Notes for PDE seminar

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Contents

Chapter 1

Convex analysis

In this chapter some of the basic theorems on convex functions are presented. Throughout this chapter X will be a real locally convex topological vector space unless specified otherwise. Thus X is a vector space and Ψ is a collection of functions defined on X such that if $\rho \in \Psi$,

$$\rho(x+y) \leq \rho(x) + \rho(y),$$

$$\rho(ax) = |a|\, \rho(x) \text{ if } a \in \mathbb{R},$$

$$\rho(x) \geq 0,$$

These functions are called seminorms because we cannot conclude $x = 0$ when $\rho(x) = 0$. We define a basis, $\mathcal{B}$, for a topology on X as follows. For A a finite subset of Ψ and $r > 0$,

$$B_A(x,r) \equiv \{y \in X : \rho(x-y) < r \text{ for all } \rho \in A\}.$$

Then $\mathcal{B} \equiv \{B_A(x,r) : x \in X, r > 0, \text{ and } A \subseteq \Psi,\ A \text{ finite}\}$.

Theorem 1.1 *$\mathcal{B}$ is the basis for a topology on X.*

Let τ be the topology consisting of unions of all subsets of $\mathcal{B}$. We call (X,τ) a locally convex topological vector space.

Theorem 1.2 *The vector space operations of addition and scalar multiplication are continuous. More precisely,*

$$+ : X \times X \to X,\ \cdot : \mathbb{R}\times X \to X$$

are continuous.

Theorem 1.3 *Let x be given and let $f_x(y) = x + y$. Then f_x is $1-1$, onto, and continuous. If $\alpha \neq 0$ and $g_\alpha(x) = \alpha x$, then g_α is also $1-1$ onto and continuous.*

Definition 1.4 *Define, for A a finite subset of Ψ,*

$$\rho_A(x) = \max\{\rho(x) : \rho \in A\}.$$

Theorem 1.5 *The following are equivalent for f, a linear function mapping X to $\mathbb{R}$.*

$$f \text{ is continuous at } 0. \tag{(1.1)}$$

For some $A \subseteq \Psi$, A finite,

$$|f(x)| \leq C\rho_A(x) \tag{(1.2)}$$

for all $x \in X$ where the constant may depend on A but is independent of x.

$$f \text{ is continuous at } x \tag{(1.3)}$$

for all x.

Proof: Clearly (1.3) implies (1.1). Suppose (1.1). Then

$$0 = f(0) \in B(0,1) \subseteq \mathbb{R}.$$

Since f is continuous at 0, $0 \in f^{-1}(B(0,1))$ and there exists an open set $V \in \tau$ such that

$$0 \in V \subseteq f^{-1}(B(0,1)).$$

Then $0 \in B_A(0,r) \subseteq V$ for some r and some $A \subseteq \Psi$, A finite. Hence

$$|f(y)| < 1 \text{ if } \rho_A(y) < r.$$

Since f is linear

$$|f(x)| \leq \frac{2}{r}\rho_A(x).$$

To see this, note that if $x \neq 0$, then

$$\frac{rx}{2\rho_A(x)} \in B_A(0,r)$$

and so

$$\frac{|f(rx)|}{2\rho_A(x)} \leq 1$$

which shows that (1.1) implies (1.2).

Now suppose (1.2) and suppose $f(x) \in V$, an open set in $\mathbb{R}$. Then

$$f(x) \in B(f(x), r) \subseteq V$$

for some $r > 0$. Suppose $\rho_A(x-y) < r(C_A+1)^{-1}$. Then

$$|f(x) - f(y)| = |f(x-y)| \leq C_A\rho_A(y-x) < r.$$

Hence

$$f\left(B_A\left(x, r(C_A+1)^{-1}\right)\right) \subseteq B(f(x), r) \subseteq V.$$

Thus f is continuous at x. This proves the theorem.

1.1 Continuity properties of convex functions

Definition 1.6 *We say a function $\phi : X \to (-\infty, \infty]$ is convex if whenever $\lambda \in [0,1]$,*

$$\phi(\lambda x + (1-\lambda) y) \leq \lambda \phi(x) + (1-\lambda) \phi(y).$$

We also define

$$epi(\phi) \equiv \{(x,a) : a \geq \phi(x)\}$$

and say ϕ is lower semi-continuous, l.s.c., if $epi(\phi)$ is a closed subset of $X \times (-\infty, \infty]$. If in addition to this, $\phi(x) < \infty$ for some $x \in X$, we say ϕ is proper. The effective domain of ϕ is defined as

$$dom(\phi) \equiv \{x \in X \text{ such that } \phi(x) < \infty\}.$$

A set, K, is convex if whenever $x, y \in K$ it follows that $x\lambda + (1-\lambda) y \in K$ for all $\lambda \in [0,1]$. We refer to the expression, $x\lambda + (1-\lambda) y$ where $\lambda \in [0,1]$ as a convex combination of x and y.

It will always be the case that ϕ is proper.

Lemma 1.7 *Suppose ϕ is convex, and proper. Also suppose*

$$\phi(y) < a$$

for all $y \in U$ where U is some open set containing x. Then ϕ is continuous at x. In fact, if $B_A(x,r) \subseteq U$ where A is a finite subset of Ψ, the set of seminorms, then for $\varepsilon \in (0, 1/2)$, and $y \in \overline{B_A(x, \varepsilon r)}$,

$$2\varepsilon\phi(x) - 2\varepsilon a \leq \phi(y) - \phi(x) \leq 2\varepsilon a - 2\varepsilon\phi(x). \tag{(1.4)}$$

Proof: Let the open set be U and let $x \in B_A(x,r) \subseteq U$ where A is a finite subset of Ψ. Let $\varepsilon \in (0, 1/2)$. If $y \in \overline{B_A(x, \varepsilon r)}$, then

$$\frac{y-x}{2\varepsilon} + x \in B_A(x,r)$$

and

$$y = 2\varepsilon\left(\frac{y-x}{2\varepsilon} + x\right) + \overbrace{(1-2\varepsilon)}^{\varepsilon<1/2} x.$$

Therefore, if $y \in \overline{B_A(x, \varepsilon r)}$,

$$\begin{aligned} \phi(y) &\leq \overbrace{2\varepsilon\phi\left(\frac{y-x}{2\varepsilon} + x\right)}^{<a} + (1-2\varepsilon)\phi(x) \\ &\leq 2\varepsilon a + (1-2\varepsilon)\phi(x). \end{aligned} \tag{(1.5)}$$

Now it is also true that

$$y = (1+2\varepsilon) x - 2\varepsilon\left(x + \frac{x-y}{2\varepsilon}\right)$$

and so

$$x = \frac{y}{1+2\varepsilon} + \frac{2\varepsilon}{1+2\varepsilon}\left(x + \frac{x-y}{2\varepsilon}\right)$$

and, therefore,

$$\begin{aligned}\phi(x) &\leq \frac{\phi(y)}{1+2\varepsilon}+\frac{2\varepsilon}{1+2\varepsilon}\overbrace{\phi\left(x+\frac{x-y}{2\varepsilon}\right)}^{<a} \\ &\leq \frac{\phi(y)}{1+2\varepsilon}+\frac{2\varepsilon a}{1+2\varepsilon}.\end{aligned} \tag{(1.6)}$$

By (1.5) and (1.6),

$$(1+2\varepsilon)\phi(x)-2a\varepsilon \leq \phi(y) \leq (1-2\varepsilon)\phi(x)+2\varepsilon a$$

and so

$$2\varepsilon\phi(x)-2a\varepsilon<\phi(y)-\phi(x)\leq 2a\varepsilon-2\varepsilon\phi(x).$$

Since ε is arbitrary, this proves (1.4) and the continuity of ϕ at x.

Corollary 1.8 *If ϕ is convex and proper, defined on $\mathbb{R}^n$, then ϕ is continuous on the interior of dom (ϕ).*

Proof: Let $\mathbf{x} \in$ interior of *dom* (ϕ). Then $\mathbf{x} \in B(\mathbf{x},r) \in$ *dom* (ϕ) for some $r>0$. Define $\mathbf{v}_i \equiv \delta\mathbf{e}_i$ for $i=1,\cdots,n$, and $\mathbf{v}_0 \equiv \mathbf{0}$. Let

$$S \equiv \left\{\sum_{i=0}^{n}\lambda_i\mathbf{v}_i : \sum_{i=0}^{n}\lambda_i=1 \text{ and } \lambda_i>0 \text{ for all } i\right\}.$$

Then S is an open set of diameter less than $\sqrt{2}\delta$ which contains the point

$$\sum_{i=0}^{n}\frac{1}{1+n}\mathbf{v}_i \equiv \mathbf{p}.$$

Let $U \equiv \mathbf{x}-\mathbf{p}+S$. Then U is an open set containing $\mathbf{x}$ which has diameter less than $\sqrt{2}\delta$ and so $\mathbf{x} \in U \subseteq B(\mathbf{x},r)$ whenever δ is small enough. Also $\mathbf{x}-\mathbf{p}+\mathbf{v}_i \in B(\mathbf{x},r)$ for each i whenever δ is small enough. (Both $\mathbf{v}_i$ and $\mathbf{p}$ are multiples of δ.) Let δ be sufficiently small. Then we may conclude

$$\phi\left(\mathbf{x}-\mathbf{p}+\sum_{i=0}^{n}\lambda_i\mathbf{v}_i\right) \leq \sum_{i=0}^{n}\lambda_i\phi(\mathbf{x}-\mathbf{p}+\mathbf{v}_i) \leq \max\{\phi(\mathbf{x}-\mathbf{p}+\mathbf{v}_i)\}_{i=0}^{n}.$$

Therefore, ϕ is continuous by Lemma 1.7.

More can be said about the continuity of ϕ near a point where ϕ is continuous. In fact the convex function is Lipschitz continuous near such a point of continuity.

Theorem 1.9 *Let X be a normed linear space and $\phi : X \to (-\infty,\infty]$ is convex and proper. If $x \in$ dom (ϕ) and ϕ is continuous at x, then ϕ is Lipschitz continuous on some open set containing x.*

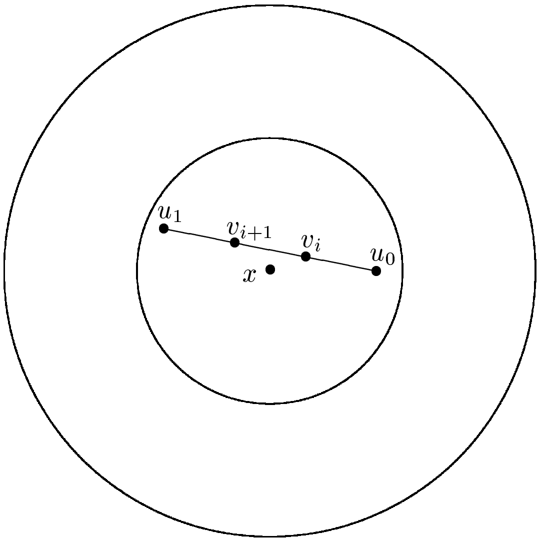
Proof: By continuity of ϕ, there exists $r_0>0$ such that for $y \in B(x,r_0)$,

$$|\phi(x)-\phi(y)|<1.$$

Consider u_0 and $u_1 \in B(x,r_0/2)$ and let $v_1,\cdots,v_m$ be points on the line segment joining u_0 and u_1 such that

$$||v_{i+1}-v_i||<\frac{r_0}{4},\ u_0=v_1,\ u_1=v_m,\ \text{and} \tag{(1.7)}$$

$$||u_0 - u_1|| = \sum_{i=1}^{m-1} ||v_{i+1} - v_i||.$$



For v_i, consider $v \to \phi(v) - \phi(v_i)$, a convex function. We have also for all $v \in \overline{B\left(v_i, \frac{r_0}{2}\right)} \subseteq B(x, r_0)$,

$$\begin{aligned} |\phi(v) - \phi(v_i)| &\leq |\phi(v) - \phi(x)| + |\phi(x) - \phi(v_i)| \\ &\leq 1 + 1 = 2. \end{aligned}$$

Therefore, by Lemma 1.7, if $\varepsilon < 1/2$, and $v \in \overline{B\left(v_i, \varepsilon\frac{r_0}{2}\right)}$,

$$2\varepsilon(0) - 2\varepsilon(2) \leq \phi(v) - \phi(v_i) \leq 2\varepsilon(2) - 2\varepsilon(0)$$

and so for such v,

$$-4\varepsilon \leq \phi(v) - \phi(v_i) \leq 4\varepsilon \tag{(1.8)}$$

Letting $\varepsilon = \frac{2}{r_0}||v_{i+1} - v_i||$, it follows from (1.7) that $\varepsilon < 1/2$ and

$$v_{i+1} \in \overline{B\left(v_i, \varepsilon\frac{r_0}{2}\right)} \subseteq B(x, r_0).$$

By Lemma 1.7, and (1.8),

$$|\phi(v_{i+1}) - \phi(v_i)| \leq 4\varepsilon = \frac{8}{r_0}||v_{i+1} - v_i||.$$

Thus,

$$\begin{aligned} |\phi(u_0) - \phi(u_1)| &\leq \sum_{i=1}^{m-1} |\phi(v_{i+1}) - \phi(v_i)| \leq \\ \frac{8}{r_0}\sum_{i=1}^{m-1} ||v_{i+1} - v_i|| &= \frac{8}{r_0}||u_0 - u_1||. \end{aligned}$$

This shows ϕ is Lipschitz on $B\left(x, \frac{r_0}{2}\right)$ as claimed.

Corollary 1.10 *Let $\phi : \mathbb{R}^n \to (-\infty, \infty]$ be convex and proper. Then ϕ is locally Lipschitz on dom (ϕ).*

The next theorem extends these results slightly. It turns out that on a Banach space, as opposed to a normed linear space, a convex function is locally Lipschitz continuous on the interior of its domain. Thus on a Banach space one gets the continuity at a point in the interior of the domain for free. We don't need to assume it as we did in the above theorem.

Theorem 1.11 *Suppose X is a Banach space and $\phi : X \to (-\infty, \infty]$ is convex and proper. Then ϕ is locally Lipschitz on the interior of dom (ϕ).*

Proof: Let $x_0 \in$ interior of *dom* (ϕ) and define $\psi(x) \equiv \phi(x_0 + x)$. Let $a > \psi(0) = \phi(x_0)$. Thus $0 \in$ interior of *dom* (ψ). Define

$$E \equiv \{x \in X \ : \psi(x) \leq a\}.$$

Then E is convex and contains 0.

Claim: $X = \cup_{n=1}^{\infty} n(E \cap (-E))$.

Proof of the claim: Let $y \in X$ be arbitrary and let $g(t) \equiv \psi(ty)$ for $t \in \mathbb{R}$. Thus $0 \in$ interior of *dom* (g) and so g is continuous at 0 by Corollary 1.8 applied to $\mathbb{R}$. Therefore, for $|t|$ small enough, $ty \in E$ and $-ty \in E$. Thus $ty \in E \cap (-E)$ and this proves the claim.

It follows from the completeness of X and the Baire category theorem that the interior of $n(E \cap (-E))$ is nonempty for some n. Therefore, the interior of $E \cap (-E)$ is also nonempty and for some p and $r > 0$, $B(p, r) \subseteq E \cap (-E)$. Then $B(-p, r) \subseteq E \cap (-E)$ also. Therefore, if $z \in B(0, r)$, we have $z - p \in B(-p, r)$ and $z + p \in B(p, r)$ and so

$$z = \frac{1}{2}((z - p) + (z + p))$$

is a convex combination of points in $E \cap (-E)$ which implies $z \in E \cap (-E)$. Thus

$$B(0, r) \subseteq E \cap (-E).$$

Therefore $\psi(x) = \phi(x + x_0) \leq a$ for all $x \in B(0, r)$ and so $\phi(y) \leq a$ for all $y \in B(x, r)$. By Lemma 1.7, this proves ϕ is continuous at x. Theorem 1.9 implies ϕ is Lipschitz on some open set containing x.

1.2 Separation theorems in locally convex topological vector spaces

A set, K, is said to be convex if whenever $x, y \in K$,

$$\lambda x + (1 - \lambda) y \in K$$

for all $\lambda \in [0, 1]$. Let U be an open convex set containing 0 and define

$$m(x) \equiv \inf\{t > 0 : x/t \in U\}.$$

This is called a Minkowski functional.

Proposition 1.12 *m is defined on X and satisfies*

$$m(x + y) \leq m(x) + m(y) \qquad ((1.9))$$

$$m(\lambda x) = \lambda m(x) \quad \text{if } \lambda > 0. \qquad ((1.10))$$

Thus, m is a gauge function on X.

Proof: Let $x \in X$ be arbitrary. There exists $A \subseteq \Psi$ such that

$$0 \in B_A(0,r) \subseteq U.$$

Then

$$\frac{rx}{2\rho_A(x)} \in B_A(0,r) \subseteq U$$

which implies

$$\frac{2\rho_A(x)}{r} \geq m(x). \tag{(1.11)}$$

Thus $m(x)$ is defined on X.

Let $x/t \in U$, $y/s \in U$. Then since U is convex,

$$\frac{x+y}{t+s} = \left(\frac{t}{t+s}\right)\left(\frac{x}{t}\right) + \left(\frac{s}{t+s}\right)\left(\frac{y}{s}\right) \in U.$$

It follows that

$$m(x+y) \leq t+s.$$

Choosing s,t such that $t-\varepsilon < m(x)$ and $s-\varepsilon < m(y)$,

$$m(x+y) \leq m(x) + m(y) + 2\varepsilon.$$

Since ε is arbitrary, this shows (1.9). It remains to show (1.10). Let $x/t \in U$. Then if $\lambda > 0$,

$$\frac{\lambda x}{\lambda t} \in U$$

and so $m(\lambda x) \leq \lambda t$. Thus $m(\lambda x) \leq \lambda m(x)$ for all $\lambda > 0$. Hence

$$m(x) = m\left(\lambda^{-1}\lambda x\right) \leq \lambda^{-1} m(\lambda x) \leq \lambda^{-1}\lambda m(x) = m(x)$$

and so

$$\lambda m(x) = m(\lambda x).$$

This proves the proposition.

Lemma 1.13 *Let U be an open convex set containing 0 and let $q \notin U$. Then there exists $f \in X'$ such that*

$$\operatorname{Re} f(q) > \operatorname{Re} f(x)$$

for all $x \in U$.

Proof: Let m be the Minkowski functional just defined and let

$$F(cq) = cm(q)$$

for $c \in \mathbb{R}$. If $c > 0$ then

$$F(cq) = m(cq)$$

while if $c \leq 0$,

$$F(cq) = cm(q) \leq 0 \leq m(cq).$$

By the Hahn Banach theorem, F has an extension, g, defined on all of X satisfying

$$g(x+y) = g(x) + g(y),\ g(cx) = cg(x)$$

for all $c \in \mathbb{R}$, and

$$g(x) \leq m(x).$$

Thus, $g(-x) \leq m(-x)$ and so

$$-m(-x) \leq g(x) \leq m(x).$$

It follows as in (1.11) that for some $A \subseteq \Psi$, A finite, and $r > 0$,

$$|g(x)| \leq m(x) + m(-x)$$

$$\leq \frac{2}{r}\rho_A(x) + \frac{2}{r}\rho_A(-x) = \frac{4}{r}\rho_A(x)$$

because

$$\rho_A(-x) = |-1|\,\rho_A(x) = \rho_A(x).$$

Hence g is continuous by Theorem 1.5. Now define

$$f(x) \equiv g(x) - ig(ix).$$

Thus f is linear and continuous so $f \in X'$ and $\operatorname{Re} f(x) = g(x)$. But for $x \in U$, Theorem 1.2 implies that $x/t \in U$ for some $t < 1$ and so $m(x) < 1$. Since U is convex and $0 \in U$, it follows $q/t \notin U$ if $t < 1$ because if it were,

$$q = t\left(\frac{q}{t}\right) + (1-t)\,0 \in U.$$

Therefore, $m(q) \geq 1$ and for $x \in U$,

$$\operatorname{Re} f(x) = g(x) \leq m(x) < 1 \leq m(q) = g(q) = \operatorname{Re} f(q)$$

and this proves the lemma.

Theorem 1.14 *Let K be closed and convex and let $p \notin K$. Then there exists a real number, c, and $f \in X'$ such that*

$$\operatorname{Re} f(p) > c > \operatorname{Re} f(k)$$

for all $k \in K$.

Proof: Since K is closed, and $p \notin K$, there exists a finite subset of Ψ, A, and a positive $r > 0$ such that

$$K \cap B_A(p, 2r) = \emptyset.$$

Pick $k_0 \in K$ and let

$$U = K + B_A(0, r) - k_0,\ q = p - k_0.$$

It follows that U is an open convex set containing 0 and $q \notin U$. Therefore, by Lemma 1.13, there exists $f \in X'$ such that

$$\operatorname{Re} f(p - k_0) = \operatorname{Re} f(q) > \operatorname{Re} f(k + e - k_0) \tag{(1.12)}$$

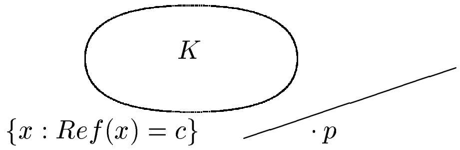
for all $k \in K$ and $e \in B_A(0, r)$. If $\operatorname{Re} f(e) = 0$ for all $e \in B_A(0, r)$, then $\operatorname{Re} f = 0$ and (1.12) could not hold. Therefore, $\operatorname{Re} f(e) > 0$ for some $e \in B_A(0, r)$ and so,

$$\operatorname{Re} f(p) > \operatorname{Re} f(k) + \operatorname{Re} f(e)$$

for all $k \in K$. Let $c_1 \equiv \sup \{\operatorname{Re} f(k) : k \in K\}$. Then for all $k \in K$,

$$\operatorname{Re} f(p) \geq c_1 + \operatorname{Re} f(e) > c_1 + \frac{\operatorname{Re} f(e)}{2} > \operatorname{Re} f(k).$$

Let $c = c_1 + \frac{\operatorname{Re} f(e)}{2}$. This proves the theorem.



There are other separation theorems which can be proved in a similar way. The next theorem considers the separation of an open convex set from a convex set.

Theorem 1.15 *Let A and B be disjoint, convex and nonempty sets with B open. Then there exists $f \in X'$ such that*

$$\operatorname{Re} f(a) < \operatorname{Re} f(b)$$

for all $a \in A$ and $b \in B$.

Proof: Let $b_0 \in B, a_0 \in A$. Then the set

$$B - A + a_0 - b_0$$

is open, convex, contains 0, and does not contain $a_0 - b_0$. By Lemma 1.13 there exists $f \in X'$ such that

$$\operatorname{Re} f(a_0 - b_0) > \operatorname{Re} f(b - a + a_0 - b_0)$$

for all $a \in A$ and $b \in B$. Therefore, for all $a \in A, b \in B$,

$$\operatorname{Re} f(b) > \operatorname{Re} f(a).$$

Before giving another separation theorem, we need a lemma.

Lemma 1.16 *If B is convex, then $int(B) \equiv$ union of all open sets contained in B is convex. Also, if $int(B) \neq \emptyset$, then $B \subseteq \overline{int(B)}$.*

Proof: Suppose $x, y \in int(B)$. Then there exists $r > 0$ and a finite set $A \subseteq \Psi$ such that

$$B_A(x, r), B_A(y, r) \subseteq B.$$

Let

$$V \equiv \cup_{\lambda \in [0,1]} \lambda B_A(x, r) + (1 - \lambda) B_A(y, r).$$

Then V is open, $V \subseteq B$, and if $\lambda \in [0,1]$, then

$$\lambda x + (1-\lambda)\, y \in V \subseteq B.$$

Therefore, $int\,(B)$ is convex as claimed.

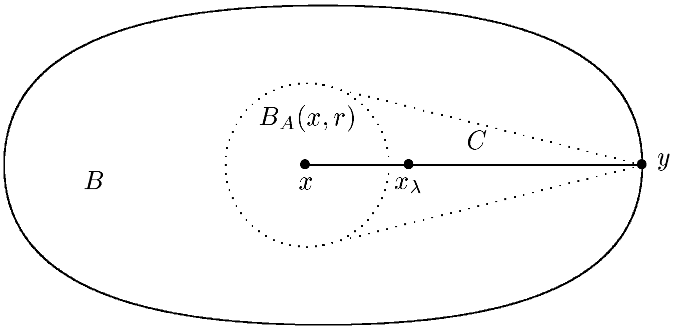
Now let $y \in B$ and $x \in int\,(B)$. Let

$$x \in B_A\,(x,r) \subseteq int\,(B)$$

and let $x_\lambda \equiv (1-\lambda)\, x + \lambda y$. Define the open cone,

$$C \equiv \cup_{\lambda \in [0,1]} B_A(x_\lambda, (1-\lambda)\, r).$$

Thus C is represented in the following picture.



We claim $C \subseteq B$ as suggested in the picture. To see this, let

$$z \in B_A\,(x_\lambda, (1-\lambda)\, r)\,, \lambda \in (0,1).$$

Then

$$\rho_A\,(z - x_\lambda) < (1-\lambda)\, r$$

and so

$$\rho_A \left(\frac{z}{1-\lambda} - x - \frac{\lambda y}{1-\lambda} \right) < r.$$

Therefore,

$$\frac{z}{1-\lambda} - \frac{\lambda y}{1-\lambda} \in B_A\,(x,r) \subseteq B.$$

It follows

$$(1-\lambda) \left(\frac{z}{1-\lambda} - \frac{\lambda y}{1-\lambda} \right) + \lambda y = z \in B$$

and so $C \subseteq B$ as claimed. Now this shows $x_\lambda \in int\,(B)$ and $\lim_{\lambda \to 1} x_\lambda = y$. Thus, $y \in \overline{int\,(B)}$ and this proves the lemma.

Corollary 1.17 *Let A, B be convex, nonempty sets. Suppose $int\,(B) \neq \emptyset$ and $A \cap int\,(B) = \emptyset$. Then there exists $f \in X'$, $f \neq 0$, such that for all $a \in A$ and $b \in B$,*

$$\operatorname{Re} f\,(b) \geq \operatorname{Re} f\,(a).$$

Proof: By Theorem 1.15, there exists $f \in X'$ such that for all $b \in int\,(B)$, and $a \in A$,

$$\operatorname{Re} f\,(b) > \operatorname{Re} f\,(a)\,.$$

Thus, in particular, $f \neq 0$. By Lemma 1.16, if $b \in B$ and $a \in A$,

$$\operatorname{Re} f\,(b) \geq \operatorname{Re} f\,(a)\,.$$

This proves the theorem.

Lemma 1.18 *If X is a topological Hausdorff space then compact implies closed.*

Proof: Let K be compact and suppose K^C is not open. Then there exists $p \in K^C$ such that

$$V_p \cap K \neq \emptyset$$

for all open sets V_p containing p. Let

$$\mathcal{C} = \{\left(\overline{V}_p\right)^C : V_p \text{ is an open set containing } p\}.$$

Then $\mathcal{C}$ is an open cover of K because if $q \in K$, there exist disjoint open sets V_p and V_q containing p and q respectively. Thus $q \in \left(\overline{V}_p\right)^C$. This is an example of an open cover of K which has no finite subcover, contradicting the assumption that K is compact. This proves the lemma.

Lemma 1.19 *If X is a locally convex topological vector space, and if every point is a closed set, then the seminorms and X' separate the points. By this we mean that if $x \neq y$, then for some $\rho \in \Psi$,*

$$\rho\,(x-y) \neq 0$$

and for some $f \in X'$,

$$f\,(x) \neq f\,(y)\,.$$

In this case, X is a Hausdorff space.

Proof: Let $x \neq y$. Then by Theorem 1.14, there exists $f \in X'$ such that $f\,(x) \neq f(y)$. Thus X' separates the points. Since $f \in X'$, Theorem 1.5 implies

$$|f\,(z)| \leq C\rho_A\,(z)$$

for some A a finite subset of Ψ. Thus

$$0 < |f\,(x-y)| \leq C\rho_A\,(x-y)$$

and so $\rho\,(x-y) \neq 0$ for some $\rho \in A \subseteq \Psi$. Now to show X is Hausdorff, let

$$0 < r < \rho\,(x-y)\,2^{-1}.$$

Then the two disjoint open sets containing x and y respectively are

$$B_\rho\,(x,r) \text{ and } B_\rho\,(y,r).$$

This proves the lemma.

1.3 Separation properties for convex functions

The separation properties in locally convex topological vector space imply corresponding separation properties for the epigraph of a convex function. These separation properties are essential in much of what follows.

Theorem 1.20 *Let $\phi : X \to (-\infty, \infty]$ be convex l.s.c. and proper. Then if $(x_0, a_0) \notin epi(\phi)$, there exists $x_0^* \in X'$ and $c \in \mathbb{R}$ such that for all $x \in X$,*

$$\phi(x) + x_0^*(x) > c > x_0^*(x_0) + a_0.$$

The proof involves a few lemmas.

Lemma 1.21 *$epi(\phi) \cap (X \times \mathbb{R})$ is closed in $X \times \mathbb{R}$.*

Proof: If $(x, a) \notin epi(\phi) \cap (X \times \mathbb{R})$, then $a < \phi(x)$ so $a \neq \infty$. Therefore, $(x, a) \in X \times \mathbb{R}$. There exists an open set U in $X \times (-\infty, \infty]$ such that $(x, a) \in U$ but $U \cap epi(\phi) = \emptyset$. Then $U \cap (X \times \mathbb{R})$ is an open set containing (x, a) which does not intersect $epi(\phi) \cap (X \times \mathbb{R})$.

Definition 1.22 *$X \times \mathbb{R}$ is a topological vector space relative to the seminorms*

$$\overline{\rho}((x, a)) \equiv \rho(x) + |a|$$

where $\rho \in \Psi$, the seminorms defined on X.

Lemma 1.23 *If $f \in (X \times \mathbb{R})'$ then there exists $x^* \in X'$ and $\beta \in \mathbb{R}$ such that*

$$f((x, a)) = x^*(x) + \beta a$$

for all $(x, a) \in X \times \mathbb{R}$.

Proof: If $f \in (X \times \mathbb{R})'$, Theorem 1.5 implies there exists a finite subset, A, of Ψ such that

$$f((x, a)) \leq \overline{\rho}_A((x, a))$$

where

$$\overline{\rho}_A(x, a) \equiv \max(\rho(x) : \rho \in A) + |a| = \max(\overline{\rho}((x, a)) : \rho \in A).$$

Now define $x^*(x) \equiv f((x, 0))$. Thus $|x^*(x)| \leq \rho_A(x)$ so $x^* \in X'$. Similarly, use the Riesz representation theorem for Hilbert space to define $\beta \in \mathbb{R}$ such that $f((0, a)) = \beta a$. Thus

$$f((x, a)) = f((x, 0)) + f((0, a)) = x^*(x) + \beta a.$$

Proof of the theorem: Let $(x_0, a_0) \notin epi(\phi)$. Then $a_0 < \phi(x_0)$ and so $(x_0, a_0) \in X \times \mathbb{R}$. By Lemmas 1.21 and 1.23 and the separation theorems in topological vector space, there exists $x_0^* \in X'$, $c \in \mathbb{R}$, and $\beta \in \mathbb{R}$ such that for all x,

$$x_0^*(x_0) + \beta a_0 < c < \beta \phi(x) + x_0^*(x).$$

If $\beta \leq 0$, a contradiction results from the assumption that $a_0 < \phi(x_0)$. It follows we may divide by β and, replacing x_0^* with x_0^*/β, the conclusion is obtained. This proves the theorem.

Recall that in locally convex topological vector spaces, if a set is closed and convex, it follows it is weakly closed also. The corresponding theorem in this context is the following corollary.

Corollary 1.24 *If $\phi : X \to (-\infty, \infty]$ is convex, l.s.c., and proper, then the same is true if X is given the weak topology.*

Proof: Let $(x_0, a_0) \notin epi(\phi)$. Then by Theorem 1.20 there exists $x^* \in X'$ and $c \in \mathbb{R}$ such that

$$x^*(x) + a > c > x^*(x_0) + a_0$$

whenever $a \geq \phi(x)$. It follows there exists U, a weakly open set containing 0 and $\eta > 0$ such that

$$c > x^*(x_0 + z) + a_0 + b$$

whenever $z \in U$ and $|b| < \eta$. Thus the set, $(U + x_0) \times (a_0 - \eta, a_0 + \eta)$, open in the topology of $(X,$weak topology$) \times \mathbb{R}$ does not intersect $epi(\phi)$ and so $epi(\phi)$ is closed. This proves the corollary.

Definition 1.25 *We say a convex, proper, l.s.c. function, ϕ, defined on a Banach space, X, is coercive if*

$$\lim_{||x|| \to \infty} \phi(x) = +\infty.$$

With Corollary 1.24, it is easy to establish the following existence theorem. This is easy because we can work with the weak topology and use the Eberlein Smulian theorem to extract a weakly convergent subsequence once an estimate has been established from the assumption of coercivity. This type of argument is quite common in the study of weak solutions to various partial differential equations.

Theorem 1.26 *Let $\phi : X \to (-\infty, \infty]$ be convex, proper, l.s.c., and coercive where X is a reflexive Banach space. Then ϕ has a global minimum.*

Proof: Let $\phi(x_0) < \infty$. By the assumption that ϕ is coercive, there exists $R > 0$ such that $x_0 \in B(0, R)$ and if $||x|| > R$, then $\phi(x) > \phi(x_0)$.

Claim: ϕ is bounded below.

Proof: If not, we can define $A_n \equiv \{x \in X : \phi(x) \leq -n\}$. Then A_n is weakly closed by Corollary 1.24 because if $x \notin A_n$, then $\phi(x) > -n$ which implies $(x, -n) \notin epi(\phi)$ and so there is a weakly open set containing x, U, such that $U \times \{-n\} \subseteq X \times \mathbb{R} \setminus epi(\phi)$ and so for all $y \in U$, $\phi(y) > -n$. Since the complement of A_n is open, it follows A_n is closed.

Let $K_n \equiv A_n \cap \overline{B(0, R)}$, a weakly compact set. For n large enough, K_n must be empty because if not, it would follow there exists a point in all the K_n, x, and therefore, $\phi(x) = -\infty$ contrary to the assumption that ϕ maps into $(-\infty, \infty]$. Thus ϕ must be bounded below.

Now let $\lambda \equiv \inf\{\phi(x) : x \in X\} > -\infty$. Let $x_n \in \overline{B(0, R)}$ be such that $\phi(x_n) \to \lambda$. By taking a subsequence, if necessary, we may also assume $x_n \to x \in \overline{B(0, R)}$ weakly. Then since $epi(\phi)$ is closed, it follows $(x, \lambda) \in epi(\phi)$. Thus $\lambda \geq \phi(x) \geq \lambda$, the second inequality following because of the definition of λ. This proves the theorem.

1.4 Conjugate functions

Definition 1.27 *Let $\phi : X \to (-\infty, \infty]$ be some function, not necessarily convex but satisfying $\phi(y) < \infty$ for some $y \in X$. We define $\phi^* : X' \to (-\infty, \infty]$ by*

$$\phi^*(x^*) \equiv \sup\{x^*(y) - \phi(y) : y \in X\}.$$

This function, ϕ^*, defined above, is called the conjugate function of ϕ or the polar of ϕ. Note $\phi^*(x^*) \neq -\infty$ because $\phi(y) < \infty$ for some y.

Theorem 1.28 *Let X be a real Banach space. Then ϕ^* is convex and l.s.c.*

Proof: Let $\lambda \in [0, 1]$. Then

$$\phi^*(\lambda x^* + (1 - \lambda) y^*) = \sup\{(\lambda x^* + (1 - \lambda) y^*)(y) - \phi(y) : y \in X\}$$

$$\sup\{\lambda(x^*(y)-\phi(y))+(1-\lambda)(y^*(y)-\phi(y)) : y\in X\}$$

$$\leq \lambda\phi^*(x^*)+(1-\lambda)\phi^*(y^*).$$

It remains to show the function is $l.s.c.$ Consider $f_y(x^*)\equiv x^*(y)-\phi(y)$. Then f_y is obviously convex. Also

$$epi(\phi^*)=\cap_{y\in X} epi(f_y).$$

Therefore, if we can show $epi(f_y)$ is closed, we are done. If $(x^*,a)\notin epi(f_y)$, then $a<x^*(y)-\phi(y)$ and, by continuity, this condition will persist for all y^* near x^*. Thus $epi(f_y)$ is closed and this proves the theorem.

Note this theorem holds with no change in the proof if X is only a locally convex topological vector space and X' is given the weak $*$ topology.

Definition 1.29 *We define ϕ^{**} on X by*

$$\phi^{**}(x)\equiv\sup\{x^*(x)-\phi^*(x^*), x^*\in X'\}.$$

Theorem 1.30 *$\phi^{**}(x)\leq\phi(x)$ for all x and if ϕ is convex and l.s.c., $\phi^{**}(x)=\phi(x)$ for all $x\in X$.*

Proof:

$$\phi^{**}(x)\equiv\sup\{x^*(x)-\sup\{x^*(y)-\phi(y) : y\in X\} : x^*\in X'\}$$

$$\leq\sup\{x^*(x)-(x^*(x)-\phi(x))\}=\phi(x).$$

Next suppose ϕ is convex and $l.s.c.$ If $\phi^{**}(x_0)<\phi(x_0)$, then by Theorem 1.20 there exists x_0^* such that for all $y\in X$,

$$x_0^*(y)+\phi(y)>c>x_0^*(x_0)+\phi^{**}(x_0). \quad ((1.13))$$

It follows from (1.13) that

$$\phi^{**}(x_0)\geq(-x_0^*)(x_0)-\phi^*(-x_0^*)=$$

$$(-x_0^*)(x_0)-\sup\{(-x_0^*)(y)-\phi(y) : y\in X\}$$

$$=(-x_0^*)(x_0)+\inf\{x_0^*(y)+\phi(y) : y\in X\}$$

$$\geq c-x_0^*(x_0)>\phi^{**}(x_0),$$

a contradiction. Hence, if ϕ is convex and $l.s.c.$, $\phi^{**}(x)=\phi(x)$ for all x and this proves the theorem.

The following corollary is descriptive of the situation just discussed. It says that to find $epi(\phi^{**})$ it suffices to take the intersection of all closed convex sets which contain $epi(\phi)$.

Corollary 1.31 *$epi(\phi^{**})$ is the smallest closed convex set containing $epi(\phi)$.*

Proof: $epi(\phi^{**})\supseteq epi(\phi)$ from Theorem 1.30. Also $epi(\phi^{**})$ is closed by the proof of Theorem 1.28. Suppose $epi(\phi)\subseteq K\subseteq epi(\phi^{**})$ and K is convex and closed. Let

$$\psi(x)\equiv\min\{a : (x,a)\in K\}.$$

($\{a : (x, a) \in K\}$ is a closed subset of $(-\infty, \infty]$ so the minimum exists.) ψ is also a convex function with $epi\,(\psi) = K$. To see ψ is convex, let $\lambda \in [0,1]$. Then, by the convexity of K,

$$\lambda\,(x, \psi\,(x)) + (1-\lambda)\,(y, \psi\,(y))$$

$$= (\lambda x + (1-\lambda)\,y, \lambda\psi\,(x) + (1-\lambda)\,\psi\,(y)) \in K.$$

It follows from the definition of ψ that

$$\psi\,(\lambda x + (1-\lambda)\,y) \leq \lambda\psi\,(x) + (1-\lambda)\,\psi\,(y).$$

Then

$$\phi^{**} \leq \psi \leq \phi$$

and so from the definitions,

$$\phi^{***} \geq \psi^{*} \geq \phi^{*}$$

which implies from the definitions and Theorem 1.30 that

$$\phi^{**} = \phi^{****} \leq \psi^{**} = \psi \leq \phi^{**}.$$

Therefore, $\psi = \phi^{**}$ and $epi\,(\phi^{**})$ is the smallest closed convex set containing $epi\,(\phi)$ as claimed.

1.5 Subgradients

Definition 1.32 *Let X be a real locally convex topological vector space. For $x \in X$, $\delta\phi\,(x) \subseteq X'$, possibly $\emptyset$. This subset of X' is defined by $y^* \in \delta\phi\,(x)$ means for all $z \in X$,*

$$y^*\,(z - x) \leq \phi\,(z) - \phi\,(x).$$

Also $x \in \delta\phi^*\,(y^*)$ means that for all $z^* \in X'$,

$$(z^* - y^*)\,(x) \leq \phi^*\,(z^*) - \phi^*\,(y^*).$$

We define $dom\,(\delta\phi) \equiv \{x : \delta\phi\,(x) \neq \emptyset\}$.

The subgradient is an attempt to generalize the derivative. For example, a function may have a subgradient but fail to be differentiable at some point. A good example is $f\,(x) = |x|$. At $x = 0$, this function fails to have a derivative but it does have a subgradient. In fact, $\delta f\,(0) = [-1, 1]$.

To begin with we consider the question of existence of the subgradient of a convex function. There is a very simple criterion for existence. It is essentially that the subgradient is nonempty at every point of the interior of the domain of ϕ.

Theorem 1.33 *Let $\phi : X \to (-\infty, \infty]$ be convex and suppose for some $u \in dom\,(\phi)$, ϕ is continuous. Then $\delta\phi\,(x) \neq \emptyset$ for all $x \in int\,(dom\,(\phi))$. Thus*

$$dom\,(\delta\phi) \supseteq int\,(dom\,(\phi)).$$

Proof: Let $x_0 \in int\,(dom\,(\phi))$ and let

$$A \equiv \{(x_0, \phi\,(x_0))\}, B \equiv epi\,(\phi) \cap X \times \mathbb{R}.$$

Then A and B are both nonempty and convex. Since ϕ is continuous at $u \in dom(\phi)$,

$$(u, \phi(u) + 1) \in int(epi\phi \cap X \times \mathbb{R}).$$

Thus $int(B) \neq \emptyset$ and also $int(B) \cap A = \emptyset$. Therefore, by Lemma 1.23 and Corollary 1.17, there exists $x^* \in X'$ and $\beta \in \mathbb{R}$ such that

$$(x^*, \beta) \neq (0, 0) \tag{(1.14)}$$

and for all $(x, a) \in B$,

$$x^*(x) + \beta a \geq x^*(x_0) + \beta\phi(x_0). \tag{(1.15)}$$

In particular, whenever $x \in dom(\phi)$,

$$x^*(x) + \beta\phi(x) \geq x^*(x_0) + \beta\phi(x_0).$$

If $\beta = 0$, this would mean $x^*(x - x_0) \geq 0$ for all $x \in dom(\phi)$. Since $x_0 \in int(dom(\phi))$, this implies $x^* = 0$, contradicting (1.14). If $\beta < 0$, we could apply (1.15) to the case when $a = \phi(x_0) + 1$ and $x = x_0$ to obtain a contradiction. It follows $\beta > 0$ and so $-x^*/\beta \in \delta\phi(x_0)$. This proves the theorem.

There is an interesting symmetry which relates $\delta\phi, \delta\phi^*, \phi$, and ϕ^*.

Theorem 1.34 *Suppose ϕ is convex, l.s.c., and proper. Then*

$$y^* \in \delta\phi(x) \text{ if and only if } x \in \delta\phi^*(y^*)$$

and, in this case,

$$y^*(x) = \phi^*(y^*) + \phi(x).$$

Proof: If $y^* \in \delta\phi(x)$ then $y^*(z - x) \leq \phi(z) - \phi(x)$ and so

$$y^*(z) - \phi(z) \leq y^*(x) - \phi(x)$$

for all $z \in X$. Therefore,

$$\phi^*(y^*) \leq y^*(x) - \phi(x) \leq \phi^*(y^*).$$

Hence

$$y^*(x) = \phi^*(y^*) + \phi(x). \tag{(1.16)}$$

Now if $z^* \in X'$ is arbitrary, (1.16) shows

$$(z^* - y^*)(x) = z^*(x) - \phi^*(y^*) - \phi(x) \leq \phi^*(z^*) - \phi^*(y^*)$$

and this shows $x \in \delta\phi^*(y^*)$.

Now suppose $x \in \delta\phi^*(y^*)$. Then for $z^* \in X'$,

$$(z^* - y^*)(x) \leq \phi^*(z^*) - \phi^*(y^*)$$

and so, taking sup over all z^*, and using Theorem 1.30,

$$\phi^{**}(x) = \phi(x) \leq y^*(x) - \phi^*(y^*) \leq \phi^{**}(x).$$

Thus

$$y^*(x) = \phi^*(y^*) + \phi^{**}(x) = \phi^*(y^*) + \phi(x) \geq y^*(z) - \phi(z) + \phi(x)$$

for all $z \in X$ and this implies for all $z \in X$,

$$\phi(z) - \phi(x) \geq y^*(z - x)$$

so $y^* \in \delta\phi(x)$ and this proves the theorem.

Definition 1.35 *If X is a Banach space, $u \in H^1(0,T;X)$ if there exists $g \in L^2(0,T;X)$ such that*

$$u(t) = u(0) + \int_0^t g(s)\,ds$$

and we define $u'(\cdot) \equiv g(\cdot)$.

The next Lemma is quite interesting for its own sake but it is also used in the next theorem.

Lemma 1.36 *Suppose $g \in L^2(0,T;X)$. Then as $h \to 0$,*

$$\frac{1}{h}\int_{(\cdot)}^{(\cdot)+h} g(s)\,ds\mathcal{X}_{[0,T-h]}(\cdot) \to g$$

in $L^2(0,T;X)$.

Proof: Let

$$\widetilde{g}(u) \equiv \begin{cases} g(u) \text{ if } u \in [0,T] \\ 0 \text{ if } u \notin [0,T] \end{cases}, \ \phi_h(r) \equiv \frac{1}{h}\mathcal{X}_{[-h,0]}(r).$$

Thus $\widetilde{g} \in L^2(\mathbb{R};X)$ and

$$\widetilde{g} * \phi_h(t) \equiv \int_{\mathbb{R}} \widetilde{g}(t-s)\,\phi_h(s)\,ds.$$

Then

$$||\widetilde{g} * \phi_h - \widetilde{g}||_{L^2(\mathbb{R};X)} \leq \left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}} ||\widetilde{g}(t) - \widetilde{g}(t-s)||_X\,\phi_h(s)\,ds\right)^2 dt\right)^{1/2}$$

which by Minkowski's inequality for integrals is no larger than

$$\leq \int_{\mathbb{R}} \phi_h(s)\left(\int_{\mathbb{R}} ||\widetilde{g}(t) - \widetilde{g}(t-s)||_X^2\,dt\right)^{1/2} ds$$

$$= \frac{1}{h}\int_{-h}^{0}\left(\int_{\mathbb{R}} ||\widetilde{g}(t) - \widetilde{g}(t-s)||_X^2\,dt\right)^{1/2} ds < \frac{1}{h}\int_{-h}^{0} \varepsilon ds = \varepsilon$$

whenever h is small enough. This follows from continuity of translation in $L^2(\mathbb{R};X)$, a consequence of the regularity of the measure. Thus, $\widetilde{g} * \phi_h \to \widetilde{g}$ in $L^2(\mathbb{R};X)$. Now

$$\widetilde{g} * \phi_h(t) - \frac{1}{h}\int_t^{t+h} g(s)\,ds\mathcal{X}_{[0,T-h]}(t)$$

$$= \begin{cases} 0 \text{ if } t \in [0,T-h] \\ \frac{1}{h}\int_t^{t+h} \widetilde{g}(u)\,du \text{ if } t \notin [0,T-h] \end{cases}$$

and therefore,

$$\left|\left|\widetilde{g} * \phi_h(\cdot) - \frac{1}{h}\int_{(\cdot)}^{(\cdot)+h} g(s)\,ds\mathcal{X}_{[0,T-h]}(\cdot)\right|\right|_{L^2(\mathbb{R};X)} =$$

$$\left(\int_{-h}^{0}\left|\left|\frac{1}{h}\int_{t}^{t+h}\widetilde{g}(u)\,du\right|\right|_X^2 dt\right)^{1/2}+\left(\int_{T-h}^{T}\left|\left|\frac{1}{h}\int_{t}^{t+h}\widetilde{g}(u)\,du\right|\right|_X^2 dt\right)^{1/2}$$

$$\leq \frac{1}{h}\left(\int_{-h}^{0}\left(\int_{-h}^{h}||\widetilde{g}(u)||_X\,du\right)^2 dt\right)^{1/2}$$

$$+\frac{1}{h}\left(\int_{T-h}^{T}\left(\int_{T-h}^{T+h}||\widetilde{g}(u)||_X\,du\right)^2 dt\right)^{1/2}$$

which by Minkowski's inequality for integrals is no larger than

$$\leq \frac{1}{h}\int_{-h}^{h}\left(\int_{-h}^{0}||\widetilde{g}(u)||_X^2\,dt\right)^{1/2}du+\frac{1}{h}\int_{T-h}^{T+h}\left(\int_{T-h}^{T}||\widetilde{g}(u)||_X^2\,dt\right)^{1/2}du$$

$$\leq \frac{1}{\sqrt{h}}\int_{-h}^{h}||\widetilde{g}(u)||_X\,du+\frac{1}{\sqrt{h}}\int_{T-h}^{T+h}||\widetilde{g}(u)||_X\,du$$

$$\leq \frac{(2h)^{1/2}}{\sqrt{h}}\left(\int_{-h}^{h}||\widetilde{g}(u)||_X^2\,du\right)^{1/2}+\frac{(2h)^{1/2}}{\sqrt{h}}\left(\int_{T-h}^{T+h}||\widetilde{g}(u)||_X^2\,du\right)^{1/2},$$

an expression which converges to zero as $h\to 0$ because of the dominated convergence theorem and the fact that $||\widetilde{g}||_X$ is in $L^1(\mathbb{R};X)$. Therefore,

$$\frac{1}{h}\int_{(\cdot)}^{(\cdot)+h}g(s)\,ds\mathcal{X}_{[0,T-h]}(\cdot)\to\widetilde{g}$$

in $L^2(\mathbb{R};X)$ and consequently in $L^2(0,T;X)$ as well. But $\widetilde{g}=g$ on $[0,T]$. This proves the lemma.

The following theorem is a form of the chain rule in which the derivative is replaced by the subgradient.

Theorem 1.37 *Suppose $u\in H^1(0,T;X)$, $z\in L^2(0,T;X')$, and $z(t)\in\delta\phi(u(t))$ a.e $t\in[0,T]$. Then the function, $t\to\phi(u(t))$ is in $L^1(0,T)$ and its weak derivative equals $z(u')$.*

Proof: Modify u on a set of measure zero such that $\delta\phi(u(t))\neq\emptyset$ for all t. Next modify z on a set of measure zero such that for $\widetilde{u}$ and $\widetilde{z}$ the modified functions, $\widetilde{z}(t)\in\delta\phi(\widetilde{u}(t))$ for all t. First we show $t\to\phi(\widetilde{u}(t))$ is in $L^1(0,T)$. Pick $t_0\in[0,T]$ and let

$$\widetilde{z}(t_0)\in\delta\phi(\widetilde{u}(t_0)).$$

Then for $t\in[0,T]$,

$$\widetilde{z}(t_0)(\widetilde{u}(t)-\widetilde{u}(t_0))+\phi(\widetilde{u}(t_0))\leq$$

$$\phi(\widetilde{u}(t))\leq\widetilde{z}(t)(\widetilde{u}(t)-\widetilde{u}(t_0))+\phi(\widetilde{u}(t_0))$$

since $\widetilde{z}(t)(\widetilde{u}(t_0)-\widetilde{u}(t)) \leq \phi(\widetilde{u}(t_0)) - \phi(\widetilde{u}(t))$. This inequality shows $t \to \phi(\widetilde{u}(t))$ is in $L^1(0,T)$ since $\widetilde{z} \in L^2(0,T;X')$ and $\widetilde{u} \in L^2(0,T;X)$. Also, for $t \in [0,T-h]$,

$$\mathcal{X}_{[0,T-h]}(t)\,\widetilde{z}(t)\left(\frac{\widetilde{u}(t+h)-\widetilde{u}(t)}{h}\right) \leq \mathcal{X}_{[0,T-h]}(t)\,\frac{\phi(\widetilde{u}(t+h))-\phi(\widetilde{u}(t))}{h}$$

$$\leq \mathcal{X}_{[0,T-h]}(t)\,\widetilde{z}(t+h)\left(\frac{\widetilde{u}(t+h)-\widetilde{u}(t)}{h}\right)$$

Now $\mathcal{X}_{[0,T-h]}(\cdot)\,\widetilde{z}(\cdot+h) \to z(\cdot)$ in $L^2(0,T;X')$ by continuity of translation. Also,

$$\mathcal{X}_{[0,T-h]}(\cdot)\,\frac{\widetilde{u}(\cdot+h)-\widetilde{u}(\cdot)}{h} = \mathcal{X}_{[0,T-h]}(\cdot)\,\frac{u(\cdot+h)-u(\cdot)}{h}$$

$$= \mathcal{X}_{[0,T-h]}(\cdot)\,\frac{1}{h}\int_{(\cdot)}^{(\cdot)+h} u'(s)\,ds$$

in $L^2(0,T;X)$ and so by Lemma 1.36,

$$\mathcal{X}_{[0,T-h]}(\cdot)\,\frac{\phi(\widetilde{u}(\cdot+h))-\phi(\widetilde{u}(\cdot))}{h} \to z(u')$$

in $L^1(0,T)$.

It follows from the definition of weak derivatives that in the sense of weak derivatives,

$$\frac{d}{dt}(\phi(u(\cdot))) = z(u') \in L^1(0,T).$$

Note that this implies that for a.e. $t \in [0,T]$, $\phi(u(t))$ is equal to a continuous function, $\phi \circ u$, and that

$$(\phi \circ u)(t) - (\phi \circ u)(0) = \int_0^t z(u')\,ds.$$

There are other rules of calculus which have a generalization to subgradients. The following theorem is on such a generalization. It generalizes the theorem which states that the derivative of a sum equals the sum of the derivatives.

Theorem 1.38 *Let ϕ_1 and ϕ_2 be convex, l.s.c. and proper. Then*

$$\delta(\lambda\phi_i)(x) = \lambda\delta\phi_i(x)\,,\ \delta(\phi_1+\phi_2)(x) \supseteq \delta\phi_1(x) + \delta\phi_2(x) \tag{(1.17)}$$

if $\lambda > 0$. If there exists $\overline{x} \in dom(\phi_1) \cap dom(\phi_2)$ and ϕ_1 is continuous at $\overline{x}$ then for all $x \in X$,

$$\delta(\phi_1+\phi_2)(x) = \delta\phi_1(x) + \delta\phi_2(x). \tag{(1.18)}$$

Proof: (1.17) is obvious so we only need to show (1.18). Suppose $\overline{x}$ is as described. It is clear (1.18) holds whenever $x \notin dom(\phi_1) \cap dom(\phi_2)$ since then both sides equal $\emptyset$. Therefore, we will assume

$$x \in dom(\phi_1) \cap dom(\phi_2)$$

in what follows. Let $x^* \in \delta(\phi_1+\phi_2)(x)$. We need to show x^* is the sum of an element of $\delta\phi_1(x)$ and $\delta\phi_2(x)$. Thus we want to show the existence of x_1^* and x_2^* such that for every y,

$$\begin{aligned} x^*(y-x) &= x_1^*(y-x) + x_2^*(y-x) \\ &\leq \phi_1(y) - \phi_1(x) + \phi_2(y) - \phi_2(x) \end{aligned}$$

and so we need

$$\phi_1(y) - \phi_1(x) - x^*(y-x) \geq \phi_2(x) - \phi_2(y).$$

Define

$$C_1 \equiv \{(y,a) \in X \times \mathbb{R} : \phi_1(y) - x^*(y-x) - \phi_1(x) \leq a\},$$

$$C_2 \equiv \{(y,a) \in X \times \mathbb{R} : a \leq \phi_2(x) - \phi_2(y)\}.$$

We will show $int(C_1) \cap C_2 = \emptyset$ and this will allow us to use Corollary 1.17 to obtain an element of X' which does something interesting. Both C_1 and C_2 are convex and nonempty. C_1 is nonempty because it contains $(\overline{x}, \phi_1(\overline{x}) - x^*(\overline{x}-x) - \phi_1(x))$. C_2 is also nonempty because it contains $(\overline{x}, \phi_2(x) - \phi_2(\overline{x}))$. In addition to this,

$$(\overline{x}, \phi_1(\overline{x}) - x^*(\overline{x}-x) - \phi_1(x) + 1) \in int(C_1)$$

due to the assumed continuity of ϕ_1 at $\overline{x}$. If $(y,a) \in int(C_1)$ then

$$\phi_1(y) - x^*(y-x) - \phi_1(x) \leq a - \varepsilon$$

whenever ε is small enough. Therefore, if (y,a) is also in C_2, the assumption that $x^* \in \delta(\phi_1 + \phi_2)(x)$ implies

$$a - \varepsilon \geq \phi_1(y) - x^*(y-x) - \phi_1(x) \geq \phi_2(x) - \phi_2(y) \geq a,$$

a contradiction. Therefore $int(C_1) \cap C_2 = \emptyset$ and so by Corollary 1.17 and Lemma 1.23, there exists $(w^*, \beta) \in X' \times \mathbb{R}$ with

$$(w^*, \beta) \neq (0,0), \tag{(1.19)}$$

and

$$w^*(y) + \beta a \geq w^*(y_1) + \beta a_1, \tag{(1.20)}$$

whenever $(y,a) \in C_1$ and $(y_1, a_1) \in C_2$.

Claim: $\beta > 0$.

Proof of claim: If $\beta < 0$ let

$$a = \phi_1(\overline{x}) - x^*(\overline{x}-x) - \phi_1(x) + 1,$$

$$a_1 = \phi_2(x) - \phi_2(\overline{x}), \text{ and } y = y_1 = \overline{x}.$$

Then

$$\beta(\phi_1(\overline{x}) - x^*(\overline{x}-x) - \phi_1(x) + 1) \geq \beta(\phi_2(x) - \phi_2(\overline{x})).$$

Dividing by β yields

$$\phi_1(\overline{x}) - x^*(\overline{x}-x) - \phi_1(x) + 1 \leq \phi_2(x) - \phi_2(\overline{x})$$

and so

$$\phi_1(\overline{x}) + \phi_2(\overline{x}) - (\phi_1(x) + \phi_2(x)) + 1 \leq x^*(\overline{x}-x)$$

$$\leq \phi_1(\overline{x}) + \phi_2(\overline{x}) - (\phi_1(x) + \phi_2(x)),$$

a contradiction. Therefore, $\beta \geq 0$.

Now suppose $\beta = 0$. Letting

$$a = \phi_1(\overline{x}) - x^*(\overline{x} - x) - \phi_1(x) + 1,$$

$$(\overline{x}, a) \in int(C_1),$$

and so there exists an open set U containing 0 and $\eta > 0$ such that

$$\overline{x} + U \times (a - \eta, a + \eta) \subseteq C_1.$$

Therefore, (1.20) applied to $(\overline{x} + z, a) \in C_1$ and $(\overline{x}, \phi_2(x) - \phi_2(\overline{x})) \in C_2$ for $z \in U$ yields

$$w^*(\overline{x} + z) \geq w^*(\overline{x})$$

for all $z \in U$. Hence $w^*(z) = 0$ on U which implies $w^* = 0$, contradicting (1.19). This proves the claim.

Now with the claim, it follows $\beta > 0$ and so, letting $z^* = w^*/\beta$, (1.20) implies

$$z^*(y) + a \geq z^*(y_1) + a_1 \quad ((1.21))$$

whenever $(y, a) \in C_1$ and $(y_1, a_1) \in C_2$. In particular,

$$(y, \phi_1(y) - x^*(y - x) - \phi_1(x)) \in C_1 \text{ and} \quad ((1.22))$$

$$(y_1, \phi_2(x) - \phi_2(y_1)) \in C_2. \quad ((1.23))$$

So letting $y = x$,

$$z^*(x) + (\phi_1(x) - x^*(x - x) - \phi_1(x)) \geq z^*(y_1) + \phi_2(x) - \phi_2(y_1).$$

Therefore,

$$z^*(y_1 - x) \leq \phi_2(y_1) - \phi_2(x)$$

for all y_1 and so $z^* \in \delta\phi_2(x)$. Now let $y_1 = x$ in (1.23) and using (1.21) and (1.22), it follows

$$z^*(y) + \phi_1(y) - x^*(y - x) - \phi_1(x) \geq z^*(x)$$

and so $x^* - z^* \in \delta\phi_1(x)$ so $x^* = z^* + (x^* - z^*) \in \delta\phi_2(x) + \delta\phi_1(x)$ and this proves the theorem.

1.6 Duality maps

Next we discuss a very important example known as the duality map from a Banach space to its dual space. Before doing so, consider a Hilbert space H. We can define a map R from H to H', called the Riesz map, by the rule

$$R(x)(y) \equiv (y, x).$$

By the Riesz representation theorem, we know this map is onto and one to one with the properties

$$R(x)(x) = ||x||^2, \text{ and } ||Rx||^2 = ||x||^2.$$

The duality map from a Banach space to its dual is an attempt to generalize this notion of Riesz map to an arbitrary Banach space.

Definition 1.39 *For X a Banach space define $F : X \to \mathcal{P}(X')$ by*

$$F(x) \equiv \left\{x^* \in X' : x^*(x) = ||x||^2, ||x^*|| \le ||x||\right\}. \tag{(1.24)}$$

Lemma 1.40 *With $F(x)$ defined as above, it follows that*

$$F(x) = \left\{x^* \in X' : x^*(x) = ||x||^2, ||x^*|| = ||x||\right\}$$

and $F(x)$ is a closed, nonempty, convex subset of X'.

Proof: If x^* is in the set described in (1.24),

$$x^*\left(\frac{x}{||x||}\right) = ||x||$$

and so $||x^*|| \ge ||x||$. Therefore

$$x^* \in \left\{x^* \in X' : x^*(x) = ||x||^2, ||x^*|| = ||x||\right\}.$$

This shows this set and the set of (1.24) are equal. It is also clear the set of (1.24) is closed and convex. It only remains to show this set is nonempty.

Define $f : \mathbb{R}x \to \mathbb{R}$ by $f(\alpha x) = \alpha ||x||^2$. Then the norm of f on $\mathbb{R}x$ is $||x||$ and $f(x) = ||x||^2$. By the Hahn Banach theorem, f has an extension to all of X, x^*, and this extension is in the set of (1.24), showing this set is nonempty as required.

The next theorem shows this duality map is the subgradient of $\frac{1}{2}||x||^2$.

Theorem 1.41 *For X a real Banach space, let $\phi(x) \equiv \frac{1}{2}||x||^2$. Then $F(x) = \delta\phi(x)$.*

Proof: Let $x^* \in F(x)$. Then

$$\begin{aligned} x^*(y-x) &= x^*(y) - x^*(x) \\ &\le ||x||\,||y|| - ||x||^2 \le \frac{1}{2}||y||^2 - \frac{1}{2}||x||^2. \end{aligned}$$

This shows $F(x) \subseteq \delta\phi(x)$. Now let $x^* \in \delta\phi(x)$. Then for all $t \in \mathbb{R}$,

$$x^*(ty) = x^*(ty + x - x) \le \frac{1}{2}\left(||x+ty||^2 - ||x||^2\right). \tag{(1.25)}$$

Now if $t > 0$, divide both sides by t. This yields

$$x^*(y) \le \frac{1}{2t}\left(2t\,||x||\,||y|| + t^2\,||x||^2\right).$$

Letting $t \to 0$, we see that

$$x^*(y) \le ||x||\,||y||. \tag{(1.26)}$$

Next suppose $t = -s$, where $s > 0$ in (1.25). Then

$$x^*(y) \ge \frac{1}{2s}\left[||x||^2 - ||x-sy||^2\right] \ge$$

$$\frac{1}{2s}\left[||x-sy||^2 - 2\,||x-sy||\,||sy|| + ||sy||^2 - ||x-sy||\right]^2. \tag{(1.27)}$$

Taking a limit as $s \to 0$ yields

$$x^*(y) \geq -||x|| \, ||y||. \tag{(1.28)}$$

It follows from (1.28) and (1.26) that

$$|x^*(y)| \leq ||x|| \, ||y||$$

and that, therefore, $||x^*|| \leq ||x||$ and $|x^*(x)| \leq ||x||^2$. Now return to (1.27) and let $y = x$. Then

$$x^*(x) \geq \frac{1}{2s}\left[||x||^2\left(1-(1-s)^2\right)\right].$$

Letting $s \to 0$,

$$x^*(x) \geq ||x||^2.$$

Since it was already shown that $|x^*(x)| \leq ||x||^2$, this shows $x^* \in F(x)$ and proves the theorem.

The next result is an easy corollary of Theorem 1.26 and gives conditions under which the subgradient is onto. By this we mean that if $y^* \in X'$, then there exists $x \in X$ such that $y^* \in \delta\phi(x)$.

Theorem 1.42 *Suppose X is a reflexive Banach space and suppose $\phi : X \to (-\infty, \infty]$ is convex, proper, l.s.c., and for all $y^* \in X'$, $x \to \phi(x) - y^*(x)$ is coercive. Then $\delta\phi$ is onto.*

Proof: The function $x \to \phi(x) - y^*(x) \equiv \psi(x)$ is convex, proper, *l.s.c.*, and coercive. Therefore, by Theorem 1.26, there exists x which minimizes this function. Therefore,

$$0 \in \delta\psi(x)$$

by Theorem 1.38, $0 \in \delta\psi(x) = \delta\phi(x) - y^*$ and this proves the theorem.

Corollary 1.43 *Suppose X is a reflexive Banach space and $\phi : X \to (-\infty, \infty]$ is convex, proper, and l.s.c. Then for each $y^* \in X'$ there exist $x \in X$, $x_1^* \in F(x)$, and $x_2^* \in \delta\phi(x)$ such that*

$$y^* = x_1^* + x_2^*.$$

Proof: Apply Theorem 1.42 to the convex function $\frac{1}{2}||x||^2 + \phi(x)$ and use Theorems 1.38 and 1.41.

Chapter 2

Set valued maps

2.1 Preliminary results

In the abstract theory of partial differential equations and variational inequalities, it is important to consider set-valued maps from a Banach space to the power set of its dual. In this section we give an introduction to this theory by proving a general result on surjectivity for a class of such operators.

To begin with, if $A : X \to \mathcal{P}(Y)$ is a set-valued map, we define the graph of A by

$$G(A) \equiv \{(x, y) : y \in Ax\}.$$

We will first consider a map A which maps $\mathbb{C}^n$ to $\mathcal{P}(\mathbb{C}^n)$ which satisfies

$$Ax \text{ is compact and convex.} \tag{(2.1)}$$

and also the condition that if O is open and $O \supseteq A\mathbf{x}$, then there exists $\delta > 0$ such that if

$$\mathbf{y} \in B(\mathbf{x}, \delta), \text{ then } A\mathbf{y} \subseteq O. \tag{(2.2)}$$

This last condition is sometimes referred to as upper semicontinuity. In words, A is upper semicontinuous and has values which are compact and convex.

Lemma 2.1 *Let A satisfy Formulas (2.1) and (2.2). Then AK is a subset of a compact set whenever K is compact. Also the graph of A is closed.*

Proof: Let $\mathbf{x} \in K$. Then $A\mathbf{x}$ is compact and contained in some open set whose closure is compact, $U_{\mathbf{x}}$. By assumption (2.2) there exists an open set $V_{\mathbf{x}}$ containing $\mathbf{x}$ such that if $\mathbf{y} \in V_{\mathbf{x}}$, then $A\mathbf{y} \subseteq U_{\mathbf{x}}$. Let $V_{\mathbf{x}_1}, \cdots, V_{\mathbf{x}_m}$ cover K. Then $AK \subseteq \cup_{k=1}^m \overline{U}_{\mathbf{x}_k}$, a compact set. To see the graph of A is closed, let $\mathbf{x}_k \to \mathbf{x}, \mathbf{y}_k \to \mathbf{y}$ where $\mathbf{y}_k \in A\mathbf{x}_k$. Then letting $O = A\mathbf{x} + B(\mathbf{0}, r)$ it follows from (2.2) that $\mathbf{y}_k \in A\mathbf{x}_k \subseteq O$ for all k large enough. Therefore, $\mathbf{y} \in A\mathbf{x} + B(\mathbf{0}, 2r)$ and since $r > 0$ is arbitrary and $A\mathbf{x}$ is closed it follows $\mathbf{y} \in A\mathbf{x}$.

The next lemma is an application of the Brouwer fixed point theorem. First we define an n simplex, denoted by $[\mathbf{x}_0, \cdots, \mathbf{x}_n]$, to be the convex hull of the $n+1$ points, $\{\mathbf{x}_0, \cdots, \mathbf{x}_n\}$ where $\{\mathbf{x}_i - \mathbf{x}_0\}_{i=1}^n$ are independent. Thus

$$[\mathbf{x}_0, \cdots, \mathbf{x}_n] \equiv \left\{ \sum_{i=1}^n t_i \mathbf{x}_i : \sum_{i=1}^n t_i = 1, \ t_i \geq 0 \right\}.$$

If $n \leq 2$, the simplex is a triangle, line segment, or point. If $n \leq 3$, it is a tetrahedron, triangle, line segment or point. We also say a collection of simplices is a tiling of $\mathbb{R}^n$ if $\mathbb{R}^n$ is contained in their union and if S_1, S_2 are two simplices in the tiling, with

$$S_j = \left[\mathbf{x}_0^j, \cdots, \mathbf{x}_n^j\right],$$

then

$$S_1 \cap S_2 = [\mathbf{x}_{k_0}, \cdots, \mathbf{x}_{k_r}]$$

where

$$\{\mathbf{x}_{k_0}, \cdots, \mathbf{x}_{k_r}\} \subseteq \{\mathbf{x}_0^1, \cdots, \mathbf{x}_n^1\} \cap \{\mathbf{x}_0^2, \cdots, \mathbf{x}_n^2\}$$

or else the two simplices do not intersect. The collection of simplices is said to be locally finite if, for every point, there exists a ball containing that point which also intersects only finitely many of the simplices in the collection. We leave it to the reader to verify that for each $\varepsilon > 0$, there exists a locally finite tiling of $\mathbb{R}^n$ which is composed of simplices which have diameters less than ε.

Lemma 2.2 *Suppose $A : \mathbb{C}^n \to \mathcal{P}(\mathbb{C}^n)$ satisfies Formulas (2.1) and (2.2) and K is a nonempty compact convex set in $\mathbb{C}^n$. Then if $\mathbf{y} \in \mathbb{C}^n$ there exists $(\mathbf{x}, \mathbf{w}) \in G(A)$ such that $\mathbf{x} \in K$ and*

$$\operatorname{Re}(\mathbf{y} - \mathbf{w}, \mathbf{z} - \mathbf{x}) \leq 0$$

for all $\mathbf{z} \in K$.

Proof: Tile $\mathbb{C}^n$ with $2n$ simplices such that the collection is locally finite and each simplex has diameter less than $\varepsilon < 1$. This collection of simplices is determined by a countable collection of vertices. For each vertex, $\mathbf{x}$, pick $A_\varepsilon \mathbf{x} \in A\mathbf{x}$ and define A_ε on all of $\mathbb{C}^n$ by the following rule. If

$$\mathbf{x} \in [\mathbf{x}_0, \cdots, \mathbf{x}_{2n}],$$

so $\mathbf{x} = \sum_{i=0}^{2n} t_i \mathbf{x}_i$, then

$$A_\varepsilon \mathbf{x} \equiv \sum_{k=0}^{2n} t_k A_\varepsilon \mathbf{x}_k.$$

Thus A_ε is a continuous map defined on $\mathbb{C}^n$ thanks to the local finiteness of the collection of simplices. Let P_K denote the projection on the convex set K. By the Brouwer fixed point theorem, there exists a fixed point, $\mathbf{x}_\varepsilon \in K$ such that

$$P_K(\mathbf{y} - A_\varepsilon \mathbf{x}_\varepsilon + \mathbf{x}_\varepsilon) = \mathbf{x}_\varepsilon.$$

This requires

$$\operatorname{Re}(\mathbf{y} - A_\varepsilon \mathbf{x}_\varepsilon, \mathbf{z} - \mathbf{x}_\varepsilon) \leq 0$$

for all $\mathbf{z} \in K$.

Suppose $\mathbf{x}_\varepsilon \in [\mathbf{x}_0^\varepsilon, \cdots, \mathbf{x}_{2n}^\varepsilon]$ so $\mathbf{x}_\varepsilon = \sum_{k=0}^{2n} t_k^\varepsilon \mathbf{x}_k^\varepsilon$. Then since $\mathbf{x}_\varepsilon$ is contained in K, a compact set, and the diameter of each simplex is less than 1, it follows that $A_\varepsilon \mathbf{x}_k^\varepsilon$ is contained in $A(\overline{K + B(\mathbf{0},1)})$, which is contained in a compact set thanks to Lemma 2.1. Taking a subsequence, we may obtain from the Heine Borel theorem that for some sequence, $\varepsilon \to 0$

$$t_k^\varepsilon \to t_k, \mathbf{x}_\varepsilon \to \mathbf{x}, A_\varepsilon \mathbf{x}_k^\varepsilon \to \mathbf{y}_k$$

for $k = 0, \cdots, 2n$. Since the diameter of the simplex containing $\mathbf{x}_\varepsilon$ converges to 0, it follows

$$\mathbf{x}_k^\varepsilon \to \mathbf{x},\ A_\varepsilon \mathbf{x}_k^\varepsilon \to \mathbf{y}_k.$$

Since the graph of A is closed and $A_\varepsilon \mathbf{x}_k^\varepsilon \in A\mathbf{x}_k^\varepsilon$, this implies $\mathbf{y}_k \in A\mathbf{x}$. Since $A\mathbf{x}$ is convex,

$$\sum_{k=1}^{2n} t_k \mathbf{y}_k \in A\mathbf{x}.$$

Hence for all $\mathbf{z} \in K$,

$$\operatorname{Re}\left(\mathbf{y}-\sum_{k=1}^{2n} t_k\mathbf{y}_k, \mathbf{z}-\mathbf{x}\right) = \lim_{\varepsilon\to 0} \operatorname{Re}\left(\mathbf{y}-\sum_{k=1}^{2n} t_k^\varepsilon A_\varepsilon \mathbf{x}_k^\varepsilon, \mathbf{z}-\mathbf{x}_\varepsilon\right)$$

$$= \lim_{\varepsilon\to 0} \operatorname{Re}\left(\mathbf{y}-A_\varepsilon\mathbf{x}_\varepsilon, \mathbf{z}-\mathbf{x}_\varepsilon\right) \le 0.$$

Let $\mathbf{w} = \sum_{k=1}^{2n} t_k\mathbf{y}_k$. This proves the lemma.

Lemma 2.3 *Suppose in addition to (2.1) and (2.2), A is coercive,*

$$\lim_{|\mathbf{x}|\to\infty} \inf\left\{\frac{\operatorname{Re}(\mathbf{y},\mathbf{x})}{|\mathbf{x}|} : \mathbf{y}\in A\mathbf{x}\right\} = \infty.$$

Then A is onto.

Proof: Let $\mathbf{y}\in\mathbb{C}^n$ and let $K_r \equiv \overline{B(\mathbf{0},r)}$. By Lemma 2.2 there exists $\mathbf{x}_r \in K_r$ and $\mathbf{w}_r \in A\mathbf{x}_r$ such that

$$\operatorname{Re}(\mathbf{y}-\mathbf{w}_r, \mathbf{z}-\mathbf{x}_r) \le 0 \tag{(2.3)}$$

for all $\mathbf{z}\in K_r$. Letting $\mathbf{z}=\mathbf{0}$,

$$\operatorname{Re}(\mathbf{w}_r,\mathbf{x}_r) \le \operatorname{Re}(\mathbf{y},\mathbf{x}_r).$$

Therefore,

$$\inf\left\{\frac{\operatorname{Re}(\mathbf{w},\mathbf{x}_r)}{|\mathbf{x}_r|} : \mathbf{w}\in A\mathbf{x}_r\right\} \le |\mathbf{y}|.$$

It follows from the assumption of coercivity that $|\mathbf{x}_r|$ is bounded independent of r. Therefore, picking r strictly larger than this bound, (2.3) implies

$$\operatorname{Re}(\mathbf{y}-\mathbf{w}_r,\mathbf{v}) \le 0$$

for all $\mathbf{v}$ in some open ball containing $\mathbf{0}$. Therefore, for all $\mathbf{v}$ in this ball

$$\operatorname{Re}(\mathbf{y}-\mathbf{w}_r,\mathbf{v}) = 0$$

and hence this holds for all $\mathbf{v}\in\mathbb{C}^n$ and so $\mathbf{y}=\mathbf{w}_r\in A\mathbf{x}_r$. This proves the lemma.

Lemma 2.4 *Let F be a finite dimensional Banach space of dimension n, and let T be a mapping from F to $\mathcal{P}(F')$ such that (2.1) and (2.2) both hold for F' in place of $\mathbb{C}^n$. Then if T is also coercive,*

$$\lim_{||\mathbf{u}||\to\infty} \inf\left\{\frac{\operatorname{Re}\mathbf{y}^*(\mathbf{u})}{||\mathbf{u}||} : \mathbf{y}^*\in T\mathbf{u}\right\} = \infty, \tag{(2.4)}$$

it follows T is onto.

Proof: Let $|\cdot|$ be an equivalent norm for F such that there is an isometry of $\mathbb{C}^n$ and F, θ. Now define $A : \mathbb{C}^n \to \mathcal{P}(\mathbb{C}^n)$ by $A\mathbf{x} \equiv \theta^* T \theta \mathbf{x}$.

$$\begin{array}{ccc} \mathcal{P}(F') & \stackrel{\theta^*}{\to} & \mathcal{P}(\mathbb{C}^n) \\ T \uparrow & \circ & \uparrow A \\ F & \stackrel{\theta}{\leftarrow} & \mathbb{C}^n \end{array}$$

Thus $\mathbf{y} \in A\mathbf{x}$ means that there exists $\mathbf{z}^* \in T\theta\mathbf{x}$ such that

$$(\mathbf{w}, \mathbf{y})_{\mathbb{C}^n} = \mathbf{z}^* (\theta \mathbf{w})$$

for all $\mathbf{w} \in \mathbb{C}^n$. Then A satisfies the conditions of Lemma 2.3 and so A is onto. Consequently T is also onto.

With these lemmas, it is possible to prove a very useful result about a class of mappings which map a reflexive Banach space to the power set of its dual space. For more theorems about these mappings and their applications, see [1]. In the discussion below, we will use the symbol, $\rightharpoonup$, to denote weak convergence.

2.2 Set valued pseudomonotone maps

Definition 2.5 *Let V be a Reflexive Banach space. We say $A : V \to \mathcal{P}(V')$ is pseudomonotone if the following conditions hold.*

$$Au \text{ is closed, nonempty, convex, and bounded.} \tag{(2.5)}$$

If F is a finite dimensional subspace of V, then if $u \in F$ and $W \supseteq Au$ for W a weakly open set in V', then there exists $\delta > 0$ such that

$$v \in B(u, \delta) \cap F \text{ implies } Av \subseteq W. \tag{(2.6)}$$

If $u_k \rightharpoonup u$ and if $u_k^* \in Au_k$ is such that

$$\limsup_{k \to \infty} \operatorname{Re} u_k^* (u_k - u) \leq 0,$$

then for all $v \in V$, there exists $u^*(v) \in Au$ such that

$$\liminf_{k \to \infty} \operatorname{Re} u_k^* (u_k - v) \geq \operatorname{Re} u^*(v)(u - v). \tag{(2.7)}$$

We say A is coercive if **??** *holds.*

Definition 2.6 *Let $A : V \to \mathcal{P}(V')$ be a set valued map. We say A is bounded if for each bounded set, $G \subseteq V$,*

$$\sup \{||z^*|| : z^* \in Ax, x \in G\} < \infty$$

Thus A is bounded if the norms of all possible elements of Ax for $x \in G$ are bounded above.

Condition (2.6) is fussy and should be avoided if possible. Fortunately, it is a special case of something that can be proved if (2.5) and (2.7) both hold and A is bounded.

Lemma 2.7 *Let $A : V \to \mathcal{P}(V')$ satisfy (2.5) and (2.7) and suppose A is also bounded. Then if $x_n \to x$ in V, and if U is a weakly open set containing Ax, then $Ax_n \subseteq U$ for all n large enough.*

Proof: We give the proof in the case when V is a real space and note that minor modifications will provide a proof in the case when V is complex. If this is not true, there exists $x_n \to x$, a weakly open set, U containing Ax and $z_n^* \in Ax_n$ but $z_n^* \notin U$. Taking a subsequence if necessary, we obtain a sequence which satisfies $z_n^* \rightharpoonup z^* \notin U$ in addition to this. Then

$$\limsup_{n\to\infty} z_n^* (x_n - x) = 0$$

so if $y \in V$ there exists $z^* (y) \in Ax$ such that

$$z^* (x - y) = \lim \inf_{n\to\infty} z_n^* (x_n - y) \geq z^* (y) (x - y) .$$

Letting $w = x - y$, this shows, since $y \in V$ is arbitrary, that the following inequality holds for every $w \in V$.

$$z^* (w) \geq z^* (x - w) (w) .$$

In particular, we may replace w with $-w$ and obtain

$$z^* (-w) \geq z^* (x + w) (-w) ,$$

which implies

$$z^* (x - w) (w) \leq z^* (w) \leq z^* (x + w) (w) .$$

Therefore, there exists

$$z_\lambda^* (y) \equiv \lambda z^* (x - w) + (1 - \lambda) z^* (x + w) \in Ax$$

such that $z^* (w) = z_\lambda^* (y) (w)$. But this is a contradiction to $z \notin Ax$ because if $z \notin Ax$ there exists $w \in V$ such that $z^* (w) > z_1^* (w)$ for all $z_1^* \in Ax$. This follows from the separation theorems in locally convex topological vector space. Therefore, $z^* \in Ax$ which contradicts the assumption that z_n^* and consequently z^* are not contained in U. This proves the lemma.

One of the nice things about pseudomonotone operators is that the sum of finitely many of them is a pseudomonotone operator. We give a proof of this in the case when the two operators satisfy (2.5) and (2.7) and are both bounded. The claim is made in [1] that it holds in the general case but I was unable to prove it.

Theorem 2.8 *Let A and B be set valued operators which are bounded and map V to $\mathcal{P}(V')$. Suppose that each of these operators satisfies (2.5) and (2.7). Then their sum is bounded and satisfies (2.5) and (2.7). Therefore, their sum is pseudomonotone.*

Proof: We assume for simplicity of presentation that V is a real space but the same result holds for complex spaces. We first must verify (2.5) for $A+B$. It is clear that $(A + B) (x)$ is convex and nonempty. It is obvious that $A + B$ is bounded. We show $(A + B) (x)$ is closed. Suppose then that $x_n^* \in Ax$ and $y_n^* \in Bx$ so that $x_n^* + y_n^* \in (A + B) (x)$. Also suppose $x_n^* + y_n^* \to z^*$. We need to show $z^* \in (A + B) (x)$. We know $\{x_n^*\}$ $\{y_n^*\}$are bounded sequences and so we can take subsequences, still denoted by n such that

$$x_n^* \rightharpoonup x^* \in A (x) , \; y_n^* \rightharpoonup y^* \in B (x) .$$

The reason $x^* \in A (x)$ is that $A (x)$ is closed and convex, hence weakly closed. Therefore, by uniqueness of limits,

$$z^* = x^* + y^* \in (A + B) (x) .$$

This verifies (2.5) for $A + B$.

Now we need to verify (2.7) for $A+B$. Suppose then that $x_n \rightharpoonup x$ and

$$\lim\sup_{n\to\infty} \left(x_n^* + y_n^*\right)(x_n - x) \leq 0 \tag{(2.8)}$$

where $x_n^* + y_n^* \in (A+B)(x_n), x_n^* \in Ax_n, y_n^* \in Bx_n$.

Claim: $\lim\sup_{n\to\infty} x_n^*(x_n - x) \leq 0$ and $\lim\sup_{n\to\infty} y_n^*(x_n - x) \leq 0$.

Proof of the claim: If this is not so, $\lim\sup_{n\to\infty} x_n^*(x_n - x) = p > 0$. Then we can take a subsequence, still denoted by n such that

$$\lim_{n\to\infty} x_n^*(x_n - x) = p > 0.$$

Then from (2.8) we must have

$$\lim\sup_{n\to\infty} y_n^*(x_n - x) < 0 \tag{(2.9)}$$

and so for any v,there exists $y^*(v) \in Bx$ such that

$$\lim\inf_{n\to\infty} y_n^*(x_n - v) \geq y^*(v)(x - v)$$

In particular, we can let $v = x$ and obtain

$$\lim\inf_{n\to\infty} y_n^*(x_n - x) \geq y^*(x)(x - x) = 0$$

contradicting (2.9). Similarly, $\lim\sup_{n\to\infty} y_n^*(x_n - x) \leq 0$.This proves the claim.

Now from the claim, if $v \in V$, there exist $x^*(v) \in Ax$ and $y^*(v) \in Bx$ such that

$$\begin{aligned} \lim\inf_{n\to\infty} x_n^*(x_n - v) &\geq x^*(v)(x - v) \\ \lim\inf_{n\to\infty} y_n^*(x_n - v) &\geq y^*(v)(x - v). \end{aligned}$$

Therefore,

$$\begin{aligned} \lim\inf_{n\to\infty} \left(x_n^* + y_n^*\right)(x_n - v) &\geq \lim\inf_{n\to\infty} x_n^*(x_n - v) + \\ \lim\inf_{n\to\infty} y_n^*(x_n - v) &\geq x^*(v)(x - v) + y^*(v)(x - v) \\ &= \left(x^*(v) + y^*(v)\right)(x - v). \end{aligned}$$

Since $x^*(v) + y^*(v) \in (A+B)(x)$, this proves the theorem.

Theorem 2.9 *Let V be a reflexive Banach space and let $A : V \to \mathcal{P}(V')$ be pseudomonotone and coercive. Then A is onto.*

Proof: Let $\mathcal{F}$ be the set of finite subspaces of V and let $F \in \mathcal{F}$. Then define A_F as

$$A_F \equiv i_F^* A i_F$$

where here i_F is the identity map from F to V. Then A_F satisfies the conditions of Lemma 2.4 and so A_F is onto $\mathcal{P}(F')$. Let $w^* \in V'$. Then since A_F is onto, there exists $u_F \in F$ such that

$$i_F^* w^* \in i_F^* A i_F u_F.$$

Thus for each finite dimensional subspace, F, there exists $u_F \in F$ such that for all $v \in F$,

$$w^*(v) = u_F^*(v),\ u_F^* \in Au_F. \tag{(2.10)}$$

Replacing v with u_F, in (2.10),

$$\frac{u_F^*(u_F)}{||u_F||} = \frac{w^*(u_F)}{||u_F||} \leq ||w^*||.$$

Therefore, the assumption that A is coercive implies $\{u_F : F \in \mathcal{F}\}$ is bounded in V. Now define

$$W_F \equiv \cup \{u_{F'} : F' \supseteq F\}.$$

Then W_F is bounded and if $\overline{W_F} \equiv$ weak closure of W_F, then

$$\left\{\overline{W_F} : F \in \mathcal{F}\right\}$$

is a collection of nonempty weakly compact (since V is reflexive) sets having the finite intersection property because $W_F \neq \emptyset$ for each F. Thus there exists

$$u \in \cap \left\{\overline{W_F} : F \in \mathcal{F}\right\}.$$

We will show $w^* \in Au$. If $w^* \notin Au$, a closed convex set, there exists $v \in V$ such that

$$\operatorname{Re} w^*(u - v) < \operatorname{Re} u^*(u - v) \tag{(2.11)}$$

for all $u^* \in Au$. This follows from the separation theorems. (These theorems imply there exists $z \in V$ such that

$$\operatorname{Re} w^*(z) < \operatorname{Re} u^*(z)$$

for all $u^* \in Au$. Define $u - v \equiv z$.) Now let $F \supseteq \{u, v\}$. Since $u \in \overline{W_F}$, a weakly sequentially compact set, there exists a sequence, $\{u_k\}$, such that

$$u_k \rightharpoonup u, \ u_k \in W_F.$$

Then since $F \supseteq \{u, v\}$, there exists $u_k^* \in Au_k$ such that

$$u_k^*(u_k - u) = w^*(u_k - u).$$

Therefore,

$$\limsup_{k\to\infty} \operatorname{Re} u_k^*(u_k - u) = \limsup_{k\to\infty} \operatorname{Re} w^*(u_k - u) = 0.$$

It follows by the assumption A is pseudomonotone; the following holds for the v defined above in (2.11).

$$\liminf_{k\to\infty} \operatorname{Re} u_k^*(u_k - v) \geq \operatorname{Re} u^*(v)(u - v), \ u^*(v) \in Au.$$

But since $v \in F, \operatorname{Re} u_k^*(u_k - v) = \operatorname{Re} w^*(u_k - v)$ and so

$$\liminf_{k\to\infty} \operatorname{Re} u_k^*(u_k - v) = \liminf_{k\to\infty} \operatorname{Re} w^*(u_k - v) = \operatorname{Re} w^*(u - v),$$

so from (2.11),

$$\operatorname{Re} w^*(u - v) = \liminf_{k\to\infty} \operatorname{Re} u_k^*(u_k - v)$$

$$\geq \operatorname{Re} u^*(v)(u - v) > \operatorname{Re} w^*(u - v),$$

a contradiction. Thus, $w^* \in Au$ and this proves the theorem.

Now we give a well known theorem about single valued pseudomonotone operators which gives an easy way to obtain examples. We emphasize the case of real spaces but everything extends to complex spaces. You just need to do everything in terms of real parts.

2.3 Monotone and hemicontinuous operators

First we state the usual definition of single valued pseudomonotone maps which is a special case of the above set valued maps.

Definition 2.10 *For V a real Banach space, we say a bounded operator, $A : V \to V'$ is a pseudomonotone map if whenever*

$$u_n \rightharpoonup u \tag{(2.12)}$$

and

$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle \leq 0 \tag{(2.13)}$$

it follows that for all $v \in V$,

$$\liminf_{n\to\infty} \langle Au_n, u_n - v\rangle \geq \langle Au, u - v\rangle. \tag{(2.14)}$$

Definition 2.11 *We say $A : V \to V'$ is monotone if for all $v, u \in V$,*

$$\langle Au - Av, u - v\rangle \geq 0,$$

and we say A is Hemicontinuous if for all $v, u \in V$,

$$\lim_{t\to 0+} \langle A\left(u + t\left(v - u\right)\right), u - v\rangle = \langle Au, u - v\rangle.$$

Theorem 2.12 *Let V be a Banach space and let $A : V \to V'$ be monotone and hemicontinuous. Then A is pseudomonotone.*

Proof: Let A be monotone and Hemicontinuous. First we prove the following claim.
Claim: If (2.12) and (2.13) hold, then $\lim_{n\to\infty}\langle Au_n, u_n - u\rangle = 0$.
Proof of the claim: Since A is monotone,

$$\langle Au_n - Au, u_n - u\rangle \geq 0$$

so

$$\langle Au_n, u_n - u\rangle \geq \langle Au, u_n - u\rangle.$$

Therefore,

$$0 = \liminf_{n\to\infty} \langle Au, u_n - u\rangle \leq \liminf_{n\to\infty} \langle Au_n, u_n - u\rangle \leq \limsup_{n\to\infty} \langle Au_n, u_n - u\rangle \leq 0.$$

Now using that A is monotone again, then letting $t > 0$,

$$\langle Au_n - A\left(u + t\left(v - u\right)\right), u_n - u + t\left(u - v\right)\rangle \geq 0$$

and so

$$\langle Au_n, u_n - u + t\left(u - v\right)\rangle \geq \langle A\left(u + t\left(v - u\right)\right), u_n - u + t\left(u - v\right)\rangle.$$

Taking the lim inf on both sides and using the claim and $t > 0$, we obtain

$$t\liminf_{n\to\infty} \langle Au_n, u - v\rangle \geq t\langle A\left(u + t\left(v - u\right)\right), \left(u - v\right)\rangle.$$

Next we divide by t and use the Hemicontinuity of A to conclude that

$$\lim\inf_{n\to\infty} \langle Au_n, u-v\rangle \geq \langle Au, u-v\rangle.$$

From the claim,

$$\begin{aligned}\lim\inf_{n\to\infty} \langle Au_n, u-v\rangle &= \lim\inf_{n\to\infty} \left(\langle Au_n, u_n - v\rangle + \langle Au_n, u-u_n\rangle\right) \\ &= \lim\inf_{n\to\infty} \langle Au_n, u_n - v\rangle \geq \langle Au, u-v\rangle.\end{aligned}$$

This proves the theorem.

Monotonicity is very important in the above proof. The next example shows that even if the operator is linear and bounded, it is not necessarily pseudomonotone.

Example 2.13 *Let H be any Hilbert space and let $A : H \to H'$ be given by*

$$\langle Ax, y\rangle \equiv (-x, y)_H .$$

Then A fails to be pseudomonotone.

Proof: Let $\{x_n\}_{n=1}^{\infty}$ be an orthonormal set of vectors in H. Then Parsevall's inequality implies

$$||x||^2 \geq \sum_{n=1}^{\infty} |(x_n, x)|^2$$

and so for any $x \in H, \lim_{n\to\infty} (x_n, x) = 0$. Thus $x_n \rightharpoonup 0 \equiv x$. Also

$$\lim\sup_{n\to\infty} \langle Ax_n, x_n - x\rangle =$$

$$\lim\sup_{n\to\infty} \langle Ax_n, x_n - 0\rangle = \lim\sup_{n\to\infty} \left(-||x_n||^2\right) = -1 \leq 0.$$

If A were pseudomonotone, we would need to be able to conclude that for all $y \in H$,

$$\lim\inf_{n\to\infty} \langle Ax_n, x_n - y\rangle \geq \langle Ax, x-y\rangle = 0.$$

However,

$$\lim\inf_{n\to\infty} \langle Ax_n, x_n - 0\rangle = -1 < 0 = \langle A0, 0-0\rangle.$$

2.4 Examples

Let Ω be a bounded open set having Lipschitz boundary. It is not necessary to assume it is bounded but in applications of most interest to me, it usually is and since the presentation is slightly easier if this assumption is made, we shall make this assumption. Let

$$a_i(\cdot,\cdot,\cdot) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R},\ i = 0, \cdots, n$$

be such that a_i is continuous in the second and third arguments and Borel measurable in the first. We will assume the following on the functions a_i.

$$\sum_{i=1}^{n} a_i(x, u, \mathbf{p}) p_i + a_0(x, u, \mathbf{p}) u \geq C_2 |\mathbf{p}|^p - k_2(x), \tag{(2.15)}$$

where $k_2 \in L^1(\Omega)$.

$$|a_j(x, u, \mathbf{p})| \leq C_1 \left(|\mathbf{p}|^{p-1} + |u|^{p-1} \right) + k_1(x), \tag{(2.16)}$$

for $k_1 \in L^{p'}(\Omega)$. And finally

$$\sum_i [a_i(x, u, \mathbf{p}) - a_i(x, u, \mathbf{q})][p_i - q_i] > 0 \tag{(2.17)}$$

whenever $\mathbf{p} \neq \mathbf{q}$.

Let V be a closed subspace of $W^{1,p}(\Omega)$, $p > 1$, and define an operator, $A : V \to V'$ as follows.

$$\langle Au, v \rangle \equiv \int_\Omega \sum_{i=1}^n a_i(x, u, \nabla u) \partial_i v + a_0(x, u, \nabla u) v dx. \tag{(2.18)}$$

Theorem 2.14 *The operator, A is pseudomonotone.*

Proof: We need to show that if

$$u_n \rightharpoonup u \text{ in } V, \tag{(2.19)}$$

and if

$$\limsup_{n\to\infty} \langle Au_n, u_n - u \rangle = \limsup_{n\to\infty} \langle Au_n - Au, u_n - u \rangle \leq 0, \tag{(2.20)}$$

then for any $v \in V$,

$$\liminf_{n\to\infty} \langle Au_n, u_n - v \rangle \geq \langle Au, u - v \rangle. \tag{(2.21)}$$

Suppose that (2.19) and (2.20) both hold and yet

$$\liminf_{n\to\infty} \langle Au_n, u_n - v \rangle < \langle Au, u - v \rangle. \tag{(2.22)}$$

Since u_n converges weakly to u in V, it follows u_n converges strongly to u in $L^p(\Omega)$ and so by taking a subsequence, still denoted by u_n we can obtain

$$\lim_{n\to\infty} \langle Au_n, u_n - v \rangle = \liminf_{n\to\infty} \langle Au_n, u_n - v \rangle$$

and

$$u_n(x) \to u(x) \text{ a.e.} \tag{(2.23)}$$

say for all $x \notin N$ where $m(N) = 0$. From now on, we will only consider $x \notin N$. Also, to save space, we sum over repeated indices.

$$\begin{aligned} p_n(x) \equiv\ & [a_i(x, u_n(x), \nabla u_n(x)) - a_i(x, u(x), \nabla u(x))][\partial_i u_n(x) - \partial_i u(x)] + \\ & [a_0(x, u_n(x), \nabla u_n(x)) - a_0(x, u(x), \nabla u(x))][[u_n(x) - u(x)]] \end{aligned}$$

$$= a_i(x, u_n(x), \nabla u_n(x)) \partial_i u_n(x) - a_i(x, u_n(x), \nabla u_n(x)) \partial_i u(x)$$

$$-a_i(x, u(x), \nabla u(x)) \partial_i u_n(x) +$$

$$a_i\left(x,u\left(x\right),\nabla u\left(x\right)\right)\partial_i u\left(x\right)+a_0\left(x,u_n\left(x\right)\nabla u_n\left(x\right)\right)u_n\left(x\right)$$

$$-a_0\left(x,u_n\left(x\right),\nabla u_n\left(x\right)\right)u\left(x\right)-a_0\left(x,u\left(x\right),\nabla u\left(x\right)\right)u_n\left(x\right)$$

$$+a_0\left(x,u\left(x\right),\nabla u\left(x\right)\right)u\left(x\right). \qquad ((2.24))$$

From the assumed estimates and routine inequalities, this is larger than an expression of the form

$$\geq \delta\left|\nabla u_n\left(x\right)\right|^p-C\left(x\right)-C\left(u\left(x\right)\right)-\left|u_n\left(x\right)\right|^p. \qquad ((2.25))$$

We will also need to write $p_n\left(x\right)$ in the following form.

$$p_n\left(x\right)\equiv$$

$$\left[a_i\left(x,u_n\left(x\right),\nabla u_n\left(x\right)\right)-a_i\left(x,u_n\left(x\right),\nabla u\left(x\right)\right)\right]\left[\partial_i u_n\left(x\right)-\partial_i u\left(x\right)\right]+$$

$$\left[a_0\left(x,u_n\left(x\right),\nabla u_n\left(x\right)\right)-a_0\left(x,u\left(x\right),\nabla u\left(x\right)\right)\right]\left[\left[u_n\left(x\right)-u\left(x\right)\right]\right]+$$

$$\left[a_i\left(x,u_n\left(x\right),\nabla u\left(x\right)\right)-a_i\left(x,u\left(x\right),\nabla u\left(x\right)\right)\right]\left[\partial_i u_n\left(x\right)-\partial_i u\left(x\right)\right]. \qquad ((2.26))$$

$$=e_n\left(x\right)+f_n\left(x\right)+g_n\left(x\right)$$

We see that $e_n\left(x\right)\geq 0$ from (2.17).

Also we see from (2.16) that g_n is uniformly integrable. To see this consider its terms. Take for example the term

$$a_i\left(x,u_n\left(x\right),\nabla u\left(x\right)\right)\partial_i u_n\left(x\right). \qquad ((2.27))$$

From the fact that $\left|\left|u_n\right|\right|_V$ is bounded, (due to the assumed weak convergence of u_n), we see from the estimate (2.16) that for S a measurable set,

$$\int_S\left|a_i\left(x,u_n\left(x\right),\nabla u\left(x\right)\right)\partial_i u_n\left(x\right)\right|dx\leq\int_S\left(C_1\left[\left|\nabla u\left(x\right)\right|^{p-1}+\left|u_n\left(x\right)\right|^{p-1}\right]+\left|k_1\left(x\right)\right|\right)\left|\partial_i u_n\left(x\right)\right|dx$$

$$\leq C\left(\int_S\left|\nabla u\right|^p+\left|u_n\right|^p+\left|k_1\right|^p\right)^{1/p'},$$

where C is a constant which depends on $\sup\left\{\left|\left|u_n\right|\right|_V\right\}$. By the Sobolev embedding theorem, $\left|u_n\right|^p$ is actually bounded in $L^r\left(\Omega\right)$ for some $r>1, r$ depending on the dimension, and so $\left|u_n\right|^p$ is uniformly integrable. It follows this expression is small whenever $m\left(S\right)$ is small enough, showing this term of $g_n\left(x\right)$ is uniformly integrable. Similar arguments apply to the other terms. Denoting by $\alpha_n\left(x\right)$, the expression $a_i\left(x,u_n\left(x\right),\nabla u\left(x\right)\right)-a_i\left(x,u\left(x\right),\nabla u\left(x\right)\right)$ and by $\beta_n\left(x\right)$ the expression $\left[\partial_i u_n\left(x\right)-\partial_i u\left(x\right)\right]$, we see that $g_n=\alpha_n\beta_n$ where $\alpha_n\left(x\right)\to 0$ a.e. and β_n is bounded in $L^p\left(\Omega\right)$. Also (2.16) implies α_n is uniformly integrable and so there exists a constant, C, depending on $\sup\left\{\left|\left|\beta_n\right|\right|_{L^p}\right\}$ such that

$$m\left[\left|\alpha_n\beta_n\right|>\varepsilon\right]\leq\frac{C}{\varepsilon}\left(\int_\Omega\left|\alpha_n\right|^p\right)^{1/p'}.$$

Now by the Vitali convergence theorem the right side converges to zero as $n \to \infty$ showing convergence in measure to zero. Therefore, by the theorem of F. Riesz, there exists a subsequence still denoted by n such that $\alpha_n(x)\beta_n(x) \to 0$ a.e. From now on, we will use this subsequence. Then from the Vitali convergence theorem, we have

$$\lim_{n\to\infty} \int_\Omega |g_n(x)|\, dx = 0. \tag{(2.28)}$$

Next we consider the term, $f_n(x)$. By Holder's inequality,

$$\int_\Omega |f_n(x)|\, dx \leq C \left(\int_\Omega |u_n(x) - u(x)|^p\, dx \right)^{1/p},$$

where C is some constant depending on $\sup\{||u_n||_V\}$. Since $u_n \to u$ in $L^p(\Omega)$, this shows

$$\lim_{n\to\infty} \int_\Omega |f_n(x)|\, dx = 0. \tag{(2.29)}$$

Claim: For all $x \notin N$,

$$\lim\inf_{n\to\infty} p_n(x) \geq 0. \tag{(2.30)}$$

Proof: If not, there is a subsequence, still called n such that

$$\lim_{n\to\infty} p_n(x) = \lim\inf_{n\to\infty} p_n(x) < 0.$$

It follows from the inequality (2.25) and

$$\lim_{n\to\infty} p_n(x) < 0$$

that $|\nabla u_n(x)|^p$ is bounded independent of n for each $x \notin N$. From (2.26), and the observation just made that $|\nabla u_n(x)|$ is bounded, the last two terms of (2.26) converge to zero and so by the hypothesis, (2.17),

$$\lim\inf_{n\to\infty} p_n(x) \geq 0$$

after all. This contradiction proves the claim.

We have just shown that for all $x \notin N$,

$$\lim_{n\to\infty} p_n(x)^- = 0. \tag{(2.31)}$$

Since $e_n \geq 0$,

$$(f_n + g_n)^- \geq p_n^- \geq 0,$$

and so by (2.28) and (2.29)

$$\lim_{n\to\infty} \int_\Omega (f_n(x) + g_n(x))^-\, dx \geq \lim_{n\to\infty} \int_\Omega p_n(x)^-\, dx = 0. \tag{(2.32)}$$

Now from (2.20) and the definition of the operator A, we see that

$$\begin{aligned} 0 &\geq \lim\sup_{n\to\infty} \int_\Omega p_n(x)\, dx = \lim\sup_{n\to\infty} \left(\int_\Omega p_n(x)^+\, dx - \int_\Omega p_n(x)^-\, dx \right) \\ &= \lim\sup_{n\to\infty} \int_\Omega p_n(x)^+\, dx - \lim\inf_{n\to\infty} \int_\Omega p_n(x)^-\, dx = \lim\sup_{n\to\infty} \int_\Omega p_n(x)^+\, dx. \end{aligned}$$

Thus from (2.32) we see that we have obtained

$$\int_\Omega p_n(x)^+ \, dx, \int_\Omega p_n(x)^- \to 0, \tag{(2.33)}$$

and hence

$$\int_\Omega |p_n(x)| \, dx \to 0. \tag{(2.34)}$$

Therefore, taking a further subsequence, we obtain

$$\lim_{n\to\infty} p_n(x) = 0 \text{ a.e.}$$

Now that we have shown that $\lim_{n\to\infty} p_n(x) = 0$, we see from (2.25) that $\nabla u_n(x)$ is bounded independent of n for a.e. x.

Claim: $\nabla u_n(x) \to \nabla u(x)$ a.e.

Proof of the claim: If not, we could get another subsequence such that $\nabla u_n(x) \to \mathbf{q} \neq \nabla u(x)$. But then from (2.26), upon taking the limit as $n \to \infty$ in this expression, we obtain for a.e. x,

$$[a_i(x, u(x), \mathbf{q}) - a_i(x, u(x), \nabla u(x))][q_i - \partial_i u(x)] = 0,$$

showing that from (2.17), $\mathbf{p} = \nabla u(x)$ after all. This establishes the claim.

Now we return to (2.22).

$$\lim_{n\to\infty} \langle Au_n, u_n - v\rangle = \lim_{n\to\infty} (\langle Au_n, u_n - u\rangle + \langle Au_n, u - v\rangle)$$

$$= \lim_{n\to\infty} \left(\int_\Omega p_n(x) \, dx + \int_\Omega a_i(x, u_n(x), \nabla u_n(x)) (\partial_i u(x) - \partial_i v(x)) \, dx \right.$$

$$\left. + \int_\Omega a_0(x, u_n(x) \nabla u_n(x)) (u(x) - v(x)) \, dx \right)$$

$$= \int_\Omega a_i(x, u(x), \nabla u(x)) (\partial_i u(x) - \partial_i v(x)) \, dx + \int_\Omega a_0(x, u(x) \nabla u(x)) (u(x) - v(x)) \, dx$$

because the integrands on the second two integrals are uniformly integrable and they converge pointwise. This equals

$$= \langle Au, u - v\rangle.$$

But this is a contradiction to the assumption that

$$\lim\inf_{n\to\infty} \langle Au_n, u_n - v\rangle < \langle Au, u - v\rangle.$$

This proves the theorem.

For a more general theorem which is proved by a similar method see [2]. Here is an example of this sort which is set valued.

Let V be a closed subspace of $H^1(\Omega)$, We define an operator, $A(\cdot)$ mapping V to V' as follows.

$$\langle A(u), v\rangle \equiv \int_\Omega (\sum_{i=1}^{3} a_i(x, u, \nabla u) \partial_i v) + a_0(x, u, \nabla u) v \, dx \,.$$

We make the following assumptions on the functions a_i.

1. $x \to a_i(x, \mathbf{z})$ is measurable while $\mathbf{z} \to a_i(x, \mathbf{z})$ is continuous.
2. There exist constants, $C_1 > 0$ and a function $k_1 \in L^2(\Omega)$ such that
$$|a_i(x, u, \mathbf{p})| \leq C(|\mathbf{p}| + |u|) + k_1(t, x),$$
and for some λ,
$$\begin{aligned} &\sum_{i=1}^{3} a_i(x, u, \mathbf{p})p_i + a_0(x, u, \mathbf{p})u \\ &\geq \quad C_1(|\mathbf{p}|^2 + |u|^2) \end{aligned} \tag{(2.35)}$$
3. For all $(u, \mathbf{p})$ and $(u, \widehat{\mathbf{p}})$, if $\mathbf{p} \neq \widehat{\mathbf{p}}$
$$\sum_{i=1}^{3}(a_i(x, u, \mathbf{p}) - a_i(x, u, \widehat{\mathbf{p}}))(p_i - \widehat{p}_i) > 0\,.$$

Now we define a set valued operator, $G : \Omega \times \mathbb{R} \to \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ as follows.

1. $G(x, u) = [g_1(x, u), g_2(x, u)]$ where $u \to G(x, u)$ is upper semi-continuous, meaning that for each x,
$$G(x, v) \subseteq [g_1(x, u) - \varepsilon, g_2(x, u) + \varepsilon]$$
whenever $|v - u|$ is small enough. Also assume $g_i : \Omega \times \mathbb{R} \to \mathbb{R}$ is Borel measurable.
2. There exist $C_3 > 0$ and $C_4 \in [0, C_1), k_3 \in L^2(\Omega)$ such that G satisfies the estimates
$$|G\,(x, u)| = \max\{|g_1\,(x, u)|\,, |g_2\,(x, u)|\} \leq C_3\,|u| + k_3\,(x)\,,$$
$$\inf\{ug_1\,(x, u)\,, ug_2\,(x, u)\} \geq -C_4\,|u|^2 \tag{(2.36)}$$
We define $w \in G(u)$ if and only if $w(x) \in G(x, u(x))$ a.e. so $G : V \to \mathcal{P}(H) \subseteq \mathcal{P}(V')$ where $H \equiv L^2(\Omega)$.

Theorem 2.15 *The operator, $A+G$ is pseudomonotone bounded and coercive from V to $\mathcal{P}(V')$. Therefore, if $f \in V'$, there exists a solution, $u \in V$ to*
$$(A + G)\,(u) \ni f$$

Proof: We know from the earlier material that A is pseudomonotone. It remains to verify that G is pseudomonotone. Then we will use Theorem 2.8 about the sum of two pseudomonotone operators. The estimates imply G is bounded and that $G\,(u)$ is a closed nonempty convex subset of H. We need to verify the limit condition for G, (2.7).

Let $u_n \rightharpoonup u$ in V and let $u_n^* \in G\,(u_n)$. Since V is embedded compactly in $H \equiv L^2\,(\Omega)$, it follows $u_n \to u$ in H. Also, we know u_n^* is bounded in H and so by compactness of the embedding of H into V' it follows we can take a subsequence for which $u_n^* \to u_\infty^*$ in V'. Taking a further subsequence, we can also have $u_n\,(x) \to u\,(x)$ pointwise a.e. We verify $u_\infty^* \in G\,(u)$. Define
$$B_\varepsilon \equiv \{x : u_\infty^*\,(x) \geq g_2\,(x, u\,(x)) + \varepsilon\}\,.$$
From the definition of G,
$$-\int_{B_\varepsilon} C_3\,|u_n\,(x)| + k_3\,(x)\,dx \leq \int_{B_\varepsilon} -u_n^*\,(x)\,dx -$$

$$\int_{B_\varepsilon} \left[C_3 \left|u_n (x)\right| + k_3 (x) - g_2 (x, u_n (x))\right] dx$$

and so taking lim sup on both sides using the assumed weak convergence of u_n^* to u_∞^*,

$$-\int_{B_\varepsilon} C_3 \left|u_n (x)\right| + k_3 (x)\, dx \leq \int_{B_\varepsilon} -u_\infty^* (x)\, dx -$$

$$\liminf \int_{B_\varepsilon} \left[C_3 \left|u_n (x)\right| + k_3 (x) - g_2 (x, u_n (x))\right] dx.$$

Now from Fatou's lemma and the upper semicontinuity of $u \to G(x,u)$, we obtain

$$-\int_{B_\varepsilon} C_3 \left|u_n (x)\right| + k_3 (x)\, dx \leq \int_{B_\varepsilon} -u_\infty^* (x)\, dx -$$

$$\int_{B_\varepsilon} \left[C_3 \left|u (x)\right| + k_3 (x) - \limsup g_2 (x, u_n (x))\right] dx$$

$$\leq \int_{B_\varepsilon} -u_\infty^* (x)\, dx - \int_{B_\varepsilon} \left[C_3 \left|u (x)\right| + k_3 (x) - g_2 (x, u (x))\right] dx,$$

which implies

$$0 \leq \int_{B_\varepsilon} g_2 (x, u (x)) - u_\infty^* (x)\, dx \leq -\varepsilon m (B_\varepsilon)$$

showing that $m(B_\varepsilon) = 0$. Therefore, letting ε be replaced with $1/n$, taking the union of $B_{1/n}$, we see that $u_\infty^* (x) \leq g_2 (x, u(x))$ a.e. and a similar argument shows $u_\infty^* (x) \geq g_1 (x, u(x))$ a.e.

We have shown that if $u_n \rightharpoonup u$ in V, $u_n^* \in G(u_n)$,there exists a subsequence, u_{n_k} such that $u_{n_k}^* \rightharpoonup u_\infty^* \in G(u)$, the weak convergence occurring in H. Therefore by the strong convergence of $u_{n_k}^*$ to u_∞^* in V', and the weak convergence of u_{n_k} to u in V, it follows that for any $v \in V$,

$$\lim_{k\to\infty} u_{n_k}^* (u_{n_k} - v) = u_\infty^* (u - v).$$

If for some $v \in V$, there exists a sequence, $u_n \rightharpoonup u$ in V such that

$$\lim \inf_{k\to\infty} u_n^* (u_n - v) < u^* (u - v)$$

for all $u^* \in G(u)$, this is a contradiction to what was just shown. Therefore, we have verified (2.7). Now by Theorem 2.8. The boundedness and coercivity of $A + G$ comes from the given assumptions. Therefore, by Theorem 2.9, we have proved the theorem.

Depending on the choice of V, we obtain weak solutions to the nonlinear inclusion,

$$-\sum_{i=1}^{3} \partial_i (a_i (x, u, \nabla u)) + a_0 (x, u, \nabla u) \in -G(x, u)$$

along with boundary conditions determined by the variational form of the equations and the choice of V and f.

2.5 Generalized gradients

Definition 2.16 *Let V be a real reflexive Banach space and let $f : V \to \mathbb{R}$ be a locally Lipschitz function, meaning that f is Lipschitz near every point of V although f need not be Lipschitz on all of V. Under these conditions,*

$$f^0(x,y) \equiv \lim_{\mu \to 0+} \sup_{h \to 0} \frac{f(x+h+\mu y) - f(x+h)}{\mu} \tag{(2.37)}$$

and $\partial f(x) \subseteq X'$ is defined by

$$\partial f(x) \equiv \left\{ x^* \in X' : x^*(y) \leq f^0(x,y) \text{ for all } y \in X \right\}. \tag{(2.38)}$$

The set just described is called the generalized gradient. In (2.37) we mean the following by the right hand side.

$$\lim_{(r,\delta)\to(0,0)} \sup \left\{ \frac{f(x+h+\mu y) - f(x+h)}{\mu} : \mu \in (0,r), h \in B(0,\delta) \right\}$$

We will show, following [1], that these generalized gradients of locally Lipschitz functions are sometimes pseudomonotone. First we give a lemma.

Lemma 2.17 *Let f be as described in the above definition. Then $\partial f(x)$ is a closed, bounded, convex, and non empty subset of V'.*

Proof: It is left as an exercise to verify the assertions that $\partial f(x)$ is closed, and convex. It follows directly from the definition. To verify this set is bounded, let $Lip_x(f)$ denote a Lipschitz constant valid near $x \in V$ and let $x^* \in \partial f(x)$. Then choosing y with $||y|| = 1$ and $x^*(y) \geq \frac{1}{2} ||x^*||$,

$$\frac{1}{2} ||x^*|| = x^*(y) \leq f^0(x,y). \tag{(2.39)}$$

Also, for small μ and h,

$$\left| \frac{f(x+h+\mu y) - f(x+h)}{\mu} \right| \leq Lip_x(f) ||y|| = Lip_x(f).$$

Therefore, $f^0(x,y) \leq Lip_x(f)$ and so (2.39) shows $||x^*|| \leq 2Lip_x(f)$.

The interesting part of this Lemma is that $\partial f(x) \neq \emptyset$. To verify this first note that the definition of f^0 implies that $y \to f^0(x,y)$ is a gauge function. Now fix $y \in V$ and define on $\mathbb{R}y$ a linear map x_0^* by

$$x_0^*(\alpha y) \equiv \alpha f^0(x,y).$$

Then if $\alpha \geq 0$,

$$x_0^*(\alpha y) = \alpha f^0(x,y) = f^0(x,\alpha y).$$

If $\alpha < 0$,

$$x_0^*(\alpha y) \equiv \alpha f^0(x,y) =$$

$$\lim_{\mu \to 0+} \inf_{h \to 0} \frac{(-\alpha) f(x+h) - (-\alpha) f(x+h+\mu y)}{\mu} =$$

$$(-\alpha)\lim_{\mu\to 0+}\inf_{h\to 0}\frac{f(x+h-\mu y)-f(x+h)}{\mu}\leq$$

$$(-\alpha)f^{0}(x,-y)=f^{0}(x,\alpha y).$$

Therefore, $x_0^*(\alpha y)\leq f^0(x,\alpha y)$ for all α. By the Hahn Banach theorem there is an extension of x_0^* to all of V, x^* which satisfies,

$$x^*(y)\leq f^0(x,y)$$

for all y. It remains to verify x^* is continuous. This follows easily from

$$|x^*(y)|=\max(x^*(-y),x^*(y))\leq$$

$$\max\left(f^0(x,y),f^0(x,-y)\right)\leq Lip_x(f)\,||y||.$$

This proves the lemma.

This lemma has verified the first condition needed in the definition of pseudomonotone. The next lemma verifies that these generalized subgradients satisfy the second of the conditions needed in the definition. In fact, we show somewhat more than is needed in the definition.

Lemma 2.18 *Let U be weakly open in V' and suppose $\partial f(x)\subseteq U$. Then $\partial f(z)\subseteq U$ whenever z is close enough to x.*

Proof: Suppose to the contrary there exists $z_n\to x$ but $z_n^*\in\partial f(z_n)\setminus U$. From the first lemma, we may assert that $||z_n^*||\leq 2Lip(f)$ for all n large enough. Therefore, there is a subsequence, still denoted by n such that z_n^* converges weakly to $z^*\notin U$.

Claim: $f^0(x,y)\geq\limsup_{n\to\infty}f^0(x_n,y)$.

Proof of the claim: There exists $\delta>0$ such that if $\mu,||h||<\delta$, then

$$\varepsilon+f^0(x,y)\geq\frac{f(x+h+\mu y)-f(x+h)}{\mu}.$$

Thus, for $||h||<\delta$,

$$\varepsilon+f^0(x,y)\geq\frac{f(x_n+(x-x_n)+h+\mu y)-f(x_n+(x-x_n)+h)}{\mu}.$$

Now let $||h'||<\frac{\delta}{2}$ and let n be so large that $||x-x_n||<\frac{\delta}{2}$. Suppose$||h'||<\frac{\delta}{2}$. Then choosing $h\equiv h'-(x-x_n)$, it follows the above inequality holds because $||h||<\delta$. Therefore, if $||h'||<\frac{\delta}{2}$, and n is sufficiently large,

$$\varepsilon+f^0(x,y)\geq\frac{f(x_n+h'+\mu y)-f(x_n+h')}{\mu}.$$

Consequently, for all n large enough,

$$\varepsilon+f^0(x,y)\geq f^0(x_n,y)$$

which proves the claim.

Now with the claim,

$$z^*(y)=\lim\sup_{n\to\infty}z_n^*(y)\leq\lim\sup_{n\to\infty}f^0(x_n,y)\leq f^0(x,y)$$

so $z^*\in\partial f(x)$ contradicting the assumption that $z^*\notin U$. This proves the lemma.

It is necessary to assume more on f^0 in order to obtain the third axiom defining pseudomonotone. The following theorem describes the situation.

Theorem 2.19 *Let $f : V \to V'$ be locally Lipschitz and suppose it satisfies the condition that whenever*

$$x_n \text{ converges weakly to } x$$

and

$$\lim\sup_{n\to\infty} f^0 (x_n, x - x_n) \geq 0$$

it follows that

$$\lim\sup_{n\to\infty} f^0 (x_n, z - x_n) \leq f^0 (x, z - x)$$

for all $z \in V$. Then ∂f is pseudomonotone.

Proof: Formulas (2.5) and (2.6) both are satisfied thanks to Lemmas 2.16 and 2.17. It remains to verify (2.7). To do so, we will adopt the convention that $x^* \in \partial f (x)$. Suppose

$$\lim\sup_{n\to\infty} x_n^* (x_n - x) \leq 0. \qquad ((2.40))$$

This implies $\lim\inf_{n\to\infty} x_n^* (x - x_n) \geq 0$. Thus,

$$0 \leq \lim\inf_{n\to\infty} x_n^* (x - x_n) \leq \lim\inf f^0 (x_n, x - x_n)$$

$$\leq \lim\sup_{n\to\infty} f^0 (x_n, x - x_n),$$

which implies, by the above assumption that for all z,

$$\lim\sup x_n^* (z - x_n) \leq \lim\sup f^0 (x_n, z - x_n) \leq f^0 (x, z - x). \qquad ((2.41))$$

In particular, this holds for $z = x$ and this implies $\lim\sup x_n^* (x - x_n) \leq 0$ which along with (2.40) yields

$$\lim_{n\to\infty} x_n^* (x_n - x) = 0 \qquad ((2.42))$$

.

Now let z be arbitrary. There exists a subsequence, n_k, depending on z such that

$$\lim_{k\to\infty} x_n^* (x_n - z) = \lim\inf x_n^* (x_n - z).$$

Now from Lemma 2.17 and its proof, the $||x_n||$ are all bounded by $Lip_x (f)$ whenever n is large enough. Therefore, there is a further subsequence, still denoted by n_k such that

$$x_{n_k}^* \text{ converges weakly to } x^* (z).$$

We need to verify that $x^* (z) \in \partial f (x)$. To do so, let y be arbitrary. Then from the definition,

$$x_n^* (y - x_n) \leq f^0 (x_n, y - x_n). \qquad ((2.43))$$

From (2.42), we can take the $\lim\sup$ of both sides and obtain, using (2.41)

$$x^* (z) (y - x) \leq \lim\sup f^0 (x_n, y - x_n) \leq f^0 (x, y - x).$$

Since y is arbitrary, this shows $x^* (z) \in \partial f (x)$ and proves the theorem.

Bibliography

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