E; $F_5(11) = 5 \cdot 3221$; $F_7(11) = 43 \cdot 45319$; $F_{11}(11) = 15797 \cdot 1806113$.

F, $11, 3221$: $F_2(3221) = 2 \cdot 3^2 \cdot 179$; $F_3(3221) = 10378063$ is unacceptable; $F_5(3221) = 5 \cdot 11 \cdot 1957650063931$ is unacceptable.

F, $11, 3221^*, 179$: $F_3(179) = 7 \cdot 4603$; $F_5(179) = 11 \cdot 93853931$ is unacceptable.

F, $11, 3221^*, 179, 4603$: $F_3(4603) = 3 \cdot 7 \cdot 1009153$; $F_5(4603) = 11 \cdot 911 \cdot 208511 \cdot 214891$.

F, $11, 3221^*, 179, 4603, 1009153$: $F_3(1009153) = 3 \cdot 739 \cdot 459355339$ is unacceptable; $F_5(1009153) = 761 \cdot 16581328831 \cdot 82191045331$ is unacceptable, so 1009153 is inadmissible.

F, $11, 3221^*, 179, 4603, 214891$: $F_3(214891) = 3 \cdot 35671 \cdot 431521$; $F_5(214891) = 5 \cdot 41 \cdot 135266041 \cdot 76901053301$ is unacceptable.

F, $11, 3221^*, 179, 4603, 214891, 431521$: $F_3(431521) = 3 \cdot 79 \cdot 785699599$ is unacceptable; $F_5(431521) = 5 \cdot 41 \cdot 71 \cdot 107833961 \cdot 22092302471$ is unacceptable, so 431521 is inadmissible.

F, $11, 45319$: $F_3(45319) = 3 \cdot 127 \cdot 5390701$, contradiction to D; $F_5(45319) = 151 \cdot 27935336611728311$ is unacceptable.

F, $11, 1806113$: $F_2(1806113) = 2 \cdot 3 \cdot 17 \cdot 17707$; $F_3(1806113) = 19 \cdot 171686630257$ is unacceptable; $F_5(1806113) = 4051 \cdot 37680901 \cdot 69710210289691$ is unacceptable.

F, $11, 1806113^*, 17707$: $F_3(17707) = 3 \cdot 7^2 \cdot 2133031$; $F_5(17707) = 98311534883794601$ is unacceptable.

F, $11, 1806113^*, 17707, 2133031$: $F_3(2133031) = 3 \cdot 1516607793331$ is unacceptable; $F_5(2133031) = 5 \cdot 131 \cdot 5641 \cdot 5602623941727635591$ is unacceptable, so 2133031 is inadmissible.

G. $7 \nmid N$.

G, 7 : $F_3(7) = 3 \cdot 19$, contradiction to E; $F_5(7) = 2801$; $F_7(7) = 29 \cdot 4733$; $F_{11}(7) = 1123 \cdot 293459$.

G, $7, 2801$: $F_2(2801) = 2 \cdot 3 \cdot 467$; $F_3(2801) = 37 \cdot 43 \cdot 4933$; $F_5(2801) = 5 \cdot 1956611 \cdot 6294091$.

G, $7, 2801^*, 467$: $F_3(467) = 19 \cdot 11503$, contradiction to E; $F_5(467) = 11 \cdot 31 \cdot 41 \cdot 3409261$, contradiction to F.

G, $7, 2801, 4933$: $F_2(4933) = 2 \cdot 2467$; $F_3(4933) = 3 \cdot 127 \cdot 193 \cdot 331$, contradiction to D; $F_5(4933) = 11 \cdot 31 \cdot 7541 \cdot 230329301$ is unacceptable.

G, $7, 2801, 4933^*, 2467$: $F_3(2467) = 3 \cdot 271 \cdot 7489$; $F_5(2467) = 11 \cdot 331 \cdot 10177286401$ is unacceptable.

G, $7, 2801, 4933^*, 2467, 271$: $F_3(271) = 3 \cdot 24571$; $F_5(271) = 5 \cdot 251 \cdot 4313591$.

G, $7, 2801, 4933^*, 2467, 271, 24571$: $F_3(24571) = 3 \cdot 201252871$ is unacceptable; $F_5(24571) = 5 \cdot 71 \cdot 1466741 \cdot 700047031$ is unacceptable, so 24571 is inadmissible.

G, $7, 2801, 4933^*, 2467, 271, 4313591$: $F_3(4313591) = 7 \cdot 13 \cdot 487 \cdot 419863069$ is unacceptable; $F_5(4313591) = 5 \cdot 71 \cdot 180701 \cdot 246941 \cdot 21856159267171$ is unacceptable, so 4313591 is inadmissible.

G, $7, 2801, 6294091$: $F_3(6294091) = 3 \cdot 13205195936791$ is unacceptable; $F_5(6294091) = 5 \cdot 1963111 \cdot 2692801 \cdot 59376284620771$ is unacceptable, so 6294091 is inadmissible.

G, $7, 4733$: $F_2(4733) = 2 \cdot 3^2 \cdot 263$; $F_3(4733) = 22406023$ is unacceptable; $F_5(4733) = 11 \cdot 41 \cdot 101 \cdot 11018941331$ is unacceptable.

G, $7, 4733^*, 263$: $F_3(263) = 7^2 \cdot 13 \cdot 109$; $F_5(263) = 4802611441$ is unacceptable.

G, $7, 4733^*, 263, 109$: $F_3(109) = 3 \cdot 7 \cdot 571$; $F_5(109) = 31 \cdot 191 \cdot 24061$, contradiction to C.

G, $7, 4733^*, 263, 109, 571$: $F_3(571) = 3 \cdot 7 \cdot 103 \cdot 151$, contradiction to B; $F_5(571) = 5 \cdot 1831 \cdot 11631811$ is unacceptable.

G, 7, 293459 : $F_3(293459) = 277{\cdot}310897033$ is unacceptable; $F_5(293459) = 751 \cdot 9875322243666178231$ is unacceptable, so 293459 is inadmissible.

H. $23 \nmid N$.

H, 23 : $F_3(23) = 7{\cdot}79$, contradiction to G; $F_5(23) = 292561$; $F_7(23) = 29 \cdot 5336717$.

H, 23, 292561 : $F_2(292561) = 2{\cdot}19{\cdot}7699$, contradiction to E; $F_3(292561) = 3{\cdot}13^2{\cdot}168820969$ is unacceptable; $F_5(292561) = 5{\cdot}11{\cdot}7151{\cdot}18626778238182061$ is unacceptable.

H, 23, 5336717 : $F_2(5336717) = 2 \cdot 3 \cdot 889453$; $F_3(5336717) = 13 \cdot 31 \cdot 70671349069$ is unacceptable; $F_5(5336717) = 811141785630879065603160541$ is unacceptable.

H, 23, 5336717*, 889453 : $F_3(889453) = 3{\cdot}37{\cdot}7127275033$ is unacceptable; $F_5(889453) = 11{\cdot}1051{\cdot}1301{\cdot}4691{\cdot}8870628156571$ is unacceptable, so 889453 is inadmissible.

I. $131 \nmid N$.

I, 131 : $F_3(131) = 17293$; $F_5(131) = 5 \cdot 61 \cdot 973001$; $F_7(131) = 127 \cdot 189967 \cdot 211093$, contradiction to D.

I, 131, 17293 : $F_2(17293) = 2 \cdot 8647$; $F_3(17293) = 3 \cdot 13 \cdot 7668337$; $F_5(17293) = 384481 \cdot 232611722621$ is unacceptable.

I, 131, 17293*, 8647 : $F_3(8647) = 3 \cdot 7 \cdot 37 \cdot 157 \cdot 613$, contradiction to G; $F_5(8647) = 41 \cdot 251 \cdot 2081 \cdot 261085291$ is unacceptable.

I, 131, 17293, 7668337 : $F_2(7668337) = 2{\cdot}23{\cdot}166703$, contradiction to H; $F_3(7668337) = 3 \cdot 801337 \cdot 24460537$ is unacceptable; $F_5(7668337) = 207331 \cdot 16677869697590861530391$ is unacceptable.

I, 131, 973001 : $F_2(973001) = 2 \cdot 3 \cdot 257 \cdot 631$; $F_3(973001) = 13 \cdot 19 \cdot 3832922749$ is unacceptable; $F_5(973001) = 5 \cdot 31 \cdot 12601 \cdot 19801 \cdot 23175534509471$ is unacceptable.

I, 131, 973001*, 257 : $F_3(257) = 61 \cdot 1087$; $F_5(257) = 11 \cdot 398137391$ is unacceptable.

I, 131, 973001*, 257, 1087 : $F_3(1087) = 3 \cdot 7 \cdot 199 \cdot 283$, contradiction to G; $F_5(1087) = 11 \cdot 31 \cdot 4097920381$ is unacceptable.

J. $37 \nmid N$.

J, 37 : $F_2(37) = 2 \cdot 19$, contradiction to E; $F_3(37) = 3 \cdot 7 \cdot 67$, contradiction to G; $F_5(37) = 11 \cdot 41 \cdot 4271$, contradiction to F.

K. $61 \nmid N$.

K, 61 : $F_2(61) = 2 \cdot 31$, contradiction to C; $F_3(61) = 3 \cdot 13 \cdot 97$; $F_5(61) = 5 \cdot 131 \cdot 21491$, contradiction to I.

K, 61, 97 : $F_2(97) = 2 \cdot 7^2$, contradiction to G; $F_3(97) = 3 \cdot 3169$; $F_5(97) = 11 \cdot 31 \cdot 262321$, contradiction to C.

K, 61, 97, 3169 : $F_2(3169) = 2 \cdot 5 \cdot 317$; $F_3(3169) = 3 \cdot 3348577$; $F_5(3169) = 61 \cdot 1653850268201$ is unacceptable.

K, 61, 97, 3169*, 317 : $F_3(317) = 7 \cdot 14401$, contradiction to G; $F_5(317) = 11 \cdot 311 \cdot 2961121$, contradiction to F.

K, 61, 97, 3169, 3348577 : $F_2(3348577) = 2 \cdot 1674289$; $F_3(3348577) = 3 \cdot 3737657091169$ is unacceptable; $F_5(3348577) = 86524531 \cdot 1453121857811448551$ is unacceptable.

K, 61, 97, 3169, 3348577*, 1674289 : $F_3(1674289) = 3 \cdot 934415109937$ is unacceptable; $F_5(1674289) = 7858179685661558917710821$ is unacceptable, so 1674289 is inadmissible.

L. $13 \nmid N$.

L, 13 : $F_2(13) = 2 \cdot 7$, contradiction to G; $F_3(13) = 3 \cdot 61$, contradiction to K; $F_5(13) = 30941$; $F_7(13) = 5229043$; $F_{11}(13) = 23 \cdot 419 \cdot 859 \cdot 18041$, contradiction to H; $F_{13}(13) = 53 \cdot 264031 \cdot 1803647$.

L, 13, 30941 : $F_2(30941) = 2 \cdot 3^4 \cdot 191$; $F_3(30941) = 157 \cdot 433 \cdot 14083$; $F_5(30941) = 5 \cdot 11 \cdot 69617111 \cdot 239371661$ is unacceptable.

L, 13, 30941*, 191 : $F_3(191) = 7 \cdot 13^2 \cdot 31$, contradiction to C; $F_5(191) = 5 \cdot 11 \cdot 1871 \cdot 13001$, contradiction to F; $F_7(191) = 127 \cdot 197 \cdot 10627 \cdot 183569$, contradiction to D.

L, 13, 30941, 433 : $F_2(433) = 2 \cdot 7 \cdot 31$, contradiction to C; $F_3(433) = 3 \cdot 37 \cdot 1693$, contradiction to J; $F_5(433) = 11 \cdot 1811 \cdot 1768661$, contradiction to F.

L, 13, 5229043 : $F_3(5229043) = 3 \cdot 31 \cdot 4051 \cdot 72577051$ is unacceptable; $F_5(5229043) = 151 \cdot 151841 \cdot 32607907713428723311$ is unacceptable, so 5229043 is inadmissible.

L, 13, 1803647 : $F_3(1803647) = 31 \cdot 104940138847$ is unacceptable; $F_5(1803647) = 41 \cdot 53738676221 \cdot 4803254465101$ is unacceptable, so 1803647 is inadmissible.

M. $3 \nmid N$.

M, 3 : $F_3(3) = 13$, contradiction to L; $F_5(3) = 11^2$, contradiction to F; $F_7(3) = 1093$, contradiction to A; $F_{11}(3) = 23 \cdot 3851$, contradiction to H; $F_{13}(3) = 797161; F_{17}(3) = 1871 \cdot 34511$; $F_{19}(3) = 1597 \cdot 363889$; $F_{31}(3) = 683 \cdot 102673 \cdot 4404047$.

M, $3, 797161$: $F_2(797161) = 2 \cdot 398581$; $F_3(797161) = 3 \cdot 61 \cdot 151 \cdot 22996651$ is unacceptable; $F_5(797161) = 5 \cdot 2671 \cdot 178601 \cdot 169299992707751$ is unacceptable.

M, $3, 797161^*, 398581$: $F_3(398581) = 3 \cdot 1621 \cdot 32668561$ is unacceptable; $F_5(398581) = 5 \cdot 1866871 \cdot 2703853428809791$ is unacceptable, so 797161 is inadmissible.

M, $3, 34511$: $F_3(34511) = 13 \cdot 19 \cdot 4822039$, contradiction to L; $F_5(34511) = 5 \cdot 11 \cdot 31 \cdot 71 \cdot 131 \cdot 89451727381$ is unacceptable.

M, $3, 1597$: $F_2(1597) = 2 \cdot 17 \cdot 47$; $F_3(1597) = 3 \cdot 43 \cdot 73 \cdot 271$; $F_5(1597) = 31 \cdot 209956826531$ is unacceptable.

M, $3, 1597^*, 47$: $F_3(47) = 37 \cdot 61$, contradiction to J; $F_5(47) = 11 \cdot 31 \cdot 14621$, contradiction to C.

M, $3, 1597, 271$: $F_3(271) = 3 \cdot 24571$; $F_5(271) = 5 \cdot 251 \cdot 4313591$.

M, $3, 1597, 271, 24571$: $F_3(24571) = 3 \cdot 201252871$ is unacceptable; $F_5(24571) = 5 \cdot 71 \cdot 1466741 \cdot 700047031$ is unacceptable, so 24571 is inadmissible.

M, $3, 1597, 271, 4313591$: $F_3(4313591) = 7 \cdot 13 \cdot 487 \cdot 419863069$ is unacceptable; $F_5(4313591) = 5 \cdot 71 \cdot 180701 \cdot 246941 \cdot 21856159267171$ is unacceptable, so 4313591 is inadmissible.

M, $3, 4404047$: $F_3(4404047) = 7 \cdot 823 \cdot 3877 \cdot 868381$, contradiction to G; $F_5(4404047) = 131 \cdot 211 \cdot 461 \cdot 2770771 \cdot 10654978292191$ is unacceptable.

N. $5 \nmid N$.

N, 5 : $F_2(5) = 2 \cdot 3$, contradiction to M; $F_3(5) = 31$, contradiction to C; $F_5(5) = 11 \cdot 71$, contradiction to F; $F_7(5) = 19531$; $F_{19}(5) = 191 \cdot 6271 \cdot 3981071$.

N, $5, 19531$: $F_3(19531) = 3 \cdot 127159831$ is unacceptable; $F_5(19531) = 5 \cdot 191 \cdot 4760281 \cdot 32009891$ is unacceptable, so 19531 is inadmissible.

N, $5, 191$: $F_3(191) = 7 \cdot 13^2 \cdot 31$, contradiction to C; $F_5(191) = 5 \cdot 11 \cdot 1871 \cdot 13001$, contradiction to F; $F_7(191) = 127 \cdot 197 \cdot 10627 \cdot 183569$, contradiction to D.

O. $29 \nmid N$.

O, 29 : $F_2(29) = 2 \cdot 3 \cdot 5$, contradiction to M; $F_3(29) = 13 \cdot 67$, contradiction to L; $F_5(29) = 732541$.

O, $29, 732541$: $F_2(732541) = 2 \cdot 47 \cdot 7793$; $F_3(732541) = 3 \cdot 43 \cdot 271 \cdot 15349897$ is unacceptable; $F_5(732541) = 5 \cdot 45121 \cdot 487681 \cdot 2617241417881$ is unacceptable.

O, $29, 732541^*, 7793$: $F_3(7793) = 7 \cdot 8676949$, contradiction to G; $F_5(7793) = 11 \cdot 71^2 \cdot 3701 \cdot 17974051$ is unacceptable.

P. $43 \nmid N$.

P, 43 : $F_3(43) = 3 \cdot 631$, contradiction to M; $F_5(43) = 3500201$; $F_7(43) = 7 \cdot 5839 \cdot 158341$, contradiction to G.

P, $43, 3500201$: $F_2(3500201) = 2 \cdot 3 \cdot 583367$, contradiction to M; $F_3(3500201) = 13 \cdot 139 \cdot 28411 \cdot 238639$, contradiction to L; $F_5(3500201) = 5 \cdot 689261 \cdot 38745181 \cdot 1124088896761$ is unacceptable.

Q. $17 \nmid N$.

Q, 17 : $F_2(17) = 2 \cdot 3^2$, contradiction to M; $F_3(17) = 307$; $F_5(17) = 88741$.

Q, $17, 307$: $F_3(307) = 3 \cdot 43 \cdot 733$, contradiction to M; $F_5(307) = 1051 \cdot 1621 \cdot 5231$.

Q, $17, 307, 5231$: $F_3(5231) = 7 \cdot 3909799$, contradiction to G; $F_5(5231) = 5 \cdot 601 \cdot 249216868661$ is unacceptable.

Q, $17, 88741$: $F_2(88741) = 2 \cdot 44371$; $F_3(88741) = 3 \cdot 7 \cdot 13 \cdot 19 \cdot 61 \cdot 24889$, contradiction to M, $F_5(88741) = 5 \cdot 11 \cdot 4451 \cdot 5441 \cdot 46558947881$ is unacceptable.

Q, $17, 88741^*, 44371$: $F_3(44371) = 3 \cdot 656276671$ is unacceptable; $F_5(44371) = 5 \cdot 60055811 \cdot 12908673431$ is unacceptable, so 44371 is inadmissible.

R. $71 \nmid N$.

R, 71 : $F_3(71) = 5113$; $F_5(71) = 5 \cdot 11 \cdot 211 \cdot 2221$, contradiction to N.

R, $71, 5113$: $F_2(5113) = 2 \cdot 2557$; $F_3(5113) = 3 \cdot 8715961$, contradiction to M; $F_5(5113) = 11 \cdot 4751 \cdot 13080080081$ is unacceptable.

R, $71, 5113^*, 2557$: $F_3(2557) = 3 \cdot 7 \cdot 13^2 \cdot 19 \cdot 97$, contradiction to M; $F_5(2557) = 11 \cdot 12011 \cdot 323683781$ is unacceptable.

S. $113 \nmid N$.

S, 113 : $F_2(113) = 2 \cdot 3 \cdot 19$, contradiction to M; $F_3(113) = 13 \cdot 991$, contradiction to L; $F_5(113) = 11 \cdot 251 \cdot 59581$, contradiction to L; $F_7(113) = 7 \cdot 44983 \cdot 6670903$, contradiction to G.

T. $197 \nmid N$.

T, 197 : $F_2(197) = 2 \cdot 3^2 \cdot 11$, contradiction to M; $F_3(197) = 19 \cdot 2053$, contradiction to E; $F_5(197) = 661 \cdot 991 \cdot 2311$; $F_7(197) = 7 \cdot 29 \cdot 97847 \cdot 2957767$, contradiction to G.

T, $197, 2311$: $F_3(2311) = 3 \cdot 883 \cdot 2017$, contradiction to M; $F_5(2311) = 5 \cdot 2591 \cdot 4001 \cdot 550531$, contradiction to N.

U. $211 \nmid N$.

U, 211 : $F_3(211) = 3 \cdot 13 \cdot 31 \cdot 37$, contradiction to M; $F_5(211) = 5 \cdot 1361 \cdot 292661$, contradiction to N.

V. $239 \nmid N$.

V, 239 : $F_3(239) = 19 \cdot 3019$, contradiction to E; $F_5(239) = 3276517921$ is unacceptable; $F_7(239) = 7 \cdot 29 \cdot 245561 \cdot 3754507$, contradiction to G.

W. $281 \nmid N$.

W, 281 : $F_2(281) = 2 \cdot 3 \cdot 47$, contradiction to M; $F_3(281) = 109 \cdot 727$; $F_5(281) = 5 \cdot 31 \cdot 271 \cdot 148961$, contradiction to N.

W, $281, 727$: $F_3(727) = 3 \cdot 176419$, contradiction to M; $F_5(727) = 14281 \cdot 19587401$ is unacceptable.

Lemma 5.1 follows from A, B, C, …, W.

Appendix C. Computer programs

The following UBASIC (extension .ub) and MAPLE (extension .mws) computer programs were used for computations.

C.1. **Cubeprog.ub.** Used to check for cubes dividing $F_r(p)$, with $2142 < r < 5000000$.

```
10   word -540
20   print "Begin checking at":
     input Start:Var=nxtprm(Start-1)
30   input "Up to";Harold:clr time
40   if Var>Harold then print
     "Not a valid interval.":end
50   P=101:Q=2*Var+1
60   while Q<10000000
```

```
 70     if prmdiv(Q)<Q then goto 2160
 80     print "Checking Q=";Q;"for R=";Var
 90     P=101
100     while P<10000000
110      if modpow(P,Var,(Q^3))=1 then
120       :print=print+"mycubes.txt"
130       :print Q;"^3 divides";P;"^";Var;"-1"
140       :print=print
150      P=nxtprm(P)
160     wend
170     Q=Q+2*Var
180    wend
190    Var=nxtprm(Var):if Var<Harold then goto 2050
200    print=print+"numchkd.txt"
210    print "Checked R=";Start;"to R=";Harold;"in";time
220    print=print
```

C.2. **Sqrprg.ub.** Used to check for cubic divisors and that only one square divides $F_r(p)$ for $1472 < r < 2142$.

```
 10    word -540
 20    input "Begin checking at R=";Start:
       Var=nxtprm(Start-1)
 30    input "Up to R=";Endint:clr time
 40    if Var>Endint then print
       "Not a valid interval.":end
 50    P=101:Q=2*Var+1
 60    while P<10000000
 70     Q=2*R+1:Numsqr=0
 80     print "Checking P=";P;"for R=";Var
 90     while Q<10000000
100      if prmdiv(Q)<Q then goto 160
110      if modpow(P,Var,(Q^2))=1 then
120       :Numsqr=Numsqr+1
130       :if modpow(P,Var,(Q^3))=1 then
140        :print=print+"mycubes.txt"
150        :print Q;"^3 divides";P;"^";Var;"-1":
```

```
          print=print
160     Q=Q+2*Var
170    wend
180    if Numsqr>1 then print=print+"mysqrs.txt"
190     :print "More than one square divides";P;
         "^";Var;"-1."
200     :print=print
210    P=nxtprm(P)
220   wend
230   Var=nxtprm(Var):if Var<Endint then goto 50
240   print=print+"numchkd.txt"
250   print "Checked R=";Start;"to R=";Endint;"in";time
260   print=print
```

C.3. **Prop7x.ub.** Used to directly check values of $F_r(p)$ with $r < 1472$.

```
 10   word -540
 15   clr time
 20   input "Enter R";R
 25   input "Enter beginning P";Pbeg
 30   input "Enter ending P";Pend
 35   P=nxtprm(Pbeg-1):V=0
 40   while P<Pend
 45    print "Checking P=";P;"for R=";R
 50    V=0
 55    if modpow(P,R,R)=1 then V=V+log(R)
 60    X=2*R+1
 70    while X<10000000
 80     if prmdiv(X)<X then goto 150
 90     if modpow(P,R,X)=1 then V=V+log(X)
100      :if modpow(P,R,X^2)=1 then V=V+log(X)
110       :if modpow(P,R,X^3)=1 then V=V+log(X)
120        :if modpow(P,R,X^4)=1 then V=V+log(X)
130         :print=print+"problem.txt":print X;
             "^4 divides";P;"^";R;"-1"
140         :print=print
150     X=X+2*R
```

```
160    wend
170    if V>(R-1)*log(P) then
180     :print=print+"prop7.txt":print "P=";P;"and R=";R
190     :print=print
200    P=nxtprm(P)
210   wend
220   print=print+"numchkd.txt":
      print "Checked R=";R;"for P=";Pbeg;"to";Pend
225   print "in";time
230   print=print
```

C.4. **Stester.ub.** Used to calculate S^* and the number of elements of S.

```
 60   word -120
 61   Scount=0
 62   Sproduct=1
 70   N=41
 75   while N<10000000
 77    if (N@3<>1 and N@5<>1) then
 85     :print N
 90     :Scount=Scount+1
100     :Sproduct=Sproduct*N/(N-1)
110    endif
115   N=nxtprm(N):wend
116   print "Scount=";Scount
117   print "Sproduct=";Sproduct
120   end
```

C.5. **Ttester.ub.** Used to calculate T^* and the number of elements of T.

```
 60   word -120
 61   Tcount=0
 62   Tproduct=1
 70   N=41
 75   while N<10000000
 77    if (N@3=1 and N@5=1) then
```

```
 85     :print N
 90     :Tcount=Tcount+1
100     :Tproduct=Tproduct*N/(N-1)
110    endif
115   N=nxtprm(N):wend
116   print "Tcount=";Tcount
117   print "Tproduct=";Tproduct
120   end
```

C.6. **Utester.mws.** Used to calculate U^* and the number of elements in U.

```
restart;
pj:=41;
with(numtheory):
uproduct:=1;
usum:=0;
placeholder:=0;
while pj<10000000 do
 placeholder:=0:
 if (pj mod 3=1 and pj mod 5<>1) then
  bob:=factorset(cyclotomic(5, pj)):
  print (fsolve(x=uproduct));
  for z in bob do
   if (z>10000000) then
    placeholder:=1
   fi
  od:
  if placeholder=1 then
   uproduct:=uproduct*pj/(pj-1):
   usum:=usum+1
  fi:
 fi;
 pj:=nextprime(pj):
od:
print(uproduct, usum);
```

C.7. **Vtester.mws.** Used to calculate V^* and the number of elements of V.

```
restart;
pj:=41;
with(numtheory):
vproduct:=1;
vsum:=0;
placeholder:=0;
while pj<10000000 do
 placeholder:=0;
 if (pj mod 3<>1 and pj mod 5=1) then
  bob:=factorset(cyclotomic(3,pj));
  for z in bob do
   if (z>10000000) then
    placeholder:=1
   fi
  od;
  if placeholder=1 then
   vproduct:=vproduct*pj/(pj-1);
   vsum:=vsum+1
  fi;
  print (fsolve(x=vproduct));
 fi;
 pj:=nextprime(pj);
od;
print(vproduct, vsum);
```

C.8. **Ptester.ub.** Used to calculate P^* and the number of elements of P.

```
60   word -120
61   Pcount=0
62   Pproduct=1.
70   P=41
75   while P<10000000
80    Pcount=Pcount+1
85    Pproduct=Pproduct*P/(P-1)
```

```
86    P=nxtprm(P)
90   wend
100  print "Pcount=";Pcount
110  print "Pproduct=";Pproduct
```

C.9. **Prop5.ub.** Used to find 2142 as a lower bound for r in Proposition 9.1.

```
5    Count=0
10   R=101
15   Q=1
20   while R<10000000
22    Q=1
23    Count=0
24    while Q<10000000
25     Q=Q+2*R
30     if prmdiv(Q)=Q then Count=Count+log(Q)/log(10)
35    wend
40    if 2*Count>=2*(R-1) then print R
50    R=nxtprm(R)
60   wend
```

C.10. **Prop6.ub.** Used to find 1472 as a lower bound for r in Proposition 9.2.

```
1010 word -540
1020 Count=0
1030 R=101
1040 Q=1
1050 while R<10000000
1060  Q=1
1070  Count=0
1080  while Q<10000000
1090   Q=Q+2*R
1100   if prmdiv(Q)=Q then Count=Count+log(Q)/log(10)
1110  wend
1120  if Count+7>=2*(R-1) then print R
1130  R=nxtprm(R)
```

```
1140 wend
```

C.11. **Prop8.ub.** Used to find $R(p)$ for $3 \leq p < 100$ in Proposition 9.4.

```
5   Count=0
7   P=3
8   print=print+"prop8.txt"
9   while P<100
10   R=1300
15   Q=1
20   while R<10000000
22    Q=1
23    Count=0
24    while Q<10000000
25     Q=Q+2*R
30     if prmdiv(Q)=Q then Count=Count+log(Q)/log(10)
35    wend
40    if Count>=log(P)/log(10)*(R-1) then print R
50    R=nxtprm(R)
60   wend
65   print "for P=";P
68   P=nxtprm(P)
70  wend
100 print=print
```

C.12. **Prop9.ub.** Used to directly check $F_r(p)$ for acceptability with $p < 100$ and $7 \leq r < R(p)$ in Proposition9.5.

```
10  V=0
20  input "Enter P:";Start
30  P=nxtprm(Start-1)
40  input "Enter R sub p";Rp
50  R=7
60  while R<Rp
65   V=0
70   if modpow(P,R,R)=1 then V=V+log(R)
80   X=2*R+1
90   while X<10000000
```

```
100      if prmdiv(X)<X then goto 120
110      if modpow(P,R,X)=1 then V=V+log(X)
120      X=X+2*R
130     wend
140     if V>=(R-1)*log(P) then
150      :print=print+"prop9.txt"
160      :print "Look at p=";P;"and r=";R
170      :print=print
180     R=nxtprm(R)
190    wend
```

C.13. **Probtest.ub.** Used to directly check $F_r(p)$ for acceptability when it is divisible by a fourth power. Only one value of $F_r(p)$ was divisible by a fifth power, $23^5|F_{11}(4330649)$; $F_{11}(4330649)$ was factored and has a prime divisor greater than 10^7.

```
 10    word -540
 20    input "Enter R";R
 25    input "Enter P";Pbeg
 30    P=nxtprm(Pbeg-1):V=0
 45    V=0
 50    if modpow(P,R,R)=1 then V=V+log(R)
 60    X=2*R+1
 70    while X<10000000
 80     if prmdiv(X)<X then goto 150
 90     if modpow(P,R,X)=1 then V=V+log(X)
100      :if modpow(P,R,X^2)=1 then V=V+log(X)
110       :if modpow(P,R,X^3)=1 then V=V+log(X)
120        :if modpow(P,R,X^4)=1 then V=V+log(X)
130         :if modpow(P,R,X^5)=1 then V=V+log(X)
140          :print X"^5 divides";P;"^";R;"-1"
150     X=X+2*R
160    wend
170    if V>(R-1)*log(P) then
180     :print "It’s an exception."
```

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Department of Mathematics, Brigham Young University, Provo, Utah 84602

E-mail address: pmj5@math.byu.edu