KLOOSTERMAN SUMS AND TRACES OF SINGULAR MODULI

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1. Introduction and statement of results

Let $j(z)$ be the modular function for $\mathrm{SL}_2(\mathbb{Z})$ defined by

$$j(z) = \frac{E_4(z)^3}{\Delta(z)} = q^{-1} + 744 + 196884q + \cdots ,$$

where $q = e^{2\pi i z}$.

Let $d \equiv 0, 3 \pmod 4$ be a positive integer, so that $-d$ is a negative discriminant. Denote by $\mathcal{Q}_d$ the set of positive definite integral binary quadratic forms $Q(x, y) = ax^2 + bxy + cy^2 = [a, b, c]$ with discriminant $-d = b^2 - 4ac$, including imprimitive forms (if such exist). We let α_Q be the unique complex number in the upper half plane $\mathfrak{H}$ which is a root of $Q(x, 1) = 0$.

Values of j at the points α_Q are known as *singular moduli*. Singular moduli are algebraic integers which play prominent roles in number theory. For example, Hilbert class fields of imaginary quadratic fields are generated by singular moduli, and isomorphism classes of elliptic curves with complex multiplication are distinguished by singular moduli. Because of the modularity of j, the singular modulus $j(\alpha_Q)$ depends only on the equivalence class of Q under the action of $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$.

We define $\omega_Q \in \{1, 2, 3\}$ as

$$\omega_Q = \begin{cases} 2 & \text{if } Q \sim_\Gamma [a, 0, a], \\ 3 & \text{if } Q \sim_\Gamma [a, a, a], \\ 1 & \text{otherwise.} \end{cases} \tag{1.1}$$

The Hurwitz-Kronecker class number $H(d)$ is the number of equivalence classes of forms of discriminant $-d$ under the action of Γ, weighted by ω_Q. Specifically, it is defined as

$$H(d) = \sum_{Q \in \mathcal{Q}_d/\Gamma} \frac{1}{\omega_Q}. \tag{1.2}$$

Following Zagier, we define the trace of the singular moduli of discriminant $-d$ as

$$\mathrm{Tr}(d) = \sum_{Q \in \mathcal{Q}_d/\Gamma} \frac{j(\alpha_Q) - 744}{\omega_Q}. \tag{1.3}$$

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If we modify the standard Hilbert class polynomial slightly and define

$$\mathcal{H}_d(X) = \prod_{Q\in\mathcal{Q}_d/\Gamma} (X - j(\alpha_Q))^{1/\omega_Q},$$

we can interpret $H(d)$ and $\mathrm{Tr}(d)$ as the first two Fourier coefficients of the logarithmic derivative of $\mathcal{H}_d(j(z))$.

Borcherds [Bo] proved a striking theorem describing the full Fourier expansion of $\mathcal{H}_d(j(z))$ in terms of the coefficients of certain nearly holomorphic weight $1/2$ modular forms. Specifically, Borcherds proved the following theorem.

Theorem 1.1 (Borcherds). *Let $d > 0, d \equiv 0, 3 \pmod 4$. Then*

$$\mathcal{H}_d(j(z)) = q^{-H(d)} \prod_{n=1}^{\infty} (1-q^n)^{A(n^2,d)},$$

where $A(D,d)$ is the coefficient of q^D in a certain nearly holomorphic modular form f_d of weight $1/2$ for the group $\Gamma_0(4)$.

Note. We will give a precise description of the f_d later in this section.

Zagier [Z] gave a new proof of Borcherds' theorem and generalized it, using formulas for traces of singular moduli and their generalizations. His proof uses a sequence of nearly holomorphic modular forms g_D of weight $3/2$, which are closely related to the f_d of weight $1/2$.

To generalize the trace of the singular moduli of discriminant $-d$, we let $j_m(z)$, for non-negative integers m, be the unique holomorphic function on $\mathfrak{H}/\Gamma$ with Fourier expansion $q^{-m} + O(q)$. The $j_m(z)$ can be written as polynomials in $j(z)$ of degree m with integer coefficients. For example, we have

$$\begin{aligned}
&j_0(z) = 1,\\
&j_1(z) = j(z) - 744 = q^{-1} + 196884q + 21493760q^2 + \cdots,\\
&j_2(z) = j(z)^2 - 1488j(z) + 159768 = q^{-2} + 42987520q + 40491909396q^2 + \cdots.
\end{aligned}$$

We can then define the mth trace $\mathrm{Tr}_m(d)$ of the singular moduli of discriminant $-d$ as

$$\mathrm{Tr}_m(d) = \sum_{Q\in\mathcal{Q}_d/\Gamma} \frac{j_m(\alpha_Q)}{\omega_Q}. \tag{1.4}$$

We can generalize the traces even further by adding a twist. Let D be a fundamental discriminant. We define the genus character χ_D to be the character assigning a quadratic form $Q = (a,b,c)$, of discriminant divisible by D, the value

$$\chi_D(Q) = \begin{cases} 0 & \text{if } (a,b,c,D) > 1,\\ \left(\frac{D}{n}\right) & \text{if } (a,b,c,D) = 1,\end{cases}$$

where n is any integer represented by Q and coprime to D. This is independent of the choice of n, and for a form Q of discriminant $-dD$, we have $\chi_D = \chi_{-d}$ if $-d$ and D

are both fundamental discriminants. Following Zagier, for fundamental discriminants $D > 0$ we then define a "twisted trace" $\mathrm{Tr}_m(D, d)$ as

$$\mathrm{Tr}_m(D,d) = \sum_{Q \in \mathcal{Q}_{dD}/\Gamma} \frac{\chi_D(Q) j_m(\alpha_Q)}{\omega_Q}. \tag{1.5}$$

We now define, again following Zagier, the two sequences of weakly holomorphic modular forms that relate to these traces. Let $M^!_{\lambda+1/2}$ be the space of weight $\lambda + 1/2$ weakly holomorphic modular forms on $\Gamma_0(4)$, with Fourier expansion

$$f(z) = \sum_{(-1)^\lambda n \equiv 0,1 \pmod 4} a(n) q^n.$$

Recall that a form is weakly holomorphic if its poles, if there are any, are supported at the cusps.

For any $0 < D \equiv 0, 1 \pmod 4$, let $g_D(z)$ be the unique element of $M^!_{3/2}$ with Fourier expansion

$$g_D(z) = q^{-D} + B(D,0) + \sum_{0<d\equiv 0,3 \pmod 4} B(D,d) q^d. \tag{1.6}$$

For $0 \le d \equiv 0, 3 \pmod 4$, let $f_d(z)$ be the unique form in $M^!_{1/2}$ with expansion

$$f_d(z) = q^{-d} + \sum_{0<D\equiv 0,1 \pmod 4} A(D,d) q^D. \tag{1.7}$$

All of the coefficients $A(D, d)$ and $B(D, d)$ of the f_d and g_D are integers.

Applying Hecke operators (for definitions, see section 3.1 of [O]), we also define

$$\begin{aligned} A_m(D,d) &= \text{the coefficient of } q^D \text{ in } f_d(z) \mid T_{\frac{1}{2}}(m^2), \\ B_m(D,d) &= \text{the coefficient of } q^d \text{ in } g_D(z) \mid T_{\frac{3}{2}}(m^2). \end{aligned}$$

Zagier proved the following statements about the relationships between the coefficients of the g_D and f_d.

Theorem 1.2 (Zagier)**.** *Assume the above notation.*

(1) *We have*

$$A_m(D,d) = -B_m(D,d).$$

(2) *If $m \ge 1$, then*

$$A_m(1,d) = \sum_{n \mid m} n A(n^2, d).$$

Zagier then proved the following theorem relating the traces of singular moduli to these modular forms; his results play a central role in his work on Borcherds' products.

Theorem 1.3 (Zagier)**.** *If $m \ge 1$ and $-d < 0$ is a discriminant, then*

$$\mathrm{Tr}_m(d) = -B_m(1,d).$$

Here we give a new proof of this theorem, using Kloosterman sums and the theory of weakly holomorphic Poincaré series. More specifically, we relate formulas given by Duke for the $\mathrm{Tr}_m(d)$ to the coefficients of certain Poincaré series computed by the author, Bruinier, and Ono that appear in computing the $B_m(1,d)$.

Remark. Our proof generalizes the proof of the $m=1$ case given by Duke in [D].

Other coefficients of these modular forms can be interpreted in terms of twisted traces in the following manner.

Theorem 1.4 (Zagier). *If $m \geq 1, -d < 0$ is a discriminant, and $D > 0$ is a fundamental discriminant, then*

$$\mathrm{Tr}_m(D,d) = A_m(D,d)\sqrt{D}.$$

Our second result is a new expression for twisted traces as an infinite series.

Theorem 1.5. *If $D, -d \equiv 0, 1 \pmod 4$ are a positive fundamental discriminant and a negative discriminant, respectively, with $D > 1$, and $m \geq 1$ is an integer, we have*

$$\mathrm{Tr}_m(D,d) = \sum_{\substack{c \equiv 0(4) \\ c>0}} S_{D,d}(m,c) \sinh\left(\frac{4\pi m\sqrt{dD}}{c}\right),$$

where

$$S_{D,d}(m,c) = \sum_{\substack{x(c) \\ x^2 \equiv -Dd(c)}} \chi_D\left(\frac{c}{4}, x, \frac{x^2+Dd}{c}\right) e\left(\frac{2mx}{c}\right).$$

Note. Here we have written $e(z) = e^{2\pi i z}$ for convenience, and have written the sum over all residue classes $\pmod c$ as $\sum_{x(c)}$. In addition, hereafter $\sum_{c\equiv 0(4)}$ will denote a sum over positive integers c divisible by 4.

2. A New Proof of Theorem 1.3

We want to give a new proof that the mth trace can be written as a coefficient of a modular form; specifically, we want to show that $\mathrm{Tr}_m(d) = -B_m(1,d)$. Duke [D] gives a "Kloosterman sum" proof of the modularity of the traces for the $m=1$ case by adapting a method of Tóth [T]. Here we generalize this proof to all integers $m \geq 1$.

We begin by defining, for $\lambda \in \mathbb{Z}$, the generalized Kloosterman sum

$$K_{\lambda+1/2}(m,n;c) = \sum_{\substack{a(c) \\ (a,c)=1}} \left(\frac{c}{a}\right)^{2\lambda+1} \varepsilon_a^{2\lambda+1} e\left(\frac{ma+n\overline{a}}{c}\right), \tag{2.1}$$

where for v odd, we define

$$\varepsilon_v = \begin{cases} 1 & \text{if } v \equiv 1 \pmod 4, \\ i & \text{if } v \equiv 3 \pmod 4. \end{cases} \tag{2.2}$$

In addition, we define the function δ_{odd} on the integers by

$$\delta_{\text{odd}}(v) = \begin{cases} 1 & \text{if } v \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Using the theory of weakly holomorphic Poincaré series, Bruinier, the author, and Ono ([BJO], Theorem 3.7) prove the following theorem.

Theorem 2.1. *Let n be a positive integer with $n \equiv 0, 1 \pmod 4$. Then the Fourier coefficient $B(n, d)$ with positive index d, where $d \equiv 0, 3 \pmod 4$, is given by*

$$B(n,d) = 24\delta_{\square,n} H(d) - (1+i) \sum_{c \equiv 0\ (4)} \left(1 + \delta_{\text{odd}}\left(\frac{c}{4}\right)\right) \frac{K_{3/2}(-n, d; c)}{\sqrt{cn}} \sinh\left(\frac{4\pi}{c}\sqrt{nd}\right).$$

Here $\delta_{\square,n} = 1$ if n is a square, and $\delta_{\square,n} = 0$ otherwise.

Duke shows ([D], Proposition 4) that

Theorem 2.2. *For any positive integer m and discriminant $-d$,*

$$\mathrm{Tr}_m(d) = -24H(d)\sigma(m) + \sum_{c \equiv 0(4)} S_d(m,c) \sinh\left(\frac{4\pi m\sqrt{d}}{c}\right),$$

where

$$S_d(m,c) = \sum_{x^2 \equiv -d(c)} e\left(\frac{2mx}{c}\right).$$

We combine these two formulas to obtain a new proof of Theorem 1.3. We require the following lemma.

Lemma 2.3. *For $d \equiv 0, 3 \pmod 4$ and $k, n, c \geq 1$ with $nc \equiv 0 \pmod 4$ and $(c, k) = 1$, we have*

$$S_d(nk, nc) = \sum_{\substack{h|n \\ hc \equiv 0(4)}} (1+i)\left(1 + \delta_{\text{odd}}\left(\frac{hc}{4}\right)\right) \frac{1}{\sqrt{hc}} K_{3/2}\left(-h^2k^2, d; hc\right).$$

Remark. This lemma can be obtained by slightly modifying work of Kohnen [K]; we give instead a more direct and elementary proof depending only on classical facts about Gauss sums.

As a preliminary to our proof of Lemma 2.3, we define the Gauss sum

$$G(a,b;c) = \sum_{x(c)} e\left(\frac{ax^2+bx}{c}\right). \tag{2.3}$$

Note that $G(a, b; c)$ vanishes if $(a, c) > 1$ unless $(a, c)|b$, in which case

$$G(a,b;c) = (a,c)G\left(\frac{a}{(a,c)}, \frac{b}{(a,c)}; \frac{c}{(a,c)}\right). \tag{2.4}$$

For $(a,c)=1$, we can evaluate ([BEW], Theorems 1.5.1, 1.5.2 and 1.5.4)

$$G(a,0;c)=\begin{cases}0 & \text{if } 0<c\equiv 2 \pmod 4,\\ \varepsilon_c\sqrt{c}\left(\frac{a}{c}\right) & \text{if } c \text{ odd},\\ (1+i)\varepsilon_a^{-1}\sqrt{c}\left(\frac{c}{a}\right) & \text{if } a \text{ odd and } 4|c.\end{cases} \tag{2.5}$$

We will also need the following basic lemma.

Lemma 2.4. *For integers a,b,c with b odd and $(a,4c)=1$, we have $G(a,b;4c)=0$.*

Proof. Replacing x with $x+2c$ in the sum defining $G(a,b;4c)$ simply rearranges the sum. But this change of variables also introduces a factor $e(b/2)=-1$, so we must have $G(a,b;4c)=0$. ☐

Proof of Lemma 2.3. Assume that $d\equiv 0,3 \pmod 4$ and $k,n,c\geq 1$ with $nc\equiv 0 \pmod 4$ and $(c,k)=1$.

Multiplied by nc, the right side of the equation we are trying to establish is

$$\sum_{\substack{h|n\\ hc\equiv 0(4)}}(1+i)\frac{n\sqrt{c}}{\sqrt{h}}\left(1+\delta_{\text{odd}}\left(\frac{hc}{4}\right)\right)K_{3/2}\left(-h^2k^2,d;hc\right)$$

$$=\sum_{\substack{h|n\\ hc\equiv 0(4)}}(1+i)\frac{n\sqrt{c}}{\sqrt{h}}K_{3/2}\left(-h^2k^2,d;hc\right)+\sum_{\substack{h|n\\ \frac{hc}{4}\text{ odd}}}(1+i)\frac{n\sqrt{c}}{\sqrt{h}}K_{3/2}\left(-h^2k^2,d;hc\right).$$

Write this as S_0+S_1 for brevity. For $hc/4$ odd, we can rewrite the Kloosterman sum $K_{3/2}(-h^2k^2,d;hc)$ as ([I], Lemma 2)

$$K_{3/2}\left(-h^2k^2,d;hc\right)=\left(\cos\frac{\pi(d-h^2k^2)}{2}-\sin\frac{\pi(d-h^2k^2)}{2}\right)(1-i)\varepsilon_{\frac{hc}{4}}S\left(-\overline{4}h^2k^2,\overline{4}d;\frac{hc}{4}\right), \tag{2.6}$$

where $S(m,n;q)=\sum_{a(q)}\left(\frac{a}{q}\right)e\left(\frac{ma+n\overline{a}}{q}\right)$ is a Salié sum. Since $d\equiv 0,3 \pmod 4$, the cosine-sine term is 1 unless $d\equiv 3 \pmod 4$ and hk is odd, in which case it is -1. We thus have

$$S_1=\sum_{\substack{h|n\\ \frac{hc}{4}\text{ odd}}}\frac{2n\sqrt{c}}{\sqrt{h}}\left(e\left(\frac{d}{2}\right)\right)^{hk}\varepsilon_{\frac{hc}{4}}S\left(-\overline{4}h^2k^2,\overline{4}d;\frac{hc}{4}\right).$$

On the other hand, we have by definition that

$$ncS_d(nk,nc)=nc\sum_{x^2\equiv -d(nc)}e\left(\frac{2kx}{c}\right)=\sum_{x(nc)}e\left(\frac{2kx}{c}\right)\sum_{a(nc)}e\left(\frac{a(x^2+d)}{nc}\right)$$

$$=\sum_{a(nc)}e\left(\frac{ad}{nc}\right)G(a,2nk;nc). \tag{2.7}$$

The properties of G in (2.4) allow this to be rewritten as

$$\sum_{A|(2nk,nc)} \sum_{(a,nc)=A} e\left(\frac{ad}{nc}\right) A{\cdot}G\left(\frac{a}{A}, \frac{2nk}{A}; \frac{nc}{A}\right) = \sum_{A|(2nk,nc)} \sum_{(b,\frac{nc}{A})=1} e\left(\frac{bd}{nc/A}\right) A{\cdot}G\left(b, \frac{2nk}{A}; \frac{nc}{A}\right).$$

Since $(c,k)=1$, we have $(2nk,nc)=n$ if c is odd, and $2n$ if c is even.

We treat first the case in which c is odd. We have

$$\begin{aligned} ncS_d(nk,nc) &= \sum_{A|n} \sum_{(b,\frac{nc}{A})=1} e\left(\frac{bd}{nc/A}\right) A\cdot G\left(b, \frac{2nk}{A}; \frac{nc}{A}\right) \\ &= \sum_{h|n} \sum_{(b,hc)=1} e\left(\frac{bd}{hc}\right) \frac{n}{h} \cdot G(b, 2hk, hc). \end{aligned}$$

We complete the square in the definition of $G(b,2hk,hc)$ to see that this equals

$$\sum_{h|n} \sum_{(b,hc)=1} e\left(\frac{bd-\bar{b}h^2k^2}{hc}\right) \frac{n}{h} \cdot G(b,0,hc),$$

and apply (2.5) to get

$$\begin{aligned} ncS_d(nk,nc) = &\sum_{\substack{h|n \\ 4|hc}} \sum_{(b,hc)=1} e\left(\frac{bd-\bar{b}h^2k^2}{hc}\right) (1+i)\varepsilon_b^{-1} \frac{n\sqrt{c}}{\sqrt{h}} \left(\frac{hc}{b}\right) \\ &+ \sum_{\substack{h'|n \\ h'c\,\text{odd}}} \sum_{(b,h'c)=1} e\left(\frac{bd-\bar{b}h'^2k^2}{h'c}\right) \varepsilon_{h'c} \frac{n\sqrt{c}}{\sqrt{h'}} \left(\frac{b}{h'c}\right). \end{aligned}$$

Writing $4h'=h$ in the second term, we get

$$\sum_{\substack{h|n \\ 4|hc}} (1+i)\frac{n\sqrt{c}}{\sqrt{h}} K_{3/2}(-h^2k^2, d; hc) + \sum_{\substack{h|n \\ \frac{hc}{4}\text{ odd}}} \sum_{(b,\frac{hc}{4})=1} e\left(\frac{bd-\overline{16b}h^2k^2}{hc/4}\right) \varepsilon_{\frac{hc}{4}} \frac{2n\sqrt{c}}{\sqrt{h}} \left(\frac{b}{hc/4}\right).$$

Replacing b by $\bar{4}b$ shows that this precisely equals S_0+S_1.

We now assume that $2|c$, so that k is odd. We have

$$ncS_d(nk,nc) = \sum_{A|2n} \sum_{(b,\frac{nc}{A})=1} e\left(\frac{bd}{nc/A}\right) A\cdot G\left(b, \frac{2nk}{A}; \frac{nc}{A}\right).$$

If $A|nk$, then $A|n$ since we also have $A|nc$; this lets us split the sum as

$$\sum_{A|2n} = \sum_{A|n} + \sum_{\substack{A|2n \\ \frac{2nk}{A}\text{ odd}}}$$

and obtain

$$ncS_d(nk,nc) = S_0 + \sum_{\substack{h'|2n \\ h'k\,\mathrm{odd}}} \sum_{(b,\frac{h'c}{2})=1} e\left(\frac{bd}{h'c/2}\right) \frac{2n}{h'} \cdot G\left(b, h'k; \frac{h'c}{2}\right),$$

where S_0 is obtained from the first term as in the previous case.

In the case that $c/2$ is odd, we can complete the square and write

$$G\left(b, h'k; \frac{h'c}{2}\right) = e\left(\frac{-\overline{4b}h'^2k^2}{h'c/2}\right) G\left(b, 0; \frac{h'c}{2}\right).$$

We again apply (2.5) to get

$$ncS_d(nk,nc) = S_0 + \sum_{\substack{h'|2n \\ h'k\,\mathrm{odd}}} \sum_{(b,\frac{h'c}{2})=1} e\left(\frac{bd - \overline{4b}h'^2k^2}{h'c/2}\right) \varepsilon_{\frac{h'c}{2}} \frac{n\sqrt{2c}}{\sqrt{h'}} \left(\frac{b}{h'c/2}\right).$$

Writing $2h' = h$ and replacing b by $\overline{4}b$ shows that this again equals $S_0 + S_1$.

If $8|c$, then $S_1 = 0$, and Lemma 2.4 shows that $ncS_d(nk,nc) = S_0$.

The remaining case is $c = 4w$, for w odd. In this case, we have

$$S_1 = \sum_{\substack{h|n \\ h\,\mathrm{odd}}} \frac{4n\sqrt{w}}{\sqrt{h}} e\left(\frac{d}{2}\right) \varepsilon_{hw} \sum_{(b,hw)=1} \left(\frac{b}{hw}\right) e\left(\frac{\overline{4}bd - \overline{4b}h^2k^2}{hw}\right),$$

$$ncS_d(nk,nc) = S_0 + \sum_{\substack{h|n \\ h\,\mathrm{odd}}} \sum_{(b,2hw)=1} e\left(\frac{bd}{2hw}\right) \frac{2n}{h} \sum_{x(2hw)} e\left(\frac{bx^2 + hkx}{2hw}\right).$$

But each b coprime to $2hw$ is equal (mod $2hw$) to $hw - 2z$ for some z coprime to hw, so we have

$$ncS_d(nk,nc) = S_0 + \sum_{\substack{h|n \\ h\,\mathrm{odd}}} \sum_{(b,hw)=1} e\left(\frac{(hw-2b)d}{2hw}\right) \frac{2n}{h} \sum_{x(2hw)} e\left(\frac{(hw-2b)x^2 + hkx}{2hw}\right).$$

Simplifying, and noting that $e\left(\frac{x^2}{2}\right) = e\left(\frac{x}{2}\right)$ for all integers x, we have

$$\begin{aligned} ncS_d(nk,nc) &= S_0 + \sum_{\substack{h|n \\ h\,\mathrm{odd}}} \sum_{(b,hw)=1} e\left(\frac{d}{2}\right) e\left(\frac{-bd}{hw}\right) \frac{2n}{h} G(-2b, hk+hw; 2hw) \\ &= S_0 + \sum_{\substack{h|n \\ h\,\mathrm{odd}}} \sum_{(b,hw)=1} e\left(\frac{d}{2}\right) e\left(\frac{-bd}{hw}\right) \frac{4n}{h} G(-b, \overline{2}hk; hw). \end{aligned}$$

Since hw is odd, we complete the square to get

$$G(-b, \overline{2}hk; hw) = e\left(\frac{\overline{16b}h^2k^2}{hw}\right) G(-b, 0; hw) = e\left(\frac{\overline{16b}h^2k^2}{hw}\right) \varepsilon_{hw}\sqrt{hw}\left(\frac{-b}{hw}\right).$$

Replacing b by $-\overline{4}b$ gives $ncS_d(nk, nc) = S_0 + S_1$, proving the lemma. □

Proof of Theorem 1.3. From Theorem 1.2, we know that $-B_m(1, d) = -\sum_{r|m} rB(r^2, d)$. Applying Theorem 2.1 with $n = r^2$, we find that

$$-B_m(1, d) = -\,24H(d)\sigma(m)+ \sum_{r|m}\sum_{n|r}\sum_{\substack{\frac{rc}{n}\equiv 0(4)\\ (c,n)=1}} (1+i)\left(1+\delta_{\text{odd}}\left(\frac{rc}{4n}\right)\right)\frac{K_{3/2}(-r^2, d; \frac{rc}{n})}{\sqrt{\frac{rc}{n}}}\sinh\left(\frac{4\pi n\sqrt{d}}{c}\right).$$

Writing $r = hn$, we get $\sum_{r|m}\sum_{n|r} = \sum_{n|m}\sum_{h|\frac{m}{n}}$. Replace n by m/n inside the sum on n, and write $\sum_{h|n}\sum_{hc\equiv 0(4),(c,\frac{m}{n})=1} = \sum_{nc\equiv 0(4),(c,\frac{m}{n})=1}\sum_{h|n,hc\equiv 0(4)}$ to get

$$-B_m(1, d) = -24H(d)\sigma(m)+ \sum_{n|m}\sum_{\substack{nc\equiv 0(4)\\ (c,\frac{m}{n})=1}}\sum_{\substack{h|n\\ hc\equiv 0(4)}} (1+i)\left(1+\delta_{\text{odd}}\left(\frac{hc}{4}\right)\right)\frac{K_{3/2}(\frac{-h^2m^2}{n^2}, d; hc)}{\sqrt{hc}}\sinh\left(\frac{4\pi m\sqrt{d}}{nc}\right).$$

We rewrite Theorem 2.2 as

$$\operatorname{Tr}_m(d) = -24H(d)\sigma(m) + \sum_{n|m}\sum_{\substack{nc\equiv 0(4)\\ (c,\frac{m}{n})=1}} S_d(m, nc)\sinh\left(\frac{4\pi m\sqrt{d}}{nc}\right). \tag{2.8}$$

Letting $m = nk$ in Lemma 2.3 shows that the coefficients of the hyperbolic sine terms in the two expressions are equal, and we have $\operatorname{Tr}_m(d) = -B_m(1, d)$, proving the theorem. □

3. New expressions for twisted traces

Going the other direction, we now derive an expression for twisted traces that is analogous to the expression in Theorem 2.2 for the standard traces. Specifically, we prove Theorem 1.5, which asserts that for discriminants D and $-d$, with $D > 1$ a fundamental discriminant, and an integer $m \geq 1$,

$$\operatorname{Tr}_m(D, d) = \sum_{c\equiv 0(4)}\left(\sum_{\substack{x(c)\\ x^2\equiv -Dd(c)}} \chi_D\left(\frac{c}{4}, x, \frac{x^2+Dd}{c}\right) e\left(\frac{2mx}{c}\right)\right)\sinh\left(\frac{4\pi m\sqrt{dD}}{c}\right).$$

Proof of Theorem 1.5. We know from Zagier's work that

$$\mathrm{Tr}_m(D,d) = A_m(D,d)\sqrt{D}. \tag{3.1}$$

We combine this with the generalization, for D fundamental, of the second part of Theorem 1.2,

$$A_m(D,d) = \sum_{n|m} n\left(\frac{D}{\frac{m}{n}}\right) A(n^2D,d),$$

and use the fact that $A_m(D,d) = -B_m(D,d)$ to see that

$$\mathrm{Tr}_m(D,d) = -\sum_{n|m} \sqrt{D}n\left(\frac{D}{\frac{m}{n}}\right) B(n^2D,d).$$

We now apply Theorem 2.1 to write $B(n^2D,d)$ as an infinite sum, and obtain

$$\mathrm{Tr}_m(D,d) = \sum_{n|m}\sum_{c\equiv 0(4)} (1+i)\left(\frac{D}{\frac{m}{n}}\right)\left(1+\delta_{\mathrm{odd}}\left(\frac{c}{4}\right)\right)\frac{1}{\sqrt{c}}K_{3/2}(-n^2D,d;c)\sinh\left(\frac{4\pi n\sqrt{dD}}{c}\right).$$

Rewriting $\sum_{c\equiv 0(4)}$ as $\sum_{r|n}\sum_{rc\equiv 0(4),(c,n/r)=1}$ and changing the order of summation as in the previous section, this becomes

$$\sum_{c\equiv 0(4)}\sum_{k|(m,\frac{c}{4})} (1+i)\left(\frac{D}{k}\right)\left(1+\delta_{\mathrm{odd}}\left(\frac{c}{4k}\right)\right)\sqrt{\frac{k}{c}}K_{3/2}\left(\frac{-m^2D}{k^2},d;\frac{c}{k}\right)\sinh\left(\frac{4\pi m\sqrt{Dd}}{c}\right).$$

We use the following identity, proved by Kohnen ([K], Proposition 5): For integers $a,m,d\geq 1$ and fundamental discriminants D satisfying $D,(-1)^\lambda d\equiv 0,1 \pmod 4$, we have

$$\sum_{\substack{b(4a)\\ b^2\equiv(-1)^\lambda Dd(4a)}} \chi_D\left(a,b,\frac{b^2-(-1)^\lambda Dd}{4a}\right)e\left(\frac{mb}{2a}\right) = \tag{3.2}$$
$$\sum_{k|(m,a)}(1-(-1)^\lambda i)\left(\frac{D}{k}\right)\left(1+\delta_{\mathrm{odd}}\left(\frac{a}{k}\right)\right)\frac{\sqrt{k}}{2\sqrt{a}}K_{\lambda+1/2}\left(\frac{(-1)^\lambda m^2D}{k^2},d;\frac{4a}{k}\right).$$

This identity is stated only for $(-1)^\lambda dD>0$ in Kohnen's paper, but the proof is virtually unchanged for the more general case.

Applying the identity with $\lambda=1$ and $a=c/4$, Theorem 1.5 follows immediately. □

Example. Let $D=5, d=3, m=1$. There are two forms of discriminant -15; they are $Q_1=[1,1,4]$ and $Q_2=[2,1,2]$, with $\chi_5(Q_1)=1$ and $\chi_5(Q_2)=-1$. We have

$$j(\alpha_{Q_1}) = \frac{-191025-85995\sqrt{5}}{2},$$
$$j(\alpha_{Q_2}) = \frac{-191025+85995\sqrt{5}}{2},$$

and therefore $\mathrm{Tr}(D,d) = j(\alpha_{Q_1}) - j(\alpha_{Q_2}) = -85995\sqrt{5}$.

If we compute the first 10 terms of the sum in Theorem 1.5 and divide by $\sqrt{5}$, we get $-85996.573\cdots$. The first 100 terms give $-85995.909\cdots$, and the first 1000 terms give $-85994.9562\cdots$.

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