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III

TIMED

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Name:________________________

Section: ________________________

Instructor:________________________

Math 303 (Engineering Mathematics II)

Quiz 2

Winter 2018 **TIME LIMIT 2.5 HOURS**

- Answer all question and show all your work to receive the full credit.
- Work on scratch paper **will not** be graded. Use the back if you need more space.
- Basic scientific calculator or testing center calculator are allowed (not graphing or symbolic ones).

For instructor use only:

#	Possible	Earned
1a	5	
1b	5	
2a	5	
2b	5	
Total	20	

1a. Consider the equation $\dfrac{dy}{dt} = y^2(3-y)$.

i) Without solving the differential equation, find all equilibrium solutions and classify each one as asymptotically stable, semistable or unstable.

ii) Sketch several graphs of solutions in the ty-plane; include solutions passing through the points $(0,-1),(0,1),(0,2),(0,2.5),(0,4)$;

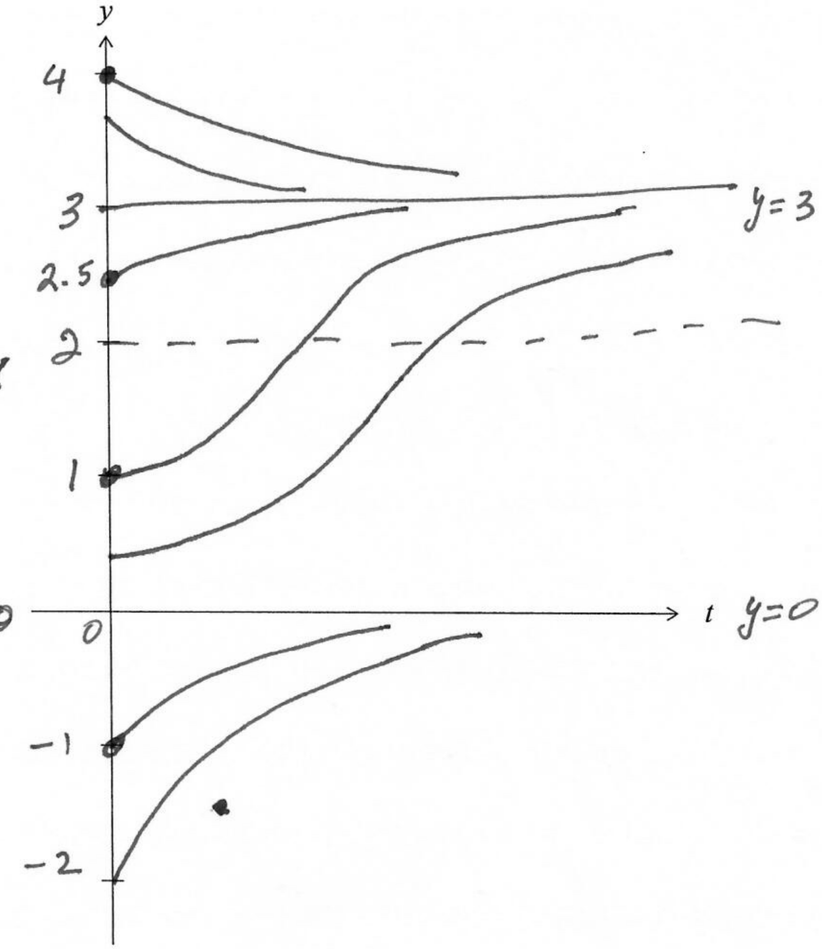
ii) determine where the graph is concave up and where it is concave down, indicating inflection points if any.

Equilibrium solns when $\dfrac{dy}{dt} = 0$

or $y(t) \equiv 0$, $y(t) \equiv 3$

$y \equiv 3$ Asympt. stable equilibrium

and $y \equiv 0$ is semistable equil.



Behavior of Solns:

a) If $y<0$, then $y' = \overset{>0}{y^2}\,\overset{>0}{(3-y)} > 0$ and $y(t)\nearrow$ increases

b) If $0<y<3$ then $y' = \overset{>0}{y^2}\,\overset{>0}{(3-y)}$ and $y \nearrow$

c) If $y>3$ then $y' = \overset{>0}{y^2}\,\overset{<0}{(3-y)} < 0$ and $y \searrow$ decreases

Then, any solution $y(t)$ such that $0<y(0)<3$ has $y''>0$, if $y<2$ Concave up and $y''<0$, if $y>2$ Concave down

Therefore, $(t^*, y(t^*)) = (t^*, 2)$ are inflect. points for each $y(t)$.

Inflection point

$y'(t) = f(y)$,

then, $y''(t) = f'(y)\cdot y' = f'(y) f(y)$

In our case,

$y''(t) = \left(2y(3-y) - y^2\right) y^2 (3-y)$

$= (6y - 3y^2) y^2 (3-y)$

Thus, $y''=0$ if $y=0$, $y=3$ and

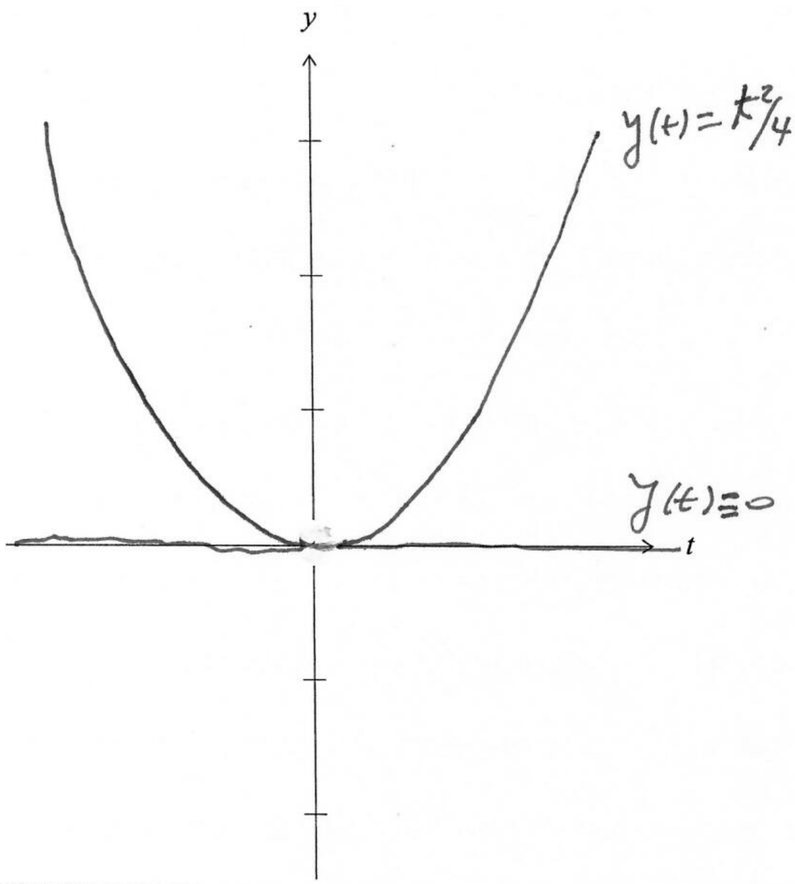
$6y-3y^2 = 0$ or $\boxed{y=2}$

$y''(t) = 3y(2-y)y^2(3-y)$

1b. Consider the initial value problem $y' = \sqrt{y}$, $y(0) = 0$.

(i) By observation, the zero function $y(t) = 0$ is a solution of this initial value problem. Find another solution of this initial value problem. Sketch both of them.

(ii) Explain why this does not violate the existence and uniqueness theorem.



i) Another Soln.

$\frac{dy}{y^{1/2}} = dt$, then $\int y^{-1/2}\,dy = \int dt$

$2y^{1/2} = t + C$, then $y^{1/2} = \frac{t+C}{2}$

or $y(t) = \left(\frac{t+C}{2}\right)^2$

Using I.C.

$0 = y(0) = \left(\frac{C}{2}\right)^2 = \frac{C^2}{4}$

and $\boxed{C=0}$

Thus,

2nd solution: $y(t) = \frac{t^2}{4}$

ii) $y' = y^{1/2}$ Not conts. as a function of t and y if y 0.

$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y}}$, discontinuous at any pair $(t, 0)$ in the ty-plane. In particular disconts at $(t, 0)$, $t \in \mathbb{R}$. and the hypothesis of the theorem does not hold.

2a. Consider differential equation $\dfrac{dy}{dx} = \dfrac{3y^2 - x^2}{2xy}$. $\quad (**)$

(i) Find all solutions of the given differential equation

It can be solved in two ways: as a homog. equ. introducing a variable $v = y/x$ or using the technique to transform an equation into an exact equation.

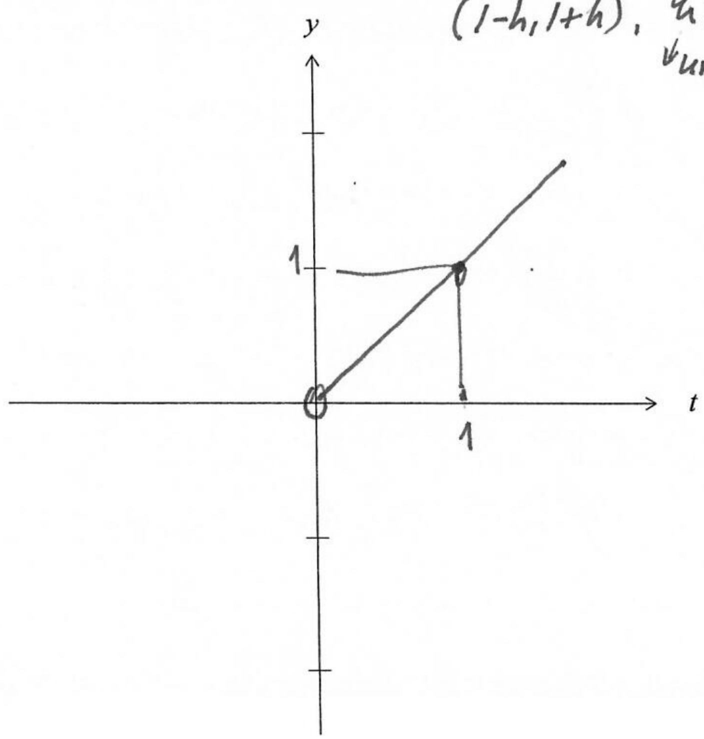
$\downarrow$ Se attached papers

(ii) Verify that initial value problem $y(1) = 1$ for this equation has a unique solution in some interval around $t = 1$

$f(y)$ and $\dfrac{\partial f}{\partial y} = \dfrac{6y(2xy) - 2y(3y^2 - x^2)}{4x^2y^2} = \dfrac{12xy - 6y^2 + 2x^2}{4x^2y}$

Since both f and $\dfrac{\partial f}{\partial y}$ are conts around $(x, y) = (1,1)$ the soln. of the diff. equ. $(**)$ exists and is unique in an interval $(1-h, 1+h)$, $h > 0$ unknown.

(iii) Find the solution subject to the initial condition $y(1) = 1$. Sketch.



2a) $$y' = \frac{3y^2 - x^2}{2xy}$$

i) $3y^2 - x^2 - 2xy\,y' = 0$ $\quad (*)$

$$M(x,y) = 3y^2 - x^2, \quad N(x,y) = -2xy$$

$$M_y = 6y, \quad N_x = -2y$$

this equ. is not exact.

Let's explore if it can be converted into an exact equation.

$$(\mu M)_y = (\mu N)_x$$

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x$$

If a) $\mu = \mu(y)$ then $\mu_x = 0$ and $\mu_y = \left(\frac{N_x - M_y}{M}\right)\mu$, | If b) $\mu = \mu(x)$, then $\mu_y = 0$ $\mu_x = \left(\frac{M_y - N_x}{N}\right)\mu$

a) If $\frac{N_x - M_y}{M} = F(y)$ we can transform the equ. into exact.

$$\frac{N_x - M_y}{M} = \frac{-2y - 6y}{3y^2 - x^2} = \frac{-8y}{3y^2 - x^2}$$ Not possible

b) If $\frac{M_y - N_x}{N} = G(x)$, It can be done

In fact, $\frac{6y + 2y}{-2xy} = \frac{-8}{2x} = \frac{-4}{x}$! ✓

2a) <u>Cont.</u> To find $\mu = \mu(x)$, we need to solve the linear equation

$$\mu'(x) = -\frac{4}{x}\mu(x), \text{ then } \frac{d\mu}{\mu} = -\frac{4}{x}\,dx$$

$$\int: \quad \ln|\mu| = -4\ln|x| + c$$

$$e^{(\ )}: \quad |\mu(x)| = \tilde{c}\left(e^{\ln|x|}\right)^{-4}$$

choose $\tilde{\tilde{c}} = 1$, then $\mu(x) = x^{-4}$

Multiplying by x^{-4}

$$\overbrace{\frac{3y^2 - x^2}{x^4}}^{=M} \overbrace{- \frac{2xy}{x^4}}^{=N} y' = 0$$

Verification $M_y = 6y/x^4$ and $N_x = -\frac{3(-2y)}{x^4} = \frac{6y}{x^4}$

To obtain soln:

$$\psi_x = \frac{3y^2}{x^4} - \frac{1}{x^2}, \quad \psi_y = \frac{-2y}{x^3}$$

$$\int dx: \boxed{\psi(x,y) = \frac{-y^2}{x^3} + \frac{1}{x} + c(y)}$$

Then $\psi_y = -2\cancel{x^3}y + c'(y) = \frac{-2\cancel{y}}{\cancel{x^3}}$

Thus, $c'(y) = 0$ and $c(y) = d$

Integral curves are

$$\psi(x,y) = \frac{-y^2}{x^3} + \frac{1}{x} = d.$$

iii) If $y(1) = 1$, then $\frac{1}{x} - \frac{y^2}{x^3} = c$ transform into $1 - 1 = c$, then $\boxed{c = 0}$

$$\boxed{y^2 = x^2}$$

The branch that passes through $(1,1)$ is $y = x$, $0 < x < +\infty$.

<u>Alternative using</u>
<u>Change of Variables $v = \frac{y}{x}$.</u>

$$\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy}$$

Dividing numerator and denominator by x^2

$$\frac{3y^2 - x^2}{2xy} = \frac{3\left(\frac{y}{x}\right)^2 - 1}{2 \cdot \frac{y}{x}} \overset{v = \frac{y}{x}}{=} \frac{3v^2 - 1}{2v}$$

If $\boxed{v(x) = \frac{y(x)}{x}} \Rightarrow \boxed{y(x) = x\,v(x)}$

$$y'(x) = v(x) + x\,v'(x)$$

then,

$$v(x) + x\,v'(x) = \frac{3v^2 - 1}{2v}$$

or

$$v' = \frac{1}{x}\left[\frac{3v^2 - 1}{2v} - v\right] = \frac{1}{x}\left[\frac{3v^2 - 1 - 2v^2}{2v}\right]$$

or $v' = \left(\frac{v^2 - 1}{2v}\right)\frac{1}{x}$

$$\frac{2v}{v^2 - 1}\,\frac{dv}{dx} = \frac{1}{x} \Rightarrow \frac{2v}{v^2 - 1}\,dv = \frac{1}{x}dx$$

$\int$: $\ln|v^2 - 1| = \ln|x| + c$

$e^{(\,)}$: $|v^2 - 1| = c|x| \Rightarrow v^2 - 1 = c|x| = cx$

or $v^2(x) = cx + 1 \Rightarrow \boxed{\frac{y^2(x)}{x^2} = cx + 1}$

Equivalent to $\boxed{\frac{y^2(x)}{x^3} = \frac{1}{x} + c}$

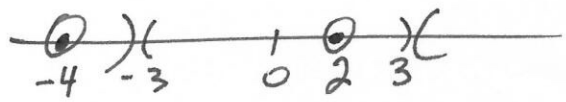
2b. Consider differential equation

$$(9-t^2)y' + 2ty = 4t^2.$$

(i) Determine (without solving the problem) an interval in which the solution of the IVP consisting of the given equation and the initial condition $y(2)=1$ is certain to exist.

$$\to \quad y' + \frac{2t}{9-t^2}\,y = \frac{4t^2}{9-t^2}, \qquad y(2)=1$$

$p(t) = \dfrac{2t}{9-t^2}$, $g(t) = \dfrac{4t^2}{9-t^2}$ both are not conts. at $t = \pm 3$



For I.C. $y(2)=1$ the longest interval where soln. exists and is unique is $(-3,3)$. because $2 \in (-3,3)$ ↳ belongs

(ii) Determine (without solving the problem) an interval in which the solution of the IVP consisting of the given equation and the initial condition $y(-4)=1$ is certain to exist.

In here, $-4 \in (-\infty,-3)$

and p and g are continuous on this interval

therefore, the longest interval where the soln. exists and is continuous is $(-\infty,-3)$