Vector-valued Modular Forms and the Seventh Order Mock Theta Functions

Nickolas Andersen

Abstract In 1988, Hickerson proved the mock theta conjectures (identities involving Ramanujan's fifth order mock theta functions) using q-series methods. In a follow-up paper he proved three analogous identities which involve Ramanujan's seventh order mock theta functions. Recently the author gave a unified proof of the mock theta conjectures using the theory of vector-valued modular forms which transform according to the Weil representation. Here we apply the method to Hickerson's seventh order identities.

1 Introduction

In his last letter to Hardy, Ramanujan introduced a new class of functions which he called mock theta functions, and he listed 17 examples [3, p. 220]. Each of these he labeled third order, fifth order, or seventh order. The seventh order mock theta functions are

$$\mathscr{F}_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n+1};q)_n},$$
$$\mathscr{F}_1(q) := \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(q^{n+1};q)_{n+1}},$$
$$\mathscr{F}_2(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^{n+1},q)_{n+1}}.$$

Here we have used the standard q-Pochhammer notation $(a;q)_n := \prod_{m=0}^{n-1}(1-aq^m)$. In Ramanujan's lost notebook there are many identities which relate linear combi-

UCLA Mathematics Department, Box 951555, Los Angeles, CA 90095-1555
e-mail: nandersen@math.ucla.edu

nations of mock theta functions to modular forms. Andrews and Garvan [2] named ten of these identities, those which involved the fifth order mock theta functions, the *mock theta conjectures*. Hickerson proved two of these identities in [11]; his proof, together with the work of Andrews and Garvan [2], established the truth of the mock theta conjectures. In a companion paper [10] immediately following [11], Hickerson proved analogous identities for the seventh order mock theta functions, namely

$$\mathscr{F}_0(q) = 2qM\left(\tfrac{1}{7}, q^7\right) + 2 - \frac{j(q^3,q^7)^2}{(q,q)_\infty}, \tag{1}$$

$$\mathscr{F}_1(q) = 2qM\left(\tfrac{2}{7}, q^7\right) + q\frac{j(q,q^7)^2}{(q,q)_\infty}, \tag{2}$$

$$\mathscr{F}_2(q) = 2qM\left(\tfrac{3}{7}, q^7\right) + \frac{j(q^2,q^7)^2}{(q,q)_\infty}. \tag{3}$$

Here (following the notation of [9])

$$M(r,q) := \sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(q^r;q)_n(q^{1-r};q)_n}$$

and

$$j(q^\rho,q^7) := (q^\rho,q^7)_\infty(q^{7-\rho},q^7)_\infty(q^7,q^7)_\infty.$$

We will refer to (1)–(3) as the *seventh order mock theta conjectures*.

Zwegers [14] showed that the mock theta functions can be completed to real analytic modular forms of weight 1/2 by multiplying by a suitable rational power of q and adding nonholomorphic integrals of certain unary theta series of weight 3/2. This allows the mock theta functions to be studied using the theory of modular forms. Recently the author [1], building on Zwegers' work and work of Bringmann–Ono [5], proved the mock theta conjectures using the theory of vector-valued modular forms. The purpose of this paper is to apply this method to prove the seventh order mock theta conjectures.

We begin by defining two nonholomorphic vectors $\boldsymbol{F}$ and $\boldsymbol{G}$ corresponding to the left-hand and right-hand sides of (1)–(3), respectively, and we establish their transformation properties using the results of [14, 5, 8]. Next, we construct a holomorphic vector-valued modular form $\mathscr{H}$ from the components of $\mathbf{F}-\mathbf{G}$ which transforms according to the Weil representation (see Lemma 4 below). There is a natural isomorphism between the space of such forms and the space $J_{1,42}$ of Jacobi forms of weight 1 and index 42. The seventh order mock theta conjectures follow from the result of Skoruppa that $J_{1,m} = \{0\}$ for all $m \geq 1$.

2 Definitions and Transformations

In this section, we describe the transformation behavior for the functions $M(\frac{a}{7},q)$ and $j(q^\rho,q^7)$ and the mock theta functions under the generators

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

of $\mathrm{SL}_2(\mathbb{Z})$. We employ the usual $\big|_k$ notation, defined for $k \in \mathbb{R}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ by

$$(f\big|_k\gamma)(z) := (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

We always take $\arg z \in (-\pi,\pi]$. It is not always the case that $f\big|_k AB = f\big|_k A\big|_k B$, but for $k \in \frac{1}{2}\mathbb{Z}$ we have

$$f\big|_k AB = \pm f\big|_k A\big|_k B, \tag{4}$$

(see [12, §2.6]). Much of the arithmetic here and throughout the paper takes place in the splitting field of the polynomial $x^6-7x^4+14x^2-7$, which has roots $\pm\kappa$, $\pm\lambda$, $\pm\mu$, where

$$\kappa := 2\sin\frac{\pi}{7}, \qquad \lambda := 2\sin\frac{2\pi}{7}, \qquad \mu := 2\sin\frac{3\pi}{7}. \tag{5}$$

The modular transformations satisfied by the mock theta functions $\mathscr{F}_0$, $\mathscr{F}_1$, and $\mathscr{F}_2$ are given in Sect. 4.3 of [14]. The nonholomorphic completions are written in terms of the nonholomorphic Eichler integral (see [14, Proposition 4.2])

$$R_{a,b}(z) := -i\int_{-\bar{z}}^{i\infty} \frac{g_{a,-b}(\tau)}{\sqrt{-i(\tau+z)}}\,d\tau,$$

where $g_{a,b}$ (see [14, §1.5]) is the unary theta function

$$g_{a,b}(z) := \sum_{\nu\in a+\mathbb{Z}} \nu e^{\pi i\nu^2 z+2\pi i\nu b}.$$

Let $q := \exp(2\pi i z)$ and $\zeta_m := \exp(2\pi i/m)$. Following §4.3 of [14] we define

$$\widetilde{\mathscr{F}}_0(z) := q^{-\frac{1}{168}}\mathscr{F}_0(q) + \zeta_{14}\left(\zeta_{12}^{-1}R_{-\frac{1}{42},\frac{1}{2}} + \zeta_{12}R_{\frac{13}{42},\frac{1}{2}}\right)(21z), \tag{6}$$

$$\widetilde{\mathscr{F}}_1(z) := q^{-\frac{25}{168}}\mathscr{F}_1(q) + \zeta_7\left(\zeta_{12}^{-1}R_{\frac{5}{42},\frac{1}{2}} + \zeta_{12}R_{\frac{19}{42},\frac{1}{2}}\right)(21z), \tag{7}$$

$$\widetilde{\mathscr{F}}_2(z) := q^{\frac{47}{168}}\mathscr{F}_2(q) + \zeta_{14}^3\left(\zeta_{12}^{-1}R_{\frac{11}{42},\frac{1}{2}} + \zeta_{12}R_{\frac{25}{42},\frac{1}{2}}\right)(21z). \tag{8}$$

Note that we have used Proposition 1.5 of [14] to slightly modify the components of $G_7(\tau)$ on p. 75 of [14]. The following is Proposition 4.5 of [14] (we have rearranged the order of the components of the vector F_7 in that proposition).

Proposition 1. *The vector*

$$\boldsymbol{F}(z) := \left(\widetilde{\mathscr{F}}_0(z), \widetilde{\mathscr{F}}_1(z), \widetilde{\mathscr{F}}_2(z)\right)^{\mathsf{T}} \tag{9}$$

satisfies the transformations

$$\boldsymbol{F}\big|_{\frac{1}{2}} T = M_T \, \boldsymbol{F} \quad \text{and} \quad \boldsymbol{F}\big|_{\frac{1}{2}} S = \zeta_8^{-1} \frac{1}{\sqrt{7}} M_S \, \boldsymbol{F},$$

where

$$M_T = \begin{pmatrix} \zeta_{168}^{-1} & 0 & 0 \\ 0 & \zeta_{168}^{-25} & 0 \\ 0 & 0 & \zeta_{168}^{47} \end{pmatrix} \quad \text{and} \quad M_S = \begin{pmatrix} \kappa & \lambda & \mu \\ \lambda & -\mu & \kappa \\ \mu & \kappa & -\lambda \end{pmatrix}.$$

Following [5, 9], we define, for $1 \leq a \leq 6$, the functions

$$M\left(\frac{a}{7}, z\right) := \sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(q^{\frac{a}{7}}; q)_n (q^{1-\frac{a}{7}}; q)_n}, \tag{10}$$

$$N\left(\frac{a}{7}, z\right) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(\zeta_7^a q; q)_n (\zeta_7^{-a} q; q)_n}. \tag{11}$$

Clearly we have $M(1-\frac{a}{7}, z) = M(\frac{a}{7}, z)$ and $N(1-\frac{a}{7}, z) = N(\frac{a}{7}, z)$. Bringmann and Ono [5] also define auxiliary functions $M(a,b,7,z)$ and $N(a,b,7,z)$ for $0 \leq a \leq 6$ and $1 \leq b \leq 6$. Together, the completed versions of these functions form a set that is closed (up to multiplication by roots of unity) under the action of $\mathrm{SL}_2(\mathbb{Z})$ (see [5, Theorem 3.4]). Garvan [8] corrected the definitions of these functions and wrote their transformation formulas more explicitly, so in what follows we reference his paper.

The nonholomorphic completions for $M(\frac{a}{7}, z)$ and $N(\frac{a}{7}, z)$ are given in terms of integrals of weight 3/2 theta functions $\Theta_1(\frac{a}{7}, z)$ and $\Theta_1(0, -a, 7, z)$ (defined in Sect. 2 of [8]). A straightforward computation shows that

$$\Theta_1(0, -a, 7, z) = 21\sqrt{3}\,\zeta_{14}^a \left(\zeta_{12}^{-1}\, g_{\frac{6a-7}{42}, -\frac{1}{2}}(3z) + \zeta_{12}\, g_{\frac{6a+7}{42}, -\frac{1}{2}}(3z)\right).$$

Following (2.5), (2.6), (3.5), and (3.6) of [8], we define

$$\begin{aligned} \widetilde{M}\left(\frac{a}{7}, z\right) &:= 2q^{\frac{3a}{14}\left(1-\frac{a}{7}\right)-\frac{1}{24}} M\left(\frac{a}{7}, z\right) \\ &\quad + \zeta_{14}^a \left(\zeta_{12}^{-1} R_{\frac{6a-7}{42}, \frac{1}{2}} + \zeta_{12} R_{\frac{6a+7}{42}, \frac{1}{2}}\right)(3z) + \begin{cases} 2q^{-\frac{1}{1176}} & \text{if } a = 1, \\ 0 & \text{if } a = 2, 3, \end{cases} \end{aligned} \tag{12}$$

$$\widetilde{N}\left(\frac{a}{7}, z\right) := \csc\left(\frac{a\pi}{7}\right) q^{-\frac{1}{24}} N\left(\frac{a}{7}, z\right) + \frac{i}{\sqrt{3}} \int_{-\bar{z}}^{i\infty} \frac{\Theta_1(\frac{a}{7}, \tau)}{\sqrt{-i(\tau + z)}}\, d\tau. \tag{13}$$

The completed functions $\widetilde{M}(a,b,z) := \mathscr{G}_2(a,b,7;z)$ and $\widetilde{N}(a,b,z) := \mathscr{G}_1(a,b,7;z)$ are defined in (3.7) and (3.8) of that paper. By Theorem 3.1 of [8] we have

$$\widetilde{M}\left(\tfrac{a}{7},z\right)\big|_{\frac{1}{2}} T^7 = \widetilde{M}\left(\tfrac{a}{7},z\right) \times \begin{cases} \zeta_{168}^{-1} & \text{if } a=1, \\ \zeta_{168}^{-25} & \text{if } a=2, \\ \zeta_{168}^{47} & \text{if } a=3, \end{cases} \tag{14}$$

$$\widetilde{N}\left(\tfrac{a}{7},z\right)\big|_{\frac{1}{2}} T = \zeta_{24}^{-1}\widetilde{N}\left(\tfrac{a}{7},z\right), \tag{15}$$

and

$$\widetilde{M}\left(\tfrac{a}{7},z\right)\big|_{\frac{1}{2}} S = \zeta_{8}^{-1}\widetilde{N}\left(\tfrac{a}{7},z\right). \tag{16}$$

The functions $j(q^{\rho},q^7)$ are essentially theta functions of weight $1/2$. It will be more convenient to work with (following [4])

$$f_\rho(z) = f_{7,\rho}(z) := q^{\frac{(7-2\rho)^2}{56}} j(q^{\rho},q^7). \tag{17}$$

The transformation properties of theta functions are well-known; for $f_\rho(z)$ we have (see e.g. [9, pp. 217-218])

$$\left(f_1,f_2,f_3\right)^{\intercal}\big|_{\frac{1}{2}} S = \zeta_8^{-1}\frac{1}{\sqrt{7}}\begin{pmatrix} \lambda & -\mu & \kappa \\ -\mu & -\kappa & \lambda \\ \kappa & \lambda & \mu \end{pmatrix}\left(f_1,f_2,f_3\right)^{\intercal}. \tag{18}$$

The mock theta conjectures (1)–(3) are implied by the corresponding completed versions:

$$\widetilde{\mathscr{F}}_0(z) = \widetilde{M}\left(\tfrac{1}{7},7z\right) - \frac{f_3^2(z)}{\eta(z)}, \tag{19}$$

$$\widetilde{\mathscr{F}}_1(z) = \widetilde{M}\left(\tfrac{2}{7},7z\right) + \frac{f_1^2(z)}{\eta(z)}, \tag{20}$$

$$\widetilde{\mathscr{F}}_2(z) = \widetilde{M}\left(\tfrac{3}{7},7z\right) + \frac{f_2^2(z)}{\eta(z)}. \tag{21}$$

Motivated by (9) and (19)–(21), we define the vector

$$\boldsymbol{G}(z) := \begin{pmatrix} \widetilde{M}\left(\tfrac{1}{7},7z\right) - \dfrac{f_3^2(z)}{\eta(z)} \\ \widetilde{M}\left(\tfrac{2}{7},7z\right) + \dfrac{f_1^2(z)}{\eta(z)} \\ \widetilde{M}\left(\tfrac{3}{7},7z\right) + \dfrac{f_2^2(z)}{\eta(z)} \end{pmatrix}. \tag{22}$$

To prove that $\boldsymbol{F} = \boldsymbol{G}$ we first show that they transform in the same way.

Proposition 2. *The vector* $\boldsymbol{G}(z)$ *defined in* (22) *satisfies the transformations*

$$\boldsymbol{G}\big|_{\frac{1}{2}} T = M_T\, \boldsymbol{G} \quad \text{and} \quad \boldsymbol{G}\big|_{\frac{1}{2}} S = \zeta_8^{-1} \frac{1}{\sqrt{7}} M_S\, \boldsymbol{G}, \tag{23}$$

where M_T *and* M_S *are as in Proposition 1.*

In order to prove Proposition 2 we require the following three identities (equivalent identities can be found on p. 220 of [9] without proof).

Lemma 1. *Let* κ*,* λ*, and* μ *be as in* (5)*. Then*

$$\begin{aligned} \widetilde{N}\left(\tfrac{1}{7}, z\right) - \left(\kappa \widetilde{M}\left(\tfrac{1}{7}, 49z\right) + \lambda \widetilde{M}\left(\tfrac{2}{7}, 49z\right) + \mu \widetilde{M}\left(\tfrac{3}{7}, 49z\right)\right) \\ = \frac{1}{\eta(7z)} \Big[\tfrac{1}{\sqrt{7}} \big(\kappa f_1(7z) + \lambda f_2(7z) + \mu f_3(7z)\big)^2 \\ - \kappa f_3^2(7z) + \lambda f_1^2(7z) + \mu f_2^2(7z) \Big]. \end{aligned} \tag{24}$$

We defer the proof of Lemma 1 to Sect. 5; here we deduce two immediate consequences. Note that the right-hand side of (24) is holomorphic; this implies that the non-holomorphic completion terms on the left-hand side sum to zero. By (11), the coefficients of $N(\frac{a}{7}, z)$ lie in $\mathbb{Q}(\zeta_7 + \zeta_7^{-1}) = \mathbb{Q}(\kappa^2)$, and the automorphisms $\kappa^2 \mapsto \lambda^2$ and $\kappa^2 \mapsto \mu^2$ map $N(\frac{1}{7}, z)$ to $N(\frac{2}{7}, z)$ and $N(\frac{3}{7}, z)$, respectively. By (13) it follows that the coefficients of both sides of (24) lie in $\mathbb{Q}(\kappa)$. Let τ_1 and τ_2 be the automorphisms

$$\begin{aligned} \tau_1 &= (\kappa \mapsto \lambda,\ \lambda \mapsto -\mu,\ \mu \mapsto \kappa), \\ \tau_2 &= (\kappa \mapsto \mu,\ \lambda \mapsto \kappa,\ \mu \mapsto -\lambda). \end{aligned}$$

Since $\sqrt{7} = \kappa\lambda\mu$, we have $\tau_1(\sqrt{7}) = \tau_2(\sqrt{7}) = -\sqrt{7}$. Applying τ_1 and τ_2 to Lemma 1 gives the following identities.

Lemma 2. *Let* κ*,* λ*, and* μ *be as in* (5)*. Then*

$$\begin{aligned} \widetilde{N}\left(\tfrac{2}{7}, z\right) - \left(\lambda \widetilde{M}\left(\tfrac{1}{7}, 49z\right) - \mu \widetilde{M}\left(\tfrac{2}{7}, 49z\right) + \kappa \widetilde{M}\left(\tfrac{3}{7}, 49z\right)\right) \\ = \frac{1}{\eta(7z)} \Big[-\tfrac{1}{\sqrt{7}} \big(\lambda f_1(7z) - \mu f_2(7z) + \kappa f_3(7z)\big)^2 \\ - \lambda f_3^2(7z) - \mu f_1^2(7z) + \kappa f_2^2(7z) \Big]. \end{aligned} \tag{25}$$

Lemma 3. *Let* κ*,* λ*, and* μ *be as in* (5)*. Then*

$$\begin{aligned} \widetilde{N}\left(\tfrac{3}{7}, z\right) - \left(\mu \widetilde{M}\left(\tfrac{1}{7}, 49z\right) + \kappa \widetilde{M}\left(\tfrac{2}{7}, 49z\right) - \lambda \widetilde{M}\left(\tfrac{3}{7}, 49z\right)\right) \\ = \frac{1}{\eta(7z)} \Big[-\tfrac{1}{\sqrt{7}} \big(\mu f_1(7z) + \kappa f_2(7z) - \lambda f_3(7z)\big)^2 \\ - \mu f_3^2(7z) + \kappa f_1^2(7z) - \lambda f_2^2(7z) \Big]. \end{aligned} \tag{26}$$

Proof (Proof of Proposition 2). The transformation with respect to T follows immediately from (14). Let $G_j(z)$ denote the j-th component of $\boldsymbol{G}(z)$. By (16), (18), and the fact that $\eta\big|_{\frac{1}{2}}S = \zeta_8^{-1}\eta$, we have

$$G_1(z)\big|_{\frac{1}{2}}S = \zeta_8^{-1}\frac{1}{\sqrt{7}}\left[\widetilde{N}\left(\frac{1}{7},\frac{z}{7}\right) - \frac{1}{\sqrt{7}}\frac{\big(\kappa f_1(z)+\lambda f_2(z)+\mu f_3(z)\big)^2}{\eta(z)}\right].$$

Applying Lemma 1 with z replaced by $\frac{z}{7}$, we find that

$$\begin{aligned} G_1(z)\big|_{\frac{1}{2}}S &= \zeta_8^{-1}\frac{1}{\sqrt{7}}\Bigg[\kappa\widetilde{M}\left(\frac{1}{7},7z\right)+\lambda\widetilde{M}\left(\frac{2}{7},7z\right)+\mu\widetilde{M}\left(\frac{3}{7},7z\right) \\ &\qquad\qquad - \frac{\kappa f_3^2(z)-\lambda f_1^2(z)-\mu f_2^2(z)}{\eta(z)}\Bigg] \\ &= \zeta_8^{-1}\frac{1}{\sqrt{7}}\big(\kappa G_1(z)+\lambda G_2(z)+\mu G_3(z)\big). \end{aligned}$$

The transformations for G_2 and G_3 are similarly obtained using Lemmas 2 and 3, respectively.

3 Vector-valued Modular Forms and the Weil Representation

In this section we define vector-valued modular forms which transform according to the Weil representation, and we construct such a form from the components of $\boldsymbol{F}-\boldsymbol{G}$. A good reference for this material is [6, Sect. 1.1].

Let $L=\mathbb{Z}$ be the lattice with associated bilinear form $(x,y)=-84xy$ and quadratic form $q(x)=-42x^2$. The dual lattice is $L'=\frac{1}{84}\mathbb{Z}$. Let $\{\mathfrak{e}_h : \frac{h}{84}\in\frac{1}{84}\mathbb{Z}/\mathbb{Z}\}$ denote the standard basis for $\mathbb{C}[L'/L]$. Let $\mathrm{Mp}_2(\mathbb{R})$ denote the metaplectic two-fold cover of $\mathrm{SL}_2(\mathbb{R})$; the elements of this group are pairs (M,ϕ), where $M=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{R})$ and $\phi^2(z)=cz+d$. Let $\mathrm{Mp}_2(\mathbb{Z})$ denote the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under the covering map; this group is generated by

$$\widetilde{T}:=(T,1) \quad\text{and}\quad \widetilde{S}:=(S,\sqrt{z}).$$

The Weil representation can be defined by its action on these generators, namely

$$\rho_L(T,1)\mathfrak{e}_h := \zeta_{168}^{-h^2}\,\mathfrak{e}_h, \tag{27}$$

$$\rho_L(S,\sqrt{z})\mathfrak{e}_h := \frac{1}{\sqrt{-84i}}\sum_{h'(84)}\zeta_{84}^{hh'}\,\mathfrak{e}_{h'}. \tag{28}$$

A holomorphic function $\mathscr{F}:\mathbb{H}\to\mathbb{C}[L'/L]$ is a vector-valued modular form of weight $1/2$ and representation ρ_L if

$$\mathscr{F}(\gamma z)=\phi(z)\rho_L(\gamma,\phi)\mathscr{F}(z) \qquad \text{for all } (\gamma,\phi)\in\mathrm{Mp}_2(\mathbb{Z}) \tag{29}$$

and if $\mathscr{F}$ is holomorphic at ∞ (i.e. if the components of $\mathscr{F}$ are holomorphic at ∞ in the usual sense). The following lemma shows how to construct such forms from vectors that transform as in Propositions 1 and 2.

Lemma 4. *Suppose that* $\boldsymbol{H} = (H_1, H_2, H_3)$ *satisfies*

$$\boldsymbol{H}\big|_{\frac{1}{2}} T = M_T \boldsymbol{H} \quad \text{and} \quad \boldsymbol{H}\big|_{\frac{1}{2}} S = \zeta_8^{-1} \frac{1}{\sqrt{7}} M_S \boldsymbol{H},$$

where M_T *and* M_S *are as in Proposition 1, and define*

$$\mathscr{H}(z) := \sum_{h=1,13,29,41} a(h) H_1(z)(\mathfrak{e}_h - \mathfrak{e}_{-h}) - \sum_{h=5,19,23,37} H_2(z)(\mathfrak{e}_h - \mathfrak{e}_{-h}) - \sum_{h=11,17,25,31} H_3(z)(\mathfrak{e}_h - \mathfrak{e}_{-h}),$$

where

$$a_h = \begin{cases} +1 & \text{if } h = 1, 41, \\ -1 & \text{if } h = 13, 29. \end{cases}$$

Then $\mathscr{H}(z)$ *satisfies* (29).

Proof. The proof is a straightforward but tedious verification involving (27) and (28) that is best carried out with the aid of a computer algebra system; the author used MATHEMATICA.

4 Proof of the Mock Theta Conjectures

Let $\boldsymbol{F}$ and $\boldsymbol{G}$ be as in Sect. 2. To prove (19)–(21) we will prove that $\boldsymbol{H} := \boldsymbol{F} - \boldsymbol{G} = 0$. It is easy to see that the nonholomorphic parts of $\boldsymbol{F}$ and $\boldsymbol{G}$ agree, as do the terms in the Fourier expansion involving negative powers of q. It follows that the function $\mathscr{H}$ defined in Lemma 4 is a vector-valued modular form of weight $1/2$ with representation ρ_L. By Theorem 5.1 of [7], the space of such forms is canonically isomorphic to the space $J_{1,42}$ of Jacobi forms of weight 1 and index 42. By a theorem of Skoruppa [13, Satz 6.1] (see also [7, Theorem 5.7]), we have $J_{1,m} = \{0\}$ for all m; therefore $\mathscr{H} = 0$. The seventh order mock theta conjectures (1)–(3) follow. □

5 Proof of Lemma 1

We begin with a lemma which describes the modular transformation properties of $f_\rho(z)$. Let ν_η denote the multiplier system for the eta function (see [4, (2.5)]). For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, define

$$\gamma_n := \begin{pmatrix} a & nb \\ c/n & d \end{pmatrix}.$$

Lemma 5. *Let* $\rho \in \{1,2,3\}$. *If*

$$\gamma \in \Gamma(7) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod 7 \right\},$$

then

$$f_\rho(\gamma z) = \nu_\eta^3(\gamma_7)\sqrt{cz+d}\, f_\rho(z). \tag{30}$$

Proof. Suppose that $\rho \in \{1,2,3\}$ and that $\gamma \in \Gamma(7)$. Lemma 2.1 of [4] gives

$$f_\rho(\gamma z) = (-1)^{\rho b + \lfloor \rho a/7 \rfloor}\, \zeta_{14}^{\rho^2 ab}\, \nu_\eta^3(\gamma_7)\sqrt{cz+d}\, f_\rho(z).$$

Writing $a = 1+7r$ and $b = 7b'$, we find that

$$(-1)^{\rho b + \lfloor \rho a/7 \rfloor}\, \zeta_{14}^{\rho^2 ab} = (-1)^{\rho(b+r+\rho br+\rho b')}. \tag{31}$$

Using the fact that $br + r \equiv 0 \pmod 2$ we find that, in each case, the right-hand side of (31) equals 1. This completes the proof.

We are now ready to prove Lemma 1. Let $L(z)$ and $R(z)$ denote the left-hand and right-hand sides of (24), respectively. Let Γ denote the congruence subgroup

$$\Gamma = \Gamma_0(49) \cap \Gamma_1(7) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \bmod 49 \text{ and } a,d \equiv 1 \bmod 7 \right\}.$$

We claim that

$$\eta(z)L(z),\ \eta(z)R(z) \in M_1(\Gamma), \tag{32}$$

where $M_k(G)$ (resp. $M_k^!(G)$) denotes the space of holomorphic (resp. weakly holomorphic) modular forms of weight k on $G \subseteq \mathrm{SL}_2(\mathbb{Z})$. We have

$$\frac{1}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma] = 14,$$

so once (32) is established it suffices to check that the first 15 coefficients of $\eta(z)L(z)$ and $\eta(z)R(z)$ agree. A computation shows that the Fourier expansion of each function begins

$$2\left(\frac{1}{\kappa} - \kappa\right) + 2\kappa q - 2\mu q^3 - 2\left(\frac{2}{\mu} - \mu - \frac{1}{\lambda}\right)q^4 - 2\kappa q^5 + 2\lambda q^6 + 2\left(\frac{2}{\kappa} - 2\kappa + \frac{1}{\mu}\right)q^7$$
$$+ 4\kappa q^8 + 2\left(k - \frac{2}{\kappa} + 2\mu - \frac{1}{\mu}\right)q^9 - 2\mu q^{10} + 2(\mu + \lambda - 2\kappa)q^{14} + \dots.$$

To prove (32), we first note that Theorem 5.1 of [8] shows that $\eta(49z)L(z) \in M_1^!(\Gamma)$; since $\eta(z)/\eta(49z) \in M_0^!(\Gamma_0(49))$ it follows that $\eta(z)L(z) \in M_1^!(\Gamma)$. Using Lemma 5 we find that $\eta(z)R(z) \in M_1^!(\Gamma)$ provided that

$$\frac{\nu_\eta(\gamma)\nu_\eta^6(\gamma_{49})}{\nu_\eta(\gamma_7)} = 1. \tag{33}$$

This follows from a computation involving the definition of ν_η [4, (2.5)].

It remains to show that $\eta(z)L(z)$ and $\eta(z)R(z)$ are holomorphic at the cusps. Using MAGMA we compute a set of Γ-inequivalent cusp representatives:

$$\Big\{\infty, 0, \tfrac{1}{7}, \tfrac{3}{17}, \tfrac{5}{28}, \tfrac{3}{16}, \tfrac{4}{21}, \tfrac{3}{14}, \tfrac{8}{35}, \tfrac{5}{21}, \\ \tfrac{2}{7}, \tfrac{5}{14}, \tfrac{18}{49}, \tfrac{13}{35}, \tfrac{8}{21}, \tfrac{19}{49}, \tfrac{11}{28}, \tfrac{3}{7}, \tfrac{4}{7}, \tfrac{13}{21}, \tfrac{9}{14}, \tfrac{5}{7}, \tfrac{11}{14}, \tfrac{6}{7}\Big\}. \quad (34)$$

Given a cusp $\mathfrak{a} \in \mathbb{P}^1(\mathbb{Q})$ and a meromorphic modular form f of weight k with Fourier expansion $f(z) = \sum_{n\in\mathbb{Q}} a(n)q^n$, the invariant order of f at $\mathfrak{a}$ is defined as

$$\operatorname{ord}(f,\infty) := \min\{n : a(n) \neq 0\},$$
$$\operatorname{ord}(f,\mathfrak{a}) := \operatorname{ord}(f\big|_k \delta_{\mathfrak{a}}, \infty),$$

where $\delta_{\mathfrak{a}} \in \mathrm{SL}_2(\mathbb{Z})$ sends ∞ to $\mathfrak{a}$. For $N \in \mathbb{N}$, we have the relation (see e.g. [4, (1.7)])

$$\operatorname{ord}(f(Nz), \tfrac{r}{s}) = \tfrac{(N,s)^2}{N} \operatorname{ord}(f, \tfrac{Nr}{s}). \quad (35)$$

We extend this definition to functions f in the set

$$S := \Big\{\widetilde{M}(\tfrac{a}{7},z), \widetilde{N}(\tfrac{a}{7},z) : a = 1,2,3\Big\} \cup \Big\{\widetilde{M}(a,b,z), \widetilde{N}(a,b,z) : 0 \le a \le 6, 1 \le b \le 6\Big\}$$

by defining the orders of these functions at ∞ to be the orders of their holomorphic parts at ∞ (see Sect. 6.2, (2.1)–(2.4), and (3.5)–(3.8) of [8]); that is,

$$\operatorname{ord}\Big(\widetilde{M}\Big(\frac{a}{7},z\Big),\infty\Big) = \operatorname{ord}\Big(\widetilde{M}(a,b,z),\infty\Big) := \begin{cases} -\frac{1}{24} & \text{if } a = 0, \\ -\frac{1}{1176} & \text{if } a = 1,6, \\ \frac{3a}{14}\Big(1 - \frac{a}{7}\Big) - \frac{1}{24} & \text{otherwise,} \end{cases} \quad (36)$$

$$\operatorname{ord}\Big(\widetilde{N}\Big(\frac{a}{7},z\Big),\infty\Big) := -\frac{1}{24}, \quad (37)$$

$$\operatorname{ord}\Big(\widetilde{N}(a,b,z),\infty\Big) := \frac{b}{7}\Big(\frac{1}{2} + k(b,7)\Big) - \frac{3b^2}{98} - \frac{1}{24}, \quad (38)$$

where

$$k(b,7) := \begin{cases} 0 & \text{if } b = 1, \\ 1 & \text{if } b = 2,3, \\ 2 & \text{if } b = 4,5, \\ 3 & \text{if } b = 6. \end{cases}$$

Lastly, for $f \in S$ we define

$$\operatorname{ord}(f,\mathfrak{a}) := \operatorname{ord}(f\big|_{\frac{1}{2}} \delta_{\mathfrak{a}}, \infty). \quad (39)$$

This is well-defined since S is closed (up to multiplication by roots of unity) under the action of $\mathrm{SL}_2(\mathbb{Z})$. By this same fact we have

$$\min_{\text{cusps } \mathfrak{a}} \operatorname{ord}(f,\mathfrak{a}) \geq \min_{g\in S} \operatorname{ord}(g,\infty) = -\frac{1}{24} \tag{40}$$

for all $f \in S$, from which it follows that

$$\operatorname{ord}(\eta f,\mathfrak{a}) \geq 0$$

for all cusps $\mathfrak{a}$ and for all $f \in S$.

To determine the order of $\eta(z)\widetilde{M}(\frac{a}{7},49z)$ at the cusps of Γ, we write

$$\eta(z)\widetilde{M}(\tfrac{a}{7},49z) = \frac{\eta(z)}{\eta(49)} m(49z), \qquad \text{where } m(z) = \eta(z)\widetilde{M}(\tfrac{a}{7},z).$$

The cusps of $\Gamma_0(49)$ are ∞ and $\frac{r}{7}$, $0 \leq r \leq 6$. By (35) the function $\eta(z)/\eta(49z)$ is holomorphic at every cusp except for those which are $\Gamma_0(49)$-equivalent to ∞ (the latter are ∞, $\frac{18}{49}$, and $\frac{19}{49}$ in (34)); there we have $\operatorname{ord}(\eta(z)/\eta(49z),\infty) = -2$. By (40), to show that $\eta(z)\widetilde{M}(\frac{a}{7},49z)$ is holomorphic at every cusp, it suffices to verify that $\operatorname{ord}(m(49z),\frac{r}{49}) \geq 2$ for $r = 18,19$. By (35), [8, Theorems 3.1 and 3.2], the fact that $\left(\begin{smallmatrix} r & r-1 \\ 1 & 1 \end{smallmatrix}\right) = T^rST$, and (4), we have

$$\begin{aligned}
\operatorname{ord}\Big(m(49z),\frac{18}{49}\Big) &= 49 \operatorname{ord}(m(z),18) \\
&= 49\Big(\frac{1}{24} + \operatorname{ord}\Big(\widetilde{M}(4a \bmod 7, 4a \bmod 7, z),\infty\Big)\Big) \\
&= \begin{cases} 46 & \text{if } a = 1, \\ 2 & \text{if } a = 2, \\ 50 & \text{if } a = 3. \end{cases}
\end{aligned}$$

A similar computation shows that $\operatorname{ord}(m(49z),\frac{19}{49}) \geq 2$. Since $L(z)$ is holomorphic on $\mathbb{H}$, we have, for each cusp $\mathfrak{a}$, the inequality

$$\begin{aligned}
&\operatorname{ord}(\eta(z)L(z),\mathfrak{a}) \\
&\geq \min\Big\{\operatorname{ord}(\eta(z)f(z),\mathfrak{a}) : f(z) = \widetilde{N}(\tfrac{1}{7},z) \text{ or } f(z) = \widetilde{M}(\tfrac{a}{7},49z), 1 \leq a \leq 3\Big\} \geq 0.
\end{aligned}$$

We turn to $\eta(z)R(z)$. Using Lemma 3.2 of [4], we find that

$$\operatorname{ord}(f_\rho,\tfrac{r}{s}) = \begin{cases} \frac{25}{56} & \text{if } 7 \mid s \text{ and } \rho r \equiv \pm 2 \pmod 7, \\ \frac{9}{56} & \text{if } 7 \mid s \text{ and } \rho r \equiv \pm 3 \pmod 7, \\ \frac{1}{56} & \text{otherwise.} \end{cases} \tag{41}$$

By (35) and (41) we have

$$\operatorname{ord}\left(\eta(z)R(z), \tfrac{r}{s}\right) \geq \frac{1}{24} - \frac{(7,s)^2}{168} + 2 \min_{\rho=1,2,3} \operatorname{ord}\left(f_\rho(7z), \tfrac{r}{s}\right) \geq 0.$$

Therefore $\eta(z)R(z) \in M_1(\Gamma)$, which proves (32) and completes the proof of Lemma 1.

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