ARITHMETIC OF MAASS FORMS OF HALF-INTEGRAL WEIGHT

BY

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DISSERTATION

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Abstract

We investigate the arithmetic properties of coefficients of Maass forms in three contexts. First, we discuss connections to invariants of real and imaginary quadratic fields, expanding on the work of Zagier and Duke-Imamoḡlu-Tóth. Next, we examine the deep relationship between sums of Kloosterman sums and Maass cusp forms, motivated by work of Kuznetsov and Sarnak-Tsimerman, among others. Finally, we focus on the classical mock theta functions of Ramanujan, and give a simple proof of the mock theta conjectures using the modern theory of harmonic Maass forms, especially work of Zwegers and Bringmann-Ono, together with the theory of vector-valued modular forms.

For Emily.

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1 Introduction

A weak Maass form of weight k is a real analytic function on the upper-half plane $\mathbb{H}$ which satisfies a modular transformation in weight k with respect to some subgroup of the modular group $\mathrm{SL}_2(\mathbb{Z})$, is an eigenfunction of the weight k hyperbolic Laplacian, and has at most linear exponential growth at the cusps.[1] Weak Maass forms with Laplace eigenvalue 0 are called harmonic Maass forms, and those which are holomorphic on $\mathbb{H}$ are called weakly holomorphic modular forms.

Each weak Maass form f has a Fourier expansion of the form

$$f(\tau) = \sum_{\substack{n\in\mathbb{Q} \\ n\gg-\infty}} a(n,y)e(nx).$$

Here, and throughout, $\tau = x + iy \in \mathbb{H}$ and $e(x) := \exp(2\pi ix)$. Motivated by the general principle that important sequences in number theory are often related to the coefficients of modular forms, a natural problem is to determine what arithmetic data, if any, is encoded in the coefficients of weak Maass forms.

In this thesis, we investigate the arithmetic properties of coefficients of Maass forms in three directions. First, we discuss connections to invariants of real and imaginary quadratic fields, expanding on the work of Zagier and Duke-Imamoḡlu-Tóth. Next, we examine the deep relationship between sums of Kloosterman sums and Maass cusp forms, motivated by work of Kuznetsov and Sarnak-Tsimerman, among others. Finally, we focus on the classical mock theta functions of Ramanujan, and give a simple proof of the mock theta conjectures using the modern theory of harmonic Maass forms, especially work of Zwegers and Bringmann-Ono, together with the theory of vector-valued modular forms.[2]

[1] There are a few different conventions used in defining Maass forms, the Laplacian, etc., even in this thesis. In the introduction we will be intentionally vague and wait until later to sort this all out.

[2] The work on sums of Kloosterman sums (Chapter 5) and the mock theta conjectures

1.1 Invariants of quadratic fields and mock modular forms

At the 1932 International Congress of Mathematicians in Zurich, David Hilbert said that "the theory of complex multiplication (of elliptic modular functions) which forms a powerful link between number theory and analysis, is not only the most beautiful part of mathematics but also of all science" (see [Sc]). The theory of complex multiplication involves special values of the modular j-invariant

$$j(\tau) = \frac{1}{q} + 744 + 196884q + \cdots, \qquad q := e^{2\pi i\tau},$$

which is a weakly holomorphic modular form of weight 0 on $\mathrm{SL}_2(\mathbb{Z})$. For any triple of integers (a, b, c) for which $a > 0$ and $d := b^2 - 4ac < 0$, the values

$$j\left(\frac{-b+\sqrt{d}}{2a}\right) \tag{1.1.1}$$

are algebraic integers called *singular moduli.* One of the most beautiful results in the theory of complex multiplication asserts that if d is a fundamental discriminant (i.e. the discriminant of $\mathbb{Q}(\sqrt{d})$) then the singular modulus (1.1.1) generates the Hilbert class field of $\mathbb{Q}(\sqrt{d})$. More generally, every abelian extension of $\mathbb{Q}(\sqrt{d})$ can be generated by the singular moduli, together with roots of unity and special values of certain elliptic functions (see [Cox, §11] for an excellent treatment of this story).

Hilbert's twelfth problem asks for the generalization of this theory to abelian extensions of any number field. Despite heroic efforts, essentially nothing is known about the next (supposedly) simplest case, that of the real quadratic fields. However, many fascinating connections between real quadratic fields and modular forms exist; for instance, the groundbreaking work of Duke, Imamoḡlu, and Tóth [DIT] connects invariants of real and imaginary quadratic fields to coefficients of harmonic Maass forms.

(Chapter 6) together represent the completion of the research proposed in the author's Dissertation Completion Fellowship proposal.

The results of Section 1.1, together with the corresponding Chapters 3 and 4, appeared in the papers [A1] and [A2].

Briefly, a harmonic Maass form is a weak Maass form which is annihilated by the hyperbolic Laplacian (i.e. has eigenvalue 0). Such forms have a canonical decomposition $f = f^+ + f^-$ into the holomorphic part f^+ (also called a mock modular form) and the nonholomorphic part f^-. There is a differential operator ξ_k which sends harmonic Maass forms of weight k to weakly holomorphic modular forms of weight $2-k$, and the modular form $\xi_k f = \xi_k f^-$ is called the shadow of f^+.

For a nonzero integer $d \equiv 0, 1 \pmod 4$, let $\mathcal{Q}_d$ denote the set of integral binary quadratic forms $Q(x,y) = [a,b,c] = ax^2 + bxy + cy^2$ with discriminant $d = b^2 - 4ac$ which are positive definite if $d < 0$. The modular group $\Gamma_1 = \mathrm{PSL}_2(\mathbb{Z})$ acts on these quadratic forms as

$$(\gamma.Q)(x,y) = Q(Dx - By, -Cx + Ay) \qquad \text{for } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_1.$$

It is well-known that the set $\Gamma_1 \backslash \mathcal{Q}_d$ forms a finite abelian group under Gaussian composition. For $Q \in \mathcal{Q}_d$, let Γ_Q denote the stabilizer of Q in Γ_1.

Suppose first that $d < 0$. If $Q \in \mathcal{Q}_d$ then $Q(\tau, 1)$ has exactly one root τ_Q in $\mathbb{H}$, namely

$$\tau_Q = \frac{-b + \sqrt{d}}{2a}.$$

Here $\Gamma_Q = \{1\}$ unless $Q \sim [a,0,a]$ or $Q \sim [a,a,a]$, in which case Γ_Q has order 2 or 3, respectively. For $f \in \mathbb{C}[j]$, we define the modular trace of f by

$$\mathrm{Tr}_d(f) := \sum_{Q \in \Gamma_1 \backslash \mathcal{Q}_d} \frac{1}{|\Gamma_Q|} f(\tau_Q). \tag{1.1.2}$$

A theorem of Zagier [Z1] states that, for $j_1 := j - 744$, the series

$$g_1(\tau) := \frac{1}{q} - 2 - \sum_{0 > d \equiv 0,1(4)} \mathrm{Tr}_d(j_1)\, q^{-d}$$

is in $M^!_{3/2}$, the space of weakly holomorphic modular forms of weight 3/2 on $\Gamma_0(4)$ which satisfy the plus space condition (see Section 3.2 for details). Zagier further showed that g_1 is the first member of a basis $\{g_D\}_{0<D\equiv 0,1(4)}$ for $M^!_{3/2}$. Each function g_D is uniquely determined by having a Fourier expansion

of the form

$$g_D(\tau) = q^{-D} - \sum_{0>d\equiv 0,1(4)} a(D,d)q^{-d}. \tag{1.1.3}$$

The coefficients $a(D,d)$ with D a fundamental discriminant are given by

$$a(D,d) = -\operatorname{Tr}_{d,D}(j_1),$$

where $\operatorname{Tr}_{d,D}$ denotes the twisted trace

$$\operatorname{Tr}_{d,D}(f) := \frac{1}{\sqrt{D}} \sum_{Q\in\Gamma_1\backslash\mathscr{Q}_{dD}} \frac{\chi_D(Q)}{|\Gamma_Q|} f(\tau_Q), \tag{1.1.4}$$

and $\chi_D : \mathscr{Q}_{dD} \to \{\pm 1\}$ is defined in (3.1.2) below.

The coefficients of the forms in Zagier's basis $\{g_D\}$ also appear as coefficients of forms in the basis $\{f_D\}_{0\geq D\equiv 0,1(4)}$ given by Borcherds in Section 14 of [Bo]. Borcherds showed that the coefficients of the f_D are the exponents in the infinite product expansions of certain meromorphic modular forms.

Suppose now that Q has positive nonsquare discriminant; then $Q(\tau,1)$ has two irrational real roots. Let S_Q denote the geodesic in $\mathbb{H}$ connecting the roots, oriented counter-clockwise if $a>0$ and clockwise if $a<0$. In this case the stabilizer Γ_Q is infinite cyclic, and $C_Q := \Gamma_Q\backslash S_Q$ defines a closed geodesic on the modular curve. In analogy with (1.1.4) we define, for $dD>0$ not a square,

$$\operatorname{Tr}_{d,D}(f) := \frac{1}{2\pi} \sum_{Q\in\Gamma_1\backslash\mathscr{Q}_{dD}} \chi_D(Q) \int_{C_Q} f(\tau)\frac{d\tau}{Q(\tau,1)}. \tag{1.1.5}$$

Let $\mathcal{M}^+_{1/2}$ denote the space of mock modular forms of weight $1/2$ on $\Gamma_0(4)$ satisfying the plus space condition (see Section 3.2 for definitions). A beautiful result of Duke, Imamoḡlu, and Tóth [DIT] shows that the twisted traces (1.1.4) and (1.1.5) appear as coefficients of mock modular forms in a basis $\{f_D\}_{D\equiv 0,1(4)}$ for $\mathcal{M}^+_{1/2}$. When $D<0$, the form f_D is a weakly holomorphic modular form, and is uniquely determined by having a Fourier expansion of the form

$$f_D(\tau) = q^D + \sum_{0<d\equiv 0,1(4)} a(d,D)q^d.$$

The coefficients $a(d,D)$ are the same as those in (1.1.3). Therefore, when D is a fundamental discriminant, they are given in terms of twisted traces.

When $D > 0$ the mock modular form f_D is uniquely determined by being holomorphic at ∞ and having shadow equal to $2g_D$. Let

$$f_D(\tau) = \sum_{0<d\equiv 0,1(4)} a(d,D)\, q^d.$$

If D is a fundamental discriminant and dD is not a square, then Theorem 3 of [DIT] shows that

$$a(d,D) = \mathrm{Tr}_{d,D}(j_1).$$

In [DIT] the coefficients $a(d,D)$ for square dD are defined as infinite series involving Kloosterman sums and the J-Bessel function. The authors leave an arithmetic or geometric interpretation of these coefficients as an open problem.

When the discriminant of Q is a square, the stabilizer Γ_Q is trivial. In this case the geodesic C_Q connects two elements of $\mathbb{P}^1(\mathbb{Q})$, but since any $f \in \mathbb{C}[j]$ has a pole at ∞ (which is Γ_1-equivalent to every element of $\mathbb{P}^1(\mathbb{Q})$), the integral

$$\int_{C_Q} f(\tau)\, \frac{d\tau}{Q(\tau,1)} \tag{1.1.6}$$

diverges. This is the obstruction to a geometric interpretation of the modular trace for square discriminants. In a recent paper, Bruinier, Funke, and Imamoḡlu [BFI] address this issue by regularizing the integral (1.1.6) and showing that the corresponding modular traces

$$\mathrm{Tr}_d(j_1) = \frac{1}{2\pi} \sum_{Q\in\Gamma_1\backslash\mathcal{Q}_d} \int_{C_Q}^{\mathrm{reg}} j_1(\tau) \frac{d\tau}{Q(\tau,1)}$$

give the coefficients of f_1. Their proof is quite different than the argument given in [DIT] for nonsquare discriminants. It involves a regularized theta lift and applies to a much more general class of modular functions (specifically, weak harmonic Maass forms of weight 0 on any congruence subgroup of Γ_1).

Here we provide an alternate definition of $\mathrm{Tr}_{d,D}$ when dD is a square and show that the corresponding coefficients of f_D are given in terms of convergent integrals of functions $j_{1,Q}$ which are related to j_1. Furthermore, using this definition we show that a suitable modification of the proof of Theorem 3 of [DIT] for nonsquare discriminants works for all discriminants.

We first define a sequence of modular functions $\{j_m\}_{m\geq 0}$ which forms a basis for the space $\mathbb{C}[j]$. We let $j_0 := 1$ and for $m \geq 1$ we define j_m to be the unique modular function of the form

$$j_m(\tau) = q^{-m} + \sum_{n>0} c_m(n) q^n.$$

Note that $j_1 = j - 744$ was already defined above.

We define the functions $j_{m,Q}$ as follows. When the discriminant of Q is a square, each root of $Q(x,y)$ corresponds to a cusp $\alpha = \frac{r}{s} \in \mathbb{P}^1(\mathbb{Q})$ with $(r,s) = 1$. Let $\gamma_\alpha := \left(\begin{smallmatrix} * & * \\ s & -r \end{smallmatrix}\right) \in \Gamma_1$ be a matrix that sends α to ∞, and define

$$j_{m,Q}(\tau) := j_m(\tau) - 2 \sum_{\alpha \in \{\text{roots of } Q\}} \sinh(2\pi m \operatorname{Im} \gamma_\alpha \tau)\, e(m \operatorname{Re} \gamma_\alpha \tau).$$

Note that there are only two terms in the sum. When $dD > 0$ is a square, we define the twisted trace of j_m by

$$\operatorname{Tr}_{d,D}(j_m) := \frac{1}{2\pi} \sum_{Q \in \Gamma_1 \backslash \mathscr{Q}_{dD}} \chi_D(Q) \int_{C_Q} j_{m,Q}(\tau) \frac{d\tau}{Q(\tau,1)}. \tag{1.1.7}$$

Remark. If α is a root of Q and $\sigma \in \Gamma_1$, then $\sigma\alpha$ is a root of σQ (see (3.1.1) below). Since $\gamma_{\sigma\alpha}\sigma = \gamma_\alpha$, we have $j_{m,\sigma Q}(\sigma\tau) = j_{m,Q}(\tau)$. Together with (3.1.3) below and the fact that $\chi_D(\sigma Q) = \chi_D(Q)$, this shows that the summands in (1.1.7) remain unchanged by $Q \mapsto \sigma Q$. Therefore $\operatorname{Tr}_{d,D}(j_m)$ is well-defined.

Theorem 1.1.1. *Suppose that $0 < d \equiv 0, 1 \pmod 4$ and that $D > 0$ is a fundamental discriminant. With $\operatorname{Tr}_{d,D}(j_1)$ defined in (1.1.5) and (1.1.7) for nonsquare and square dD, respectively, the function*

$$f_D(\tau) = \sum_{0<d\equiv 0,1(4)} \operatorname{Tr}_{d,D}(j_1)\, q^d$$

is a mock modular form of weight $1/2$ for $\Gamma_0(4)$ with shadow $2g_D$.

It is instructive to consider the special case $d = D = 1$. In this case, there is one quadratic form $Q = [0,1,0]$ with roots 0 and ∞, so C_Q is the upper half of the imaginary axis. Then

$$j_{m,Q}(iy) = j_m(iy) - 2\sinh(2\pi m y) - 2\sinh(2\pi m/y),$$

and we have

$$\lim_{y\to 0^+}\frac{j_{m,Q}(iy)}{y}=-4\pi m.$$

Since $j_{m,Q}(iy)/y=O(1/y^2)$ as $y\to\infty$, the integral

$$\mathrm{Tr}_{1,1}(j_m)=\frac{1}{2\pi}\int_0^\infty j_{m,Q}(iy)\frac{dy}{y} \tag{1.1.8}$$

converges. Theorem 1.1.1 shows that $\mathrm{Tr}_{1,1}(j_1)=-16.028\ldots$ is the coefficient of q in the mock modular form f_1.

Remark. The regularization in [BFI, eq. (1.10)] of the integral (1.1.6) essentially amounts to replacing the divergent integral

$$\int_1^\infty e^{2\pi y}\,\frac{dy}{y}=\int_{-2\pi}^{-\infty} e^{-t}\,\frac{dt}{t}$$

by $-107.47\ldots$, which is the Cauchy principal value of the integral

$$\int_{-2\pi}^{\infty} e^{-t}\frac{dt}{t}.$$

This is equivalent to assigning the value 0 to the integral

$$\int_0^\infty (2\sinh(2\pi y)+2\sinh(2\pi/y))\,\frac{dy}{y},$$

so the values of $\mathrm{Tr}_{1,1}(j_1)$ in [BFI] and (1.1.8) agree.

The modular traces $\mathrm{Tr}_{d,D}(j_m)$ for $m>1$ are also related to the coefficients $a(D,d)$. With the modular trace now defined when dD is a square, we obtain Theorem 3 of [DIT] with the condition "dD not a square" removed. Theorem 1.1.1 follows as a corollary.

Theorem 1.1.2. *Let $a(D,d)$ be the coefficients defined above. For $0<d\equiv 0,1 \pmod 4$ and $D>0$ a fundamental discriminant we have*

$$\mathrm{Tr}_{d,D}(j_m)=\sum_{n|m}\left(\frac{D}{m/n}\right) n\,a(n^2D,d). \tag{1.1.9}$$

It is natural to ask whether there are other spaces of harmonic Maass forms which have bases whose coefficients encode real and imaginary quadratic data. This is indeed the case; here we turn our attention to $H^!_{5/2}(\chi)$, the space of harmonic Maass forms of weight $\frac{5}{2}$ which transform as

$$f(\gamma\tau) = \chi(\gamma)(c\tau + d)^{\frac{5}{2}} f(\tau) \qquad \text{for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}).$$

Here χ is the multiplier system defined by

$$\chi(\gamma) := \frac{\eta(\gamma\tau)}{\sqrt{c\tau + d}\,\eta(\tau)} \qquad \text{for any } \tau \in \mathbb{H},$$

where η is the Dedekind eta-function

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{k \geq 1} (1 - q^k). \tag{1.1.10}$$

We will show that the coefficients of forms in $H^!_{5/2}(\chi)$ are given by traces of singular invariants, that is, sums of CM values of certain weight 0 weak Maass forms on $\Gamma_0(6)$ in the imaginary quadratic case, or cycle integrals of those forms over geodesics on the modular curve

$$X_0(6) := \widehat{\Gamma_0(6)\backslash\mathbb{H}}$$

in the real quadratic case.

In [AA1], the author and Ahlgren constructed an infinite basis $\{h_m\}_{m \equiv 1(24)}$ for $H^!_{5/2}(\chi)$. For $m < 0$ the h_m are holomorphic, while for $m > 0$ they are nonholomorphic harmonic Maass forms whose shadows are weakly holomorphic modular forms of weight $-\frac{1}{2}$. The first element, $\boldsymbol{P} := h_1$, has Fourier expansion[3]

$$\boldsymbol{P}(\tau) = i\, q^{\frac{1}{24}} + \sum_{0 < n \equiv 1(24)} n\, p(1, n) q^{\frac{n}{24}} \\ - i\beta(-y) q^{\frac{1}{24}} + \sum_{0 > n \equiv 1(24)} |n|\, p(1, n) \beta(|n| y) q^{\frac{n}{24}}, \tag{1.1.11}$$

where $\beta(y)$ is the normalized incomplete gamma function

$$\beta(y) := \frac{\Gamma\left(-\frac{3}{2}, \frac{\pi y}{6}\right)}{\Gamma(-\frac{3}{2})} = \frac{3}{4\sqrt{\pi}} \int_{\frac{\pi y}{6}}^{\infty} e^{-t} t^{-\frac{5}{2}} dt.$$

[3]Note that here, and in (1.1.17) and (1.1.18), we have renormalized the functions $\boldsymbol{P}$ and h_m from [AA1]. See (4.4.1) below.

The function $\boldsymbol{P}(\tau)$ is of particular interest since $\xi_{5/2}(\boldsymbol{P})$ is proportional to

$$\eta^{-1}(\tau) = q^{-\frac{1}{24}} + \sum_{0>n\equiv 1(24)} p\left(\tfrac{1-n}{24}\right) q^{\frac{|n|}{24}},$$

where $p\left(\frac{1-n}{24}\right)$ is the ordinary partition function. Corollary 2 of [AA1] shows that that for negative $n \equiv 1 \pmod{24}$ we have $p(1,n) = \sqrt{|n|}\, p\left(\frac{1-n}{24}\right)$.

Building on work of Hardy and Ramanujan [HR], Rademacher [R1, R2] proved an exact formula for $p(k)$ (see Section 1.2). Using Rademacher's formula, Bringmann and Ono [BO1] showed that $p(k)$ can be written as a sum of CM values of a certain nonholomorphic Maass-Poincaré series on $\Gamma_0(6)$. Later Bruinier and Ono [BrO] refined the results of [BO1]; by applying a theta lift to a certain weight -2 modular form, they obtained a new formula for $p(k)$ as a finite sum of algebraic numbers.

The formula of Bruinier and Ono involves a certain nonholomorphic, $\Gamma_0(6)$-invariant function $P(\tau)$, which the authors define in the following way. Let

$$F(\tau) := \frac{1}{2} \cdot \frac{E_2(\tau) - 2E_2(2\tau) - 3E_2(3\tau) + 6E_2(6\tau)}{(\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2} = q^{-1} - 10 - 29q - \ldots,$$

where E_2 denotes the weight 2 quasi-modular Eisenstein series

$$E_2(\tau) := 1 - 24\sum_{n=1}^{\infty}\sum_{d|n} dq^n.$$

The function $F(\tau)$ is in $M^!_{-2}(\Gamma_0(6), 1, -1)$, the space of weakly holomorphic modular forms of weight -2 on $\Gamma_0(6)$, having eigenvalues 1 and -1 under the Atkin-Lehner involutions W_6 and W_2, respectively (see Section 4.1 for definitions). The function $P(\tau)$ is the weak Maass form given by

$$P(\tau) := \frac{1}{4\pi} R_{-2} F(\tau) = -\left(q\frac{d}{dq} + \frac{1}{2\pi y}\right) F(\tau),$$

where R_{-2} is the Maass raising operator of weight -2 (see Section 4.1).

For each $n \equiv 1 \pmod{24}$ and $r \in \{1, 5, 7, 11\}$, let

$$\mathcal{Q}_n^{(r)} := \left\{[a,b,c] \in \mathscr{Q}_n : 6 \mid a \text{ and } b \equiv r \pmod{12}\right\}.$$

Here $[a, b, c]$ denotes the quadratic form $ax^2 + bxy + cy^2$. The group $\Gamma_6 := \Gamma_0(6)/\{\pm 1\}$ acts on $\mathcal{Q}_n^{(r)}$, and for each r we have the canonical isomorphism (see [GKZ, Section I])

$$\Gamma_6 \backslash \mathcal{Q}_n^{(r)} \xrightarrow{\sim} \Gamma_1 \backslash \mathscr{Q}_n. \tag{1.1.12}$$

Bruinier and Ono [BrO, Theorem 1.1] proved the finite algebraic formula

$$p\left(\tfrac{1-n}{24}\right) = \frac{1}{|n|} \sum_{Q \in \Gamma_6 \backslash \mathcal{Q}_n^{(1)}} P(\tau_Q), \tag{1.1.13}$$

for each $0 > n \equiv 1 \pmod{24}$. Therefore the coefficients of the nonholomorphic part of $\boldsymbol{P}(\tau)$ are given in terms of singular invariants (note the similarity with the expression (1.1.2)).

For the holomorphic part of $\boldsymbol{P}(\tau)$ one might suspect by analogy with (1.1.5) that the traces

$$\sum_{Q \in \Gamma_6 \backslash \mathcal{Q}_n^{(1)}} \int_{C_Q} P(\tau) \frac{d\tau}{Q(\tau, 1)} \tag{1.1.14}$$

give the coefficients $p(1, n)$. However, as we show in the remarks following Proposition 4.4.1 below, the traces (1.1.14) are identically zero whenever n is not a square. In order to describe the arithmetic nature of the coefficients $p(1, n)$, we must introduce the function $P(\tau, s)$, of which $P(\tau)$ is the specialization at $s = 2$ (see Section 4.1 for the definition of $P(\tau, s)$). Similarly, for each $Q \in \mathcal{Q}_n^{(1)}$ whose discriminant is a square, we will define $P_Q(\tau, s)$, a dampened version of $P(\tau, s)$ (see Section 4.2). These are analogues of the functions $j_{1,Q}(\tau)$ and are defined in (4.2.13) below. Then, for each $n \equiv 1 \pmod{24}$ we define the trace

$$\mathrm{Tr}(n) := \begin{cases} \dfrac{1}{\sqrt{|n|}} \displaystyle\sum_{Q \in \Gamma_6 \backslash \mathcal{Q}_n^{(1)}} P(\tau_Q) & \text{if } n < 0, \\ \dfrac{1}{2\pi} \displaystyle\sum_{Q \in \Gamma_6 \backslash \mathcal{Q}_n^{(1)}} \int_{C_Q} \left[\frac{\partial}{\partial s} P(\tau, s) \Big|_{s=2} \right] \frac{d\tau}{Q(\tau, 1)} & \begin{array}{l}\text{if } n > 0 \\ \text{is not a square,}\end{array} \\ \dfrac{1}{2\pi} \displaystyle\sum_{Q \in \Gamma_6 \backslash \mathcal{Q}_n^{(1)}} \int_{C_Q} \left[\frac{\partial}{\partial s} P_Q(\tau, s) \Big|_{s=2} \right] \frac{d\tau}{Q(\tau, 1)} & \begin{array}{l}\text{if } n > 0 \\ \text{is a square.}\end{array} \end{cases} \tag{1.1.15}$$

We have the following theorem which provides an arithmetic interpretation for the coefficients of $\boldsymbol{P}(\tau)$ by relating them to the traces (1.1.15). It is a special case of Theorem 1.1.4 below.

Theorem 1.1.3. *For each* $n \equiv 1 \pmod{24}$ *we have*

$$p(1,n) = \mathrm{Tr}(n). \tag{1.1.16}$$

Remark. Since $p(1,n) = \sqrt{|n|}\, p\left(\frac{1-n}{24}\right)$ for $0 > n \equiv 1 \pmod{24}$, Theorem 1.1.3 recovers the algebraic formula (1.1.13) of Bruinier and Ono. However, our proof is quite different than the proof given in [BrO].

Recall that $\boldsymbol{P} = h_1$ is the first element of an infinite basis $\{h_m\}$ for $H_{5/2}(\chi)$. We turn to the other harmonic Maass forms h_m for general $m \equiv 1 \pmod{24}$. For negative m, we have the Fourier expansion (see [AA1, Theorem 1])

$$h_m(\tau) = |m|^{\frac{3}{2}} q^{\frac{m}{24}} - \sum_{0<n\equiv 1(24)} |mn|\, p(m,n) q^{\frac{n}{24}}. \tag{1.1.17}$$

These forms are holomorphic on $\mathbb{H}$ and can be constructed using $\eta(\tau)$ and $j'(\tau) := -q\frac{d}{dq} j(\tau)$. We list a few examples here.

$$\begin{aligned}
23^{-\frac{3}{2}} h_{-23} &= \eta\, j' \\
&= q^{-\frac{23}{24}} - q^{\frac{1}{24}} - 196\,885\, q^{\frac{25}{24}} - \dots, \\
47^{-\frac{3}{2}} h_{-47} &= \eta\, j'(j - 743) \\
&= q^{-\frac{47}{24}} - 2\, q^{\frac{1}{24}} - 21\,690\,645\, q^{\frac{25}{24}} - \dots, \\
71^{-\frac{3}{2}} h_{-71} &= \eta\, j'(j^2 - 1487j + 355\,910) \\
&= q^{-\frac{71}{24}} - 3\, q^{\frac{1}{24}} - 886\,187\,500\, q^{\frac{25}{24}} - \dots.
\end{aligned}$$

For positive m, we have the Fourier expansion (see [AA1, Theorem 1])

$$\begin{aligned}
h_m = {} & i\, m^{\frac{3}{2}} q^{\frac{m}{24}} + \sum_{0<n\equiv 1(24)} mn\, p(m,n) q^{\frac{n}{24}} \\
& - i\, m^{\frac{3}{2}} \beta(-my) q^{\frac{m}{24}} + \sum_{0>n\equiv 1(24)} |mn|\, p(m,n) \beta(|n|y) q^{\frac{n}{24}}.
\end{aligned} \tag{1.1.18}$$

For these m, the shadow of h_m is proportional to the weight $-\frac{1}{2}$ form

$$g_m(\tau) := m^{-\frac{3}{2}} q^{-\frac{m}{24}} + \sum_{0>n\equiv 1(24)} |mn|^{-\frac{1}{2}}\, p(n,m) q^{\frac{|n|}{24}},$$

with $p(n,m)$ as in (1.1.17). When $mn<0$, we have the relation $p(n,m) = -p(m,n)$. The first few examples of these forms are

$$\begin{aligned}
g_1 &= \eta^{-1} \\
&= q^{-\frac{1}{24}} + q^{\frac{23}{24}} + 2\,q^{\frac{47}{24}} + 3\,q^{\frac{71}{24}} + 5\,q^{\frac{95}{24}} + 7\,q^{\frac{119}{24}} + \dots, \\
5^3 g_{25} &= \eta^{-1}(j-745) \\
&= q^{-\frac{25}{24}} + 196\,885\ q^{\frac{23}{24}} + 21\,690\,645\,q^{\frac{47}{24}} + \dots, \\
7^3 g_{49} &= \eta^{-1}(j^2 - 1489j + 160\,511) \\
&= q^{-\frac{49}{24}} + 42\,790\,636\,q^{\frac{23}{24}} + 40\,513\,206\,272\,q^{\frac{47}{24}} + \dots.
\end{aligned}$$

We also have the relation $p(m,n) = p(n,m)$ when $m,n>0$ (see [AA1, Corollary 2]).

In order to give an arithmetic interpretation for the coefficients $p(m,n)$ for general m, we require a family of functions $\{P_v(\tau,s)\}_{v\in\mathbb{N}}$ whose first member is $P_1(\tau,s) = P(\tau,s)$. We construct these functions in Section 4.1 using nonholomorphic Maass-Poincaré series. The specializations $P_v(\tau) := P_v(\tau,2)$ can be obtained by raising the elements of a certain basis $\{F_v\}_{v\in\mathbb{N}}$ of $M_{-2}(\Gamma_0(6),1,-1)$ to weight 0. The functions F_v are uniquely determined by having Fourier expansion $F_v(\tau) = q^{-v} + O(1)$. They are easily constructed from F and the $\Gamma_0(6)$-Hauptmodul

$$J_6(\tau) := \left(\frac{\eta(\tau)\eta(2\tau)}{\eta(3\tau)\eta(6\tau)}\right)^4 + \left(3\frac{\eta(3\tau)\eta(6\tau)}{\eta(\tau)\eta(2\tau)}\right)^4 = q^{-1} - 4 + 79q + 352q^2 + \dots.$$

For example, $F_1 = F$ and

$$\begin{aligned}
F_2 &= F(J_6+14) = q^{-2} - 50 - 832\,q - \dots, \\
F_3 &= F(J_6^2 + 18J_6 + 27) = q^{-3} - 190 - 7371\,q - \dots, \\
F_4 &= F(J_6^3 + 22J_6^2 + 20J_6 - 1160) = q^{-4} - 370 - 48\,640\,q - \dots.
\end{aligned}$$

We also require functions $P_{v,Q}(\tau,s)$ for each quadratic form with square discriminant. These are dampened versions of the functions $P_v(\tau,s)$, and are constructed in Section 4.2.

With $\chi_m : \mathcal{Q}_{mn}^{(1)} \to \{-1,0,1\}$ as in (4.2.7) below, we define the general twisted traces for $m, n \equiv 1 \pmod{24}$ by

$$\mathrm{Tr}_v(m,n) := \begin{cases} \dfrac{1}{\sqrt{|mn|}} \displaystyle\sum_{Q \in \Gamma_6 \backslash \mathcal{Q}_{mn}^{(1)}} \chi_m(Q) P_v(\tau_Q) & \text{if } mn < 0, \\ \dfrac{1}{2\pi} \displaystyle\sum_{Q \in \Gamma_6 \backslash \mathcal{Q}_{mn}^{(1)}} \chi_m(Q) \int_{C_Q} \left[\frac{\partial}{\partial s} P_v(\tau, s) \Big|_{s=2} \right] \frac{d\tau}{Q(\tau,1)} & \text{if } mn > 0 \text{ is not a square,} \\ \dfrac{1}{2\pi} \displaystyle\sum_{Q \in \Gamma_6 \backslash \mathcal{Q}_{mn}^{(1)}} \chi_m(Q) \int_{C_Q} \left[\frac{\partial}{\partial s} P_{v,Q}(\tau, s) \Big|_{s=2} \right] \frac{d\tau}{Q(\tau,1)} & \text{if } mn > 0 \text{ is a square.} \end{cases} \tag{1.1.19}$$

The following theorem relates the coefficients $p(m, n)$ to these twisted traces, giving an arithmetic interpretation of the coefficients $p(m, n)$ for all m, n.

Theorem 1.1.4. *Suppose that $m, n \equiv 1 \pmod{24}$ and that m is squarefree. For each $v \geq 1$ coprime to 6 we have*

$$\mathrm{Tr}_v(m,n) = \sum_{d|v} d \left(\frac{m}{v/d} \right) \left(\frac{12}{d} \right) p(d^2 m, n). \tag{1.1.20}$$

Remark. The construction of the harmonic Maass forms h_m for $m > 0$ relies heavily on the fact that the space of cusp forms of weight $5/2$ with multiplier system χ is trivial (see [AA1, §3.3]). It would be interesting to generalize Theorem 1.1.4 to other weights; however, the presence of cusp forms in higher weights would add significant complications.

In the proof of Theorem 1.1.4 we will encounter the Kloosterman sum

$$K(a, b; c) := \sum_{\substack{d \bmod c \\ (d,c)=1}} e^{\pi i s(d,c)} e\left(\frac{\overline{d}a + db}{c} \right), \tag{1.1.21}$$

where $\overline{d}$ denotes the inverse of d modulo c. Here $s(d, c)$ is the Dedekind sum

$$s(d,c) := \sum_{r=1}^{c-1} \frac{r}{c} \left(\frac{dr}{c} - \left\lfloor \frac{dr}{c} \right\rfloor - \frac{1}{2} \right) \tag{1.1.22}$$

which appears in the transformation formula for $\eta(z)$ (see Section 2.8 of [I2]). The presence of the factor $e^{\pi i s(d,c)}$ makes the Kloosterman sum quite difficult to evaluate. The following formula, proved by Whiteman in [Wh], is attributed to Selberg and gives an evaluation in the special case $a = 0$:

$$K(0,b;c) = \sqrt{\frac{c}{3}} \sum_{\substack{\ell \bmod 2c \\ (3\ell^2+\ell)/2 \equiv b(c)}} (-1)^\ell \cos\left(\frac{6\ell+1}{6c}\pi\right). \tag{1.1.23}$$

Theorem 1.1.5 below is a generalization of (1.1.23) which is of independent interest. For each v with $(v,6) = 1$ and for each pair $m, n \equiv 1 \pmod{24}$ with m squarefree, define

$$\mathfrak{S}_v(m,n;24c) := \sum_{\substack{b \bmod 24c \\ b^2 \equiv mn(24c)}} \left(\frac{12}{b}\right) \chi_m\left(\left[6c, b, \tfrac{b^2-mn}{24c}\right]\right) e\left(\frac{bv}{12c}\right). \tag{1.1.24}$$

It is not difficult to show (see (4.3.3) below) that the right-hand side of (1.1.23) is equal to

$$\frac{1}{4}\,\mathfrak{S}_1(1, 24b+1; 24c).$$

Theorem 1.1.5. *Suppose that $n = 24n' + 1$ and that $M = v^2 m = 24M' + 1$, where m is squarefree and $(v,6) = 1$. Then*

$$K(M',n';c) = 4\sqrt{\frac{c}{3}} \left(\frac{12}{v}\right) \sum_{u|(v,c)} \mu(u) \left(\frac{m}{u}\right) \mathfrak{S}_{v/u}(m,n;24c/u). \tag{1.1.25}$$

Using Theorem 1.1.5 we obtain a bound for the size of the individual Kloosterman sums $K(a,b,c)$ which is reminiscent of Weil's bound [W]

$$|S(m,n,c)| \le \tau(c)(m,n,c)^{\frac{1}{2}}\sqrt{c} \tag{1.1.26}$$

for the ordinary Kloosterman sums (see Section 1.2), where $\tau(c)$ is the number of divisors of c.

Corollary 1.1.6. *Assuming the notation of Theorem 1.1.5, we have*

$$K(M',n';c) \ll \tau((v,c))\tau(c)(mn,c)^{\frac{1}{2}}\sqrt{c}. \tag{1.1.27}$$

1.2 Kloosterman sums and Maass cusp forms of half-integral weight

The ordinary Kloosterman sum

$$S(m,n,c) = \sum_{\substack{d \bmod c \\ (d,c)=1}} e\left(\frac{m\bar{d}+nd}{c}\right)$$

plays a leading part in analytic number theory. Indeed, many problems can be reduced to estimates for sums of Kloosterman sums (see, for example, [HB] or [Sa]). The sums $S(m,n,c)$ arise in the theory of modular forms of even integral weight on $\mathrm{SL}_2(\mathbb{Z})$ with trivial multiplier. Kloosterman sums with general weights and multipliers on subgroups of $\mathrm{SL}_2(\mathbb{Z})$ have been studied by Bruggeman [Br2], Goldfeld-Sarnak [GS], and Pribitkin [Pr], among others. Here we focus on the multiplier system χ for the Dedekind eta function and the associated Kloosterman sums

$$S(m,n,c,\chi) = \sum_{\substack{0\le a,d<c \\ \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})}} \overline{\chi}\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right) e\left(\frac{\widetilde{m}a+\widetilde{n}d}{c}\right), \qquad \widetilde{m} := m - \frac{23}{24}. \tag{1.2.1}$$

In this section we follow the notation of [GS]; in the notation of the previous section, we have $S(m,n,c,\chi) = \sqrt{i}\,K(m-1,n-1;c)$. The sums $S(m,n,c,\chi)$ are intimately connected to the partition function $p(n)$, and we begin by discussing an application of our main theorem to a classical problem.

In the first of countless important applications of the circle method, Hardy and Ramanujan [HR] proved the asymptotic formula

$$p(n) \sim \frac{1}{4\sqrt{3}\,n} e^{\pi\sqrt{2n/3}}$$

(and in fact developed an asymptotic series for $p(n)$). Perfecting their method,

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The results of Section 1.2, together with those of Chapter 5, are joint with Scott Ahlgren and appear in the paper [AA2].

Rademacher [R1, R2] proved that

$$p(n) = \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \sum_{c=1}^{\infty} \frac{A_c(n)}{c} I_{\frac{3}{2}} \left(\frac{\pi\sqrt{24n-1}}{6c} \right), \tag{1.2.2}$$

where $I_{\frac{3}{2}}$ is an I-Bessel function,

$$A_c(n) := \sum_{\substack{d \bmod c \\ (d,c)=1}} e^{\pi i s(d,c)} e\left(-\frac{dn}{c}\right),$$

and $s(d,c)$ is a Dedekind sum (see (2.1.4) below). The Kloosterman sum $A_c(n)$ is a special case of (1.2.1); in particular we have (see §2.3.7 below)

$$A_c(n) = \sqrt{-i} S(1, 1-n, c, \chi). \tag{1.2.3}$$

The series (1.2.2) converges rapidly. For example, the first four terms give

$$\begin{aligned} p(100) &\approx 190\,568\,944.783 + 348.872 - 2.598 + 0.685 \\ &= 190\,569\,291.742, \end{aligned}$$

while the actual value is $p(100) = 190\,569\,292$.

A natural problem is to estimate the error which results from truncating the series (1.2.2) after the Nth term, or in other words to estimate the quantity $R(n,N)$ defined by

$$p(n) = \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \sum_{c=1}^{N} \frac{A_c(n)}{c} I_{\frac{3}{2}} \left(\frac{\pi\sqrt{24n-1}}{6c} \right) + R(n,N). \tag{1.2.4}$$

Since $I_\nu(x)$ grows exponentially as $x \to \infty$, one must assume that $N \gg \sqrt{n}$ in order to obtain reasonable estimates. For $\alpha > 0$, Rademacher [R1, (8.1)] showed that

$$R(n, \alpha\sqrt{n}) \ll_\alpha n^{-\frac{1}{4}}.$$

Lehmer [L1, Theorem 8] proved the sharp Weil-type bound

$$|A_c(n)| < 2^{\omega_o(c)}\sqrt{c}, \tag{1.2.5}$$

where $\omega_o(c)$ is the number of distinct odd primes dividing c. Shortly thereafter

[L2] he used this bound to prove that

$$R(n, \alpha\sqrt{n}) \ll_\alpha n^{-\frac{1}{2}} \log n. \tag{1.2.6}$$

In 1938, values of $p(n)$ had been tabulated for $n \leq 600$ [Gu]. Using (1.2.5), Lehmer showed [L1, Theorem 13] that $p(n)$ is the nearest integer to the series (1.2.2) truncated at $\frac{2}{3}\sqrt{n}$ for all $n > 600$.

Building on work of Selberg and Whiteman [Wh], Rademacher [R3] later simplified Lehmer's treatment of the sums $A_c(n)$. Rademacher's book [R4, Chapter IV] gives a relatively simple derivation of the error bound $n^{-\frac{3}{8}}$ using these ideas.

Using equidistribution results for Heegner points on the modular curve $X_0(6)$, Folsom and Masri [FM] improved Lehmer's estimate. They proved that if $24n-1$ is squarefree, then

$$R\left(n, \sqrt{\frac{n}{6}}\right) \ll n^{-\frac{1}{2}-\delta} \quad \text{for some } \delta > 0. \tag{1.2.7}$$

Using the estimates for sums of Kloosterman sums below, we obtain an improvement in the exponent, and we (basically) remove the assumption that $24n-1$ is squarefree. For simplicity in stating the results, we will assume that

$$24n - 1 \text{ is not divisible by } 5^4 \text{ or } 7^4. \tag{1.2.8}$$

As the proof will show (c.f. Section 5.6) the exponent 4 in (1.2.8) can be replaced by any positive integer m; such a change would have the effect of changing the implied constants in (1.2.9) and (1.2.10).

Theorem 1.2.1. *Suppose that $\alpha > 0$. For $n \geq 1$ satisfying* (1.2.8) *we have*

$$R\big(n, \alpha\, n^{\frac{1}{2}}\big) \ll_{\alpha,\epsilon} n^{-\frac{1}{2}-\frac{1}{168}+\epsilon}. \tag{1.2.9}$$

Our method optimizes when N is slightly larger with respect to n.

Theorem 1.2.2. *Suppose that $\alpha > 0$. For $n \geq 1$ satisfying* (1.2.8) *we have*

$$R\big(n, \alpha\, n^{\frac{1}{2}+\frac{5}{252}}\big) \ll_{\alpha,\epsilon} n^{-\frac{1}{2}-\frac{1}{28}+\epsilon}. \tag{1.2.10}$$

Remark. Suppose, for example, that $24n-1$ is divisible by 5^4. Then n has the form $n = 625m + 599$, so that $p(n) \equiv 0 \pmod{625}$ by work of

Ramanujan [BeO, Section 22]. A sharp estimate for $R(n, N)$ for such n is less important since $p(n)$ can be determined by showing that $|R(n, N)| < 312.5$. The situation is similar for any power of 5 or 7.

Remark. The estimates in Theorems 1.2.1 and 1.2.2 depend on progress toward the Ramanujan–Lindelöf conjecture for coefficients of certain Maass cusp forms of weight 1/2. Assuming the conjecture, we can use the present methods to prove

$$R(n, \alpha n^{\frac{1}{2}}) \ll_{\alpha,\epsilon} n^{-\frac{1}{2}-\frac{1}{16}+\epsilon}.$$

The analogue of Linnik and Selberg's conjecture ((1.2.12) below) would give

$$R(n, \alpha n^{\frac{1}{2}}) \ll_{\alpha,\epsilon} n^{-\frac{3}{4}+\epsilon}.$$

Computations suggest that this bound would be optimal.

Theorems 1.2.1 and 1.2.2 follow from estimates for weighted sums of the Kloosterman sums (1.2.1). For the ordinary Kloosterman sum $S(m, n, c)$, Linnik [Li] and Selberg [Se] conjectured that there should be considerable cancellation in the sums

$$\sum_{c \le x} \frac{S(m, n, c)}{c}. \tag{1.2.11}$$

Sarnak and Tsimerman [ST] studied these weighted sums for varying m, n and put forth the following modification of Linnik and Selberg's conjecture with an "ϵ-safety valve" in m, n:

$$\sum_{c \le x} \frac{S(m, n, c)}{c} \ll_\epsilon (|mn|x)^\epsilon. \tag{1.2.12}$$

Using the Weil bound [W]

$$|S(m, n, c)| \le \tau(c)(m, n, c)^{\frac{1}{2}}\sqrt{c}, \tag{1.2.13}$$

where $\tau(c)$ is the number of divisors of c, one obtains

$$\sum_{c \le x} \frac{S(m, n, c)}{c} \ll \tau((m, n))x^{\frac{1}{2}} \log x.$$

Thus the conjecture (1.2.12) represents full square-root cancellation.

The best current bound in the x-aspect for the sums (1.2.11) was obtained by Kuznetsov [Ku], who proved for $m, n > 0$ that

$$\sum_{c \leq x} \frac{S(m,n,c)}{c} \ll_{m,n} x^{\frac{1}{6}} (\log x)^{\frac{1}{3}}.$$

Recently, Sarnak and Tsimerman [ST] refined Kuznetsov's method, making the dependence on m and n explicit. They proved that for $m,\ n > 0$ we have

$$\sum_{c \leq x} \frac{S(m,n,c)}{c} \ll_{\varepsilon} \left(x^{\frac{1}{6}} + (mn)^{\frac{1}{6}} + (m+n)^{\frac{1}{8}} (mn)^{\frac{\theta}{2}} \right) (mnx)^{\varepsilon},$$

where θ is an admissible exponent toward the Ramanujan–Petersson conjecture for coefficients of weight 0 Maass cusp forms. By work of Kim and Sarnak [Ki, Appendix 2], the exponent $\theta = 7/64$ is available. A generalization of their results to sums taken over c which are divisible by a fixed integer q is given by Ganguly and Sengupta [GSe].

We study sums of the Kloosterman sums $S(m,n,c,\chi)$ defined in (1.2.1). In view of (1.2.3) we consider the case when m and n have mixed sign (as will be seen in the proof, this brings up a number of difficulties which are not present in the case when $m,\ n > 0$).

Theorem 1.2.3. *For $m > 0$, $n < 0$ we have*

$$\sum_{c \leq x} \frac{S(m,n,c,\chi)}{c} \ll_{\varepsilon} \left(x^{\frac{1}{6}} + |mn|^{\frac{1}{4}} \right) |mn|^{\epsilon} \log x.$$

While Theorem 1.2.3 matches [ST] in the x aspect, it falls short in the mn aspect. As we discuss in more detail below, this is due to the unsatisfactory Hecke theory in half-integral weight. This bound is also insufficient to improve the estimate (1.2.6) for $R(n,N)$; for this we sacrifice the bound in the x-aspect for an improvement in the n-aspect. The resulting theorem, which for convenience we state in terms of $A_c(n)$, leads to the estimates of Theorems 1.2.1 and 1.2.2.

Theorem 1.2.4. *Fix $0 < \delta < 1/2$. Suppose that $n \geq 1$ satisfies* (1.2.8). *Then*

$$\sum_{c \leq x} \frac{A_c(n)}{c} \ll_{\delta,\varepsilon} n^{\frac{1}{4} - \frac{1}{56} + \epsilon} x^{\frac{3}{4}\delta} + \left(n^{\frac{1}{4} - \frac{1}{168} + \epsilon} + x^{\frac{1}{2} - \delta} \right) \log x.$$

As in [Ku] and [ST], the basic tool is a version of Kuznetsov's trace formula. This relates sums of the Kloosterman sums $S(m,n,c,\chi)$ weighted by a suitable test function to sums involving Fourier coefficients of Maass cusp forms of weight 1/2 and multiplier χ. Proskurin [P2] proved such a formula for general weight and multiplier when $mn > 0$; in Section 5.2 we give a proof in the case $mn < 0$. (Blomer [Bl] has recorded this formula for twists of the theta-multiplier by a Dirichlet character.)

The Maass cusp forms of interest[4] are functions which transform like $\mathrm{Im}(\tau)^{\frac{1}{4}}\eta(\tau)$ and which are eigenfunctions of the weight 1/2 Laplacian $\Delta_{1/2}$ (see Section 2.3 for details). To each Maass cusp form F we attach an eigenvalue λ and a spectral parameter r which are defined via

$$\Delta_{\frac{1}{2}}F + \lambda F = 0$$

and

$$\lambda = \frac{1}{4} + r^2.$$

We denote the space spanned by these Maass cusp forms by $\mathcal{S}_{\frac{1}{2}}(1, \chi)$. Denote by $\{u_j(\tau)\}$ an orthonormal basis of Maass cusp forms for this space, and let r_j denote the spectral parameter attached to each u_j. Then (recalling the notation $\widetilde{n} = n - \frac{23}{24}$), u_j has a Fourier expansion of the form

$$u_j(\tau) = \sum_{n\neq 0} \rho_j(n) W_{\frac{\mathrm{sgn}(n)}{4}, ir_j}(4\pi|\widetilde{n}|y)e(\widetilde{n}x),$$

where $W_{\kappa,\mu}(y)$ is the W-Whittaker function (see §2.3.2). In Section 5.2 we will prove that

$$\sum_{c>0} \frac{S(m,n,c,\chi)}{c}\phi\left(\frac{4\pi\sqrt{\widetilde{m}|\widetilde{n}|}}{c}\right) = 8\sqrt{i}\sqrt{\widetilde{m}|\widetilde{n}|}\sum_{r_j} \frac{\overline{\rho_j(m)}\rho_j(n)}{\mathrm{ch}\,\pi r_j}\check{\phi}(r_j), \tag{1.2.14}$$

where ϕ is a suitable test function and $\check{\phi}$ is the K-Bessel transform

$$\check{\phi}(r) := \mathrm{ch}\,\pi r \int_0^\infty K_{2ir}(u)\phi(u)\frac{du}{u}. \tag{1.2.15}$$

In Section 5.3 we introduce theta lifts (as in Niwa [N], Shintani [Sh]) which give a Shimura-type correspondence

$$\mathcal{S}_{\frac{1}{2}}(N, \psi\chi, r) \to \mathcal{S}_0(6N, \psi^2, 2r). \tag{1.2.16}$$

[4]These are a different kind of Maass form than those of the previous section.

Here $\mathcal{S}_k(N, \nu, r)$ denotes the space of Maass cusp forms of weight k on $\Gamma_0(N)$ with multiplier ν and spectral parameter r. These lifts commute with the action of the Hecke operators on the respective spaces, and in particular allow us to control the size of the weight 1/2 Hecke eigenvalues, which becomes important in Section 5.6. The existence of these lifts shows that there are no small eigenvalues; this fact is used in Sections 5.4, 5.5, and 5.7. In particular, from (1.2.16) it will follow that if $\mathcal{S}_{1/2}(1, \chi, r) \neq 0$ then either $r = i/4$ or $r > 1.9$.

To estimate the right-hand side of (1.2.14), we first obtain a mean value estimate for the sums

$$|\widetilde{n}| \sum_{0<r_j\leq x} \frac{|\rho_j(n)|^2}{\operatorname{ch} \pi r_j}.$$

In analogy with Kuznetsov's result [Ku, Theorem 6] in weight 0, we prove in Section 5.1 a general mean value result for weight $\pm 1/2$ Maass cusp forms which has the following as a corollary.

Theorem 1.2.5. *With notation as above, we have*

$$|\widetilde{n}| \sum_{0<r_j\leq x} \frac{|\rho_j(n)|^2}{\operatorname{ch} \pi r_j} = \begin{cases} \dfrac{x^{\frac{5}{2}}}{5\pi^2} + O_\epsilon\left(x^{\frac{3}{2}} \log x + |n|^{\frac{1}{2}+\epsilon} x^{\frac{1}{2}}\right) & \text{if } n < 0, \\ \dfrac{x^{\frac{3}{2}}}{3\pi^2} + O_\epsilon\left(x^{\frac{1}{2}} \log x + n^{\frac{1}{2}+\epsilon}\right) & \text{if } n > 0. \end{cases}$$

We then require uniform estimates for the Bessel transform (1.2.15), which are made subtle by the oscillatory nature of $K_{ir}(x)$ for small x and by the transitional range of the K-Bessel function. In Section 5.4 we obtain estimates for $\check{\phi}(r)$ which, together with Theorem 1.2.5, suffice to prove Theorem 1.2.3.

To prove Theorem 1.2.4 we require a second estimate for the Fourier coefficients $\rho_j(n)$. In [ST] such an estimate is obtained via the simple relationship

$$a(n) = \lambda(n)a(1)$$

satisfied by the coefficients $a(n)$ of a Hecke eigenform with eigenvalue $\lambda(n)$. This relationship is not available in half-integral weight (the best which one can do is to relate coefficients of index m^2n to those of index n). As a substitute, we employ an average version of a theorem of Duke [D1].

Duke proved that if $a(n)$ is the n-th coefficient of a normalized Maass cusp form in $\mathcal{S}_{1/2}(N, \left(\frac{D}{\bullet}\right) \nu_\theta, r)$ (ν_θ is the theta-multiplier defined in §2.1) then for squarefree n we have

$$|a(n)| \ll_\varepsilon |n|^{-\frac{2}{7}+\varepsilon}|r|^{\frac{5}{2}-\frac{\operatorname{sgn}(n)}{4}} \operatorname{ch}\left(\tfrac{\pi r}{2}\right).$$

This is not strong enough in the r-aspect for our purposes (Baruch and Mao [BM] have obtained a bound which is stronger in the n-aspect, but even weaker in the r-aspect).

Here we modify Duke's argument to obtain an average version of his result. We prove in Theorem 5.6.1 that if the u_j as above are eigenforms of the Hecke operators $T(p^2)$, $p \nmid 6$, then we have

$$\sum_{0<r_j\leq x} \frac{|\rho_j(n)|^2}{\operatorname{ch}\pi r_j} \ll_\varepsilon |n|^{-\frac{4}{7}+\varepsilon} x^{5-\frac{\operatorname{sgn} n}{2}}, \tag{1.2.17}$$

for any collection of values of n such that $24n-23$ is not divisible by arbitrarily large powers of 5 or 7.

1.3 The mock theta conjectures

In his last letter to Hardy, dated three months before his death in early 1920, Ramanujan briefly described a new class of functions which he called mock theta functions, and he listed 17 examples [BR, p. 220]. These he separated into three groups: four of third order, ten of fifth order, and three of seventh order. The fifth order mock theta functions he further divided into two groups[5]; for example, four of these fifth order functions are

$$f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n}, \qquad f_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q)_n},$$
$$F_0(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n}, \qquad F_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}}.$$

Here we have used the q-Pochhammer notation $(a;q)_n := \prod_{m=0}^{n-1}(1-aq^m)$.

The mock theta conjectures are ten identities found in Ramanujan's lost

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The results of Section 1.3 and Chapter 6 appear in the paper [A3].

[5]While Ramanujan reused the letters f, ϕ, ψ, χ, and F in each group, the usual convention is to write those in the first group with a subscript '0' and those in the second group with a subscript '1'.

notebook, each involving one of the fifth-order mock theta functions. The identities for the four mock theta functions listed above are (following the notation of [GM1, p. 206], and correcting a sign error in the fourth identity in that paper; see also [AG, GM2])

$$f_0(q) = -2q^2 M\left(\frac{1}{5}, q^{10}\right) + \theta_4(0, q^5)G(q), \tag{1.3.1}$$

$$f_1(q) = -2q^3 M\left(\frac{2}{5}, q^{10}\right) + \theta_4(0, q^5)H(q), \tag{1.3.2}$$

$$F_0(q) - 1 = qM\left(\frac{1}{5}, q^5\right) - q\psi(q^5)H(q^2), \tag{1.3.3}$$

$$F_1(q) = qM\left(\frac{2}{5}, q^5\right) + \psi(q^5)G(q^2). \tag{1.3.4}$$

Here

$$M(r, q) := \sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(q^r; q)_n (q^{1-r}; q)_n},$$

the functions $\theta_4(0, q)$ and $\psi(q)$ are theta functions, and $G(q)$ and $H(q)$ are the Rogers-Ramanujan functions (see Section 6.1 for definitions). Andrews and Garvan [AG] showed that the mock theta conjectures fall naturally into two families of five identities each (according to Ramanujan's original grouping), and that within each family the truth of any one of the identities implies the truth of the others via straightforward q-series manipulations. Shortly thereafter Hickerson [H] proved the mock theta conjectures by establishing the identities involving $f_0(q)$ and $f_1(q)$. According to Gordon and McIntosh [GM2, p. 106], the mock theta conjectures together form "one of the fundamental results of the theory of [mock theta functions]" and Hickerson's proof is a "tour de force."

In his PhD thesis [Zw], Zwegers showed that the mock theta functions can be completed to real analytic modular forms of weight 1/2 by multiplying by a suitable rational power of q and adding nonholomorphic integrals of certain unary theta series of weight 3/2. This allows the mock theta functions to be studied using the theory of harmonic Maass forms. Bringmann, Ono, and Rhoades remark in [BOR, p. 1087] that their Theorem 1.1, together with the work of Zwegers, reduces the proof of the mock theta conjectures to "the verification of two simple identities for classical weakly holomorphic modular forms." Zagier makes a similar comment in [Z2, §6]. Following their approach, Folsom [Fo] reduced the proof of the $\chi_0(q)$ and $\chi_1(q)$ mock theta conjectures to the verification of two identities in the space of modular forms of weight

$1/2$ for the subgroup $G = \Gamma_1(144 \cdot 10^2 \cdot 5^4)$. Since $[\mathrm{SL}_2(\mathbb{Z}) : G] \geq 5 \times 10^{13}$, this computation is currently infeasible.

In Chapter 6 we provide a conceptual, unified proof of the mock theta conjectures that relies neither on computational verification nor on the work Andrews and Garvan [AG]. Our method proves four of the ten mock theta conjectures simultaneously; two from each family (namely the identities (1.3.1)–(1.3.4) above). Four of the remaining six conjectures can be proved using the same method, and the remaining two follow easily from the others (see Section 6.4).

To accomplish our goal, we recast the mock theta conjectures in terms of an equality between two nonholomorphic vector-valued modular forms $\boldsymbol{F}$ and $\boldsymbol{G}$ of weight $1/2$ on $\mathrm{SL}_2(\mathbb{Z})$ which transform according to the Weil representation (see Lemma 6.2.1 below), and we show that the difference $\boldsymbol{F} - \boldsymbol{G}$ is a holomorphic vector-valued modular form. Employing a natural isomorphism between the space of such forms and the space $J_{1,60}$ of Jacobi forms of weight 1 and index 60, together with the result of Skoruppa that $J_{1,m} = \{0\}$ for all m, we conclude that $\boldsymbol{F} = \boldsymbol{G}$.

2 Background

In this chapter, we give some background information on multiplier systems, holomorphic modular forms, and two types of nonholomorphic modular forms: harmonic Maass forms and Maass cusp forms. In the literature it is typical to see two different definitions of Maass forms. In this thesis, the first definition is used in Chapters 3, 4, and 6, while the second is used in Chapter 5. We will remind the reader of the local definitions at the beginning of each chapter.

While we are primarily interested in Maass forms of integral or half-integral weight, it is often no more complicated to describe the general case. Thus, until stated otherwise, k is a real number.

2.1 Multiplier systems

Suppose that Γ is a congruence subgroup of level N for some $N \geq 1$; i.e. Γ contains the subgroup

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

We will primarily work with the congruence subgroups

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}$$
$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

We say that $\nu : \Gamma \to \mathbb{C}^\times$ is a multiplier system of weight k if

(i) $|\nu| = 1$,

(ii) $\nu(-I) = e^{-\pi i k}$, and

(iii) $\nu(\gamma_1\gamma_2)\, J(\gamma_1\gamma_2,\tau)^k = \nu(\gamma_1)\nu(\gamma_2)\, J(\gamma_2,\tau)^k J(\gamma_1,\gamma_2\tau)^k$ for all $\gamma_1,\gamma_2 \in \Gamma$, where $J(\gamma,\tau) = c\tau + d$ for $\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$.

If ν is a multiplier system of weight k, then it is also a multiplier system of weight k' for any $k' \equiv k \pmod 2$, and the conjugate $\overline{\nu}$ is a multiplier system of weight $-k$. If ν_1 and ν_2 are multiplier systems of weights k_1 and k_2 for the same group Γ, then their product $\nu_1\nu_2$ is a multiplier system of weight $k_1 + k_2$ for Γ.

When k is an integer, a multiplier system of weight k for Γ is simply a character of Γ which satisfies $\nu(-I) = e^{-\pi i k}$. If ψ is an even (resp. odd) Dirichlet character modulo N, we can extend ψ to a multiplier system of even (resp. odd) integral weight for Γ by setting $\psi(\gamma) := \psi(d)$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Given a cusp $\mathfrak{a}$ let $\Gamma_{\mathfrak{a}} := \{\gamma \in \Gamma : \gamma\mathfrak{a} = \mathfrak{a}\}$ denote its stabilizer in Γ, and let $\sigma_{\mathfrak{a}}$ denote the unique (up to translation by $\pm\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ on the right) matrix in $\mathrm{SL}_2(\mathbb{R})$ satisfying $\sigma_{\mathfrak{a}}\infty = \mathfrak{a}$ and $\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \Gamma_\infty$. Define $\alpha_{\nu,\mathfrak{a}} \in [0,1)$ by the condition

$$\nu\left(\sigma_{\mathfrak{a}}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\sigma_{\mathfrak{a}}^{-1}\right) = e(-\alpha_{\nu,\mathfrak{a}}).$$

We say that $\mathfrak{a}$ is singular with respect to ν if ν is trivial on $\Gamma_{\mathfrak{a}}$, that is, if $\alpha_{\nu,\mathfrak{a}} = 0$. For convenience we write α_ν for $\alpha_{\nu,\mathfrak{a}}$ when $\mathfrak{a} = \infty$. Note that if $\alpha_\nu > 0$ then

$$\alpha_{\overline{\nu}} = 1 - \alpha_\nu. \tag{2.1.1}$$

We are primarily interested in the multiplier system χ of weight $1/2$ on $\mathrm{SL}_2(\mathbb{Z})$ (and its conjugate $\overline{\chi}$ of weight $-1/2$) where

$$\eta(\gamma\tau) = \chi(\gamma)\sqrt{c\tau + d}\,\eta(\tau), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \tag{2.1.2}$$

and η is the Dedekind eta function defined in (1.1.10). From condition (ii) above we have $\chi(-I) = -i$. From the definition of $\eta(\tau)$ we have $\chi(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)) = e(1/24)$; it follows that

$$\alpha_\chi = \frac{23}{24} \quad \text{and} \quad \alpha_{\overline{\chi}} = \frac{1}{24},$$

so the unique cusp ∞ of $\mathrm{SL}_2(\mathbb{Z})$ is singular neither with respect to χ nor with respect to $\overline{\chi}$.

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c > 0$ there are two useful formulas for χ. Rademacher (see, for instance, (74.11), (74.12), and (71.21) of [R4]) showed that

$$\chi(\gamma) = \sqrt{-i}\, e^{-\pi i s(d,c)}\, e\left(\frac{a+d}{24c}\right), \tag{2.1.3}$$

where $s(d,c)$ is the Dedekind sum

$$s(d,c) = \sum_{r=1}^{c-1} \frac{r}{c}\left(\frac{dr}{c} - \left\lfloor \frac{dr}{c} \right\rfloor - \frac{1}{2}\right). \tag{2.1.4}$$

On the other hand, Petersson (see, for instance, §4.1 of [K]) showed that

$$\chi(\gamma) = \begin{cases} \left(\frac{d}{c}\right) e\Big(\frac{1}{24}\left[(a+d)c - bd(c^2-1) - 3c\right]\Big) & \text{if } c \text{ is odd,} \\ \left(\frac{c}{d}\right) e\Big(\frac{1}{24}\left[(a+d)c - bd(c^2-1) + 3d - 3 - 3cd\right]\Big) & \text{if } c \text{ is even.} \end{cases} \tag{2.1.5}$$

We will also encounter the multiplier system ν_θ of weight $1/2$ on $\Gamma_0(4N)$ defined by

$$\theta(\gamma\tau) = \nu_\theta(\gamma)\sqrt{c\tau+d}\,\theta(\tau), \qquad \gamma \in \Gamma_0(4N),$$

where

$$\theta(\tau) := \sum_{n\in\mathbb{Z}} e(n^2\tau).$$

Explicitly, we have

$$\nu_\theta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \left(\frac{c}{d}\right)\varepsilon_d^{-1}, \tag{2.1.6}$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the extension of the Kronecker symbol given e.g. in [S] and

$$\varepsilon_d = \left(\frac{-1}{d}\right)^{\frac{1}{2}} = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

Let $\Gamma_0(N,M)$ denote the subgroup of $\Gamma_0(N)$ consisting of matrices whose upper-right entry is divisible by M. Equation (2.1.5) shows that

$$\chi(\gamma) = \left(\frac{c}{d}\right) e\left(\frac{d-1}{8}\right) \qquad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(24,24).$$

This implies that

$$\chi\left(\begin{pmatrix} a & 24b \\ c/24 & d \end{pmatrix}\right) = \left(\frac{12}{d}\right)\left(\frac{c}{d}\right)\varepsilon_d^{-1} \qquad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(576), \tag{2.1.7}$$

which allows one to relate Maass forms with multiplier χ on $\mathrm{SL}_2(\mathbb{Z})$ to those with multiplier $\left(\frac{12}{\bullet}\right)\nu_\theta$ on $\Gamma_0(576)$.

2.2 Harmonic Maass forms

Harmonic Maass forms are usually defined in such a way that weakly holomorphic modular forms are also harmonic Maass forms. The definitions of the slash operator and the Laplacian are local to this section, and will be changed in the next section.

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q})$ (the subgroup of $\mathrm{GL}_2(\mathbb{Q})$ consisting of matrices with positive determinant) we define the weight k slash operator $\big|_k$ by

$$\left(f\big|_k\gamma\right)(\tau) := (\det\gamma)^{k/2}(c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right).$$

A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight k and multiplier ν on Γ if it satisfies the following conditions:

(T) For all $\gamma \in \Gamma$ we have

$$f\big|_k\gamma = \nu(\gamma)f. \tag{2.2.1}$$

(C) For each cusp $\mathfrak{a}$ of Γ, the function $f(\sigma_\mathfrak{a}\tau)$ has a Fourier expansion of the form

$$f(\sigma_\mathfrak{a}\tau) = \sum_{\substack{m\in\mathbb{Z}-\alpha_{\nu,\mathfrak{a}} \\ m\geq 0}} c(m)\, q^m.$$

If the condition $m \geq 0$ in (C) is replaced by $m > 0$ then f is called a cusp form; if it is replaced by $m \gg -\infty$ then f is called a weakly holomorphic modular form.

A weak Maass form of weight k, multiplier ν, and Laplace eigenvalue λ is a smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying

(T) $f\big|_k\gamma = \nu(\gamma)f$ for all $\gamma \in \Gamma$,

(H) $\Delta_k f = \lambda f$, where

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+iky\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right) \tag{2.2.2}$$

is the weight k hyperbolic Laplacian, and

(C) f has at most linear exponential growth at each cusp of Γ.

If $\lambda = 0$, we say that f is a harmonic Maass form. The differential operator

$$\xi_k := 2iy^k\frac{\overline{\partial}}{\partial\overline{\tau}}$$

plays an important role in the theory of harmonic Maass forms. It commutes with the slash operator; that is,

$$\xi_k\left(f\big|_k\gamma\right) = (\xi_k f)\big|_{2-k}\gamma.$$

Thus, if f has weight k then $\xi_k f$ has weight $2-k$. The Laplacian Δ_k decomposes as

$$\Delta_k = -\xi_{2-k}\circ\xi_k,$$

which shows that ξ_k maps harmonic Maass forms to weakly holomorphic modular forms.

If f is a harmonic Maass form of weight $k \neq 1$, then there is a canonical decomposition

$$f(\tau) = f^+(\tau) + f^-(\tau)$$

where f^+ is the holomorphic part (also called a mock modular form)

$$f^+(\tau) = \sum_{\substack{n\in\mathbb{Z}-\alpha_\nu\\ n\gg-\infty}} a(n)q^n$$

and f^- is the nonholomorphic part

$$f^-(\tau) = b(0)y^{1-k} + \sum_{\substack{n\in\mathbb{Z}+\alpha_\nu\\ n\gg-\infty, n\neq 0}} b(n)\Gamma(1-k,4\pi ny)q^{-n}.$$

Here $\Gamma(s,y)$ is the incomplete gamma function which, for $y>0$, is defined by the integral

$$\Gamma(s,y) := \int_y^\infty e^{-t} t^s \, \frac{dt}{t}.$$

The function $\xi_k f = \xi_k f^-$ is called the shadow of f^+.

2.3 Maass cusp forms

We follow the convention of, e.g., [DFI] in defining Maass forms. As in the last section, the definitions of the slash operator and Laplacian below are local to this section. Section 2.3.3 explains the relation between Maass cusp forms and holomorphic modular forms.

2.3.1 Eigenfunctions of the Laplacian

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ we define the weight k slash operator by

$$f\big|_k \gamma := j(\gamma,\tau)^{-k} f(\gamma\tau), \qquad j(\gamma,\tau) := \frac{c\tau+d}{|c\tau+d|} = e^{i\arg(c\tau+d)},$$

where we always choose the argument in $(-\pi, \pi]$.

The weight k Laplacian

$$\Delta_k := y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - iky\frac{\partial}{\partial x}$$

commutes with the weight k slash operator for every $\gamma \in \mathrm{SL}_2(\mathbb{R})$. The operator Δ_k can be written as

$$\Delta_k = -R_{k-2}L_k - \frac{k}{2}\left(1 - \frac{k}{2}\right), \tag{2.3.1}$$

$$\Delta_k = -L_{k+2}R_k + \frac{k}{2}\left(1 + \frac{k}{2}\right), \tag{2.3.2}$$

where R_k is the Maass raising operator

$$R_k := \frac{k}{2} + 2iy\frac{\partial}{\partial \tau} = \frac{k}{2} + iy\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$

and L_k is the Maass lowering operator

$$L_k := \frac{k}{2} + 2iy\frac{\partial}{\partial \overline{\tau}} = \frac{k}{2} + iy\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

These raise and lower the weight by 2: we have

$$R_k\big(f\big|_k\gamma\big) = (R_k f)\big|_{k+2}\gamma \quad \text{and} \quad L_k\big(f\big|_k\gamma\big) = (L_k f)\big|_{k-2}\gamma.$$

From (2.3.1) and (2.3.2) we obtain the relations

$$R_k\Delta_k = \Delta_{k+2}R_k \quad \text{and} \quad L_k\Delta_k = \Delta_{k-2}L_k. \tag{2.3.3}$$

A real analytic function $f : \mathbb{H} \to \mathbb{C}$ is an eigenfunction of Δ_k with eigenvalue λ if

$$\Delta_k f + \lambda f = 0. \tag{2.3.4}$$

If f satisfies (2.3.4), then for notational convenience we write

$$\lambda = \frac{1}{4} + r^2,$$

and we refer to r as the spectral parameter of f. From (2.3.3) it follows that if f is an eigenfunction of Δ_k with eigenvalue λ then $R_k f$ (resp. $L_k f$) is an eigenfunction of Δ_{k+2} (resp. Δ_{k-2}) with eigenvalue λ.

2.3.2 Maass forms

A function $f : \mathbb{H} \to \mathbb{C}$ is automorphic of weight k and multiplier ν for $\Gamma = \Gamma_0(N)$ if

$$f\big|_k\gamma = \nu(\gamma)f \qquad \text{for all } \gamma \in \Gamma. \tag{2.3.5}$$

Let $\mathcal{A}_k(N,\nu)$ denote the space of all such functions. A smooth, automorphic function which is also an eigenfunction of Δ_k and which satisfies the growth condition

$$f(\tau) \ll y^\sigma + y^{1-\sigma} \qquad \text{for some } \sigma, \text{ for all } \tau \in \mathbb{H}, \tag{2.3.6}$$

is called a Maass form. We let $\mathcal{A}_k(N,\nu,r)$ denote the vector space of Maass forms with spectral parameter r. From the preceeding discussion, the raising (resp. lowering) operator maps $\mathcal{A}_k(N,\nu,r)$ into $\mathcal{A}_{k+2}(N,\nu,r)$ (resp. $\mathcal{A}_{k-2}(N,\nu,r)$). Also, complex conjugation $f \to \overline{f}$ gives a bijection $\mathcal{A}_k(N,\nu,r) \longleftrightarrow \mathcal{A}_{-k}(N,\overline{\nu},r)$.

If $f \in \mathcal{A}_k(n,\nu,r)$, then it satisfies $f(\tau+1) = e(-\alpha_\nu)f(\tau)$. For $n \in \mathbb{Z}$ define

$$n_\nu := n - \alpha_\nu. \tag{2.3.7}$$

Then f has a Fourier expansion of the form

$$f(\tau) = \sum_{n=-\infty}^{\infty} a(n,y)e(n_\nu x). \tag{2.3.8}$$

By imposing condition (2.3.4) on the Fourier expansion we find that for $n_\nu \neq 0$, the function $a(n,y)$ satisfies

$$\frac{a''(n,y)}{(4\pi|n_\nu|y)^2} + \left(\frac{1/4 + r^2}{(4\pi|n_\nu|y)^2} + \frac{k\,\mathrm{sgn}(n_\nu)}{8\pi|n_\nu|y} - \frac{1}{4}\right) a(n,y) = 0.$$

The Whittaker functions $M_{\kappa,\mu}(y)$ and $W_{\kappa,\mu}(y)$ are the two linearly independent solutions to Whittaker's equation [DL, §13.14]

$$W'' + \left(\frac{1/4 - \mu^2}{y^2} + \frac{\kappa}{y} - \frac{1}{4}\right) W = 0.$$

As $y \to \infty$, the former solution grows exponentially, while the latter decays exponentially. Since f satisfies the growth condition (2.3.6), we must have

$$a(n,y) = a(n) W_{\frac{k}{2}\,\mathrm{sgn}\, n_\nu, ir}(4\pi|n_\nu|y) \tag{2.3.9}$$

for some constant $a(n)$. For $n_\nu = 0$ we have

$$a(n,y) = a(0)y^{\frac{1}{2}+ir} + a'(0)y^{\frac{1}{2}-ir}.$$

We call the numbers $a(n)$ (and $a'(0)$) the Fourier coefficients of f. For $\mathrm{Re}(\mu - \kappa + 1/2) > 0$ we have the integral representation [DL, (13.14.3), (13.4.4)]

$$W_{\kappa,\mu}(y) = \frac{e^{-y/2}\, y^{\mu+1/2}}{\Gamma\left(\mu - \kappa + \frac{1}{2}\right)} \int_0^\infty e^{-yt}\, t^{\mu-\kappa-\frac{1}{2}} (1+t)^{\mu+\kappa-\frac{1}{2}}\, dt. \tag{2.3.10}$$

When $\kappa = 0$ we have [DL, (13.18.9)]

$$W_{0,\mu}(y) = \frac{\sqrt{y}}{\sqrt{\pi}} K_\mu(y/2),$$

where K_μ is a K-Bessel function. For Maass forms of weight 0, many authors normalize the Fourier coefficients in (2.3.9) so that $a(n)$ is the coefficient of $\sqrt{y}\, K_{ir}(2\pi|n_\nu|y)$, which has the effect of multiplying $a(n)$ by $2|n_\nu|^{1/2}$.

By [DL, (13.15.26)], we have

$$y\frac{d}{dy}W_{\kappa,\mu}(y) = \left(\frac{y}{2}-\kappa\right)W_{\kappa,\mu}(y) - W_{\kappa+1,\mu}(y)$$
$$= \left(\kappa-\frac{y}{2}\right)W_{\kappa,\mu}(y) + \left(\kappa+\mu-\frac{1}{2}\right)\left(\kappa-\mu-\frac{1}{2}\right)W_{\kappa-1,\mu}(y).$$

This, together with [DL, (13.15.11)] shows that

$$L_k\Big(W_{\frac{k}{2}\operatorname{sgn}(n_\nu),ir}(4\pi|n_\nu|y)e(n_\nu x)\Big)$$
$$= W_{\frac{k-2}{2}\operatorname{sgn}(n_\nu),ir}(4\pi|n_\nu|y)e(n_\nu x) \times \begin{cases} -\left(r^2+\frac{(k-1)^2}{4}\right) & \text{if } n_\nu > 0, \\ 1 & \text{if } n_\nu < 0, \end{cases}$$

from which it follows that if $f_L := L_k f$ has coefficients $a_L(n)$, then

$$a_L(n) = \begin{cases} -\left(r^2+\frac{(k-1)^2}{4}\right)a(n) & \text{if } n_\nu > 0, \\ a(n) & \text{if } n_\nu < 0. \end{cases} \tag{2.3.11}$$

Similarly, for the coefficients $a_R(n)$ of $f_R := R_k f$, we have

$$a_R(n) = \begin{cases} -a(n) & \text{if } n_\nu > 0, \\ \left(r^2+\frac{(k+1)^2}{4}\right)a(n) & \text{if } n_\nu < 0. \end{cases} \tag{2.3.12}$$

From the integral representation (2.3.10) we find that $\overline{W_{\kappa,\mu}(y)} = W_{\kappa,\overline{\mu}}(y)$ when $y, \kappa \in \mathbb{R}$, so $W_{\kappa,\mu}(y)$ is real when $\mu \in \mathbb{R}$ or (by [DL, (13.14.31)]) when μ is purely imaginary. Suppose that f has multiplier ν and that $\alpha_\nu \neq 0$. Then by (2.1.1) we have $-n_\nu = (1-n)_{\overline{\nu}}$, so the coefficients $a_c(n)$ of $f_c := \overline{f}$ satisfy

$$a_c(n) = \overline{a(1-n)}. \tag{2.3.13}$$

2.3.3 The spectrum of Δ_k

Let $\mathcal{L}_k(N,\nu)$ denote the L^2-space of automorphic functions with respect to the Petersson inner product

$$\langle f, g\rangle := \int_{\Gamma\backslash\mathbb{H}} f(\tau)\overline{g(\tau)}\,d\mu, \qquad d\mu := \frac{dx\,dy}{y^2}. \tag{2.3.14}$$

Let $\mathcal{B}_k(N,\nu)$ denote the subspace of $\mathcal{L}_k(N,\nu)$ consisting of smooth functions f such that f and $\Delta_k f$ are bounded on $\mathbb{H}$. By (2.3.1), (2.3.2), and Green's formula, we have the relations (see also §3 of [Ro])

$$\langle f, -\Delta_k g\rangle = \langle R_k f, R_k g\rangle - \frac{k}{2}\Big(1+\frac{k}{2}\Big)\langle f, g\rangle \tag{2.3.15}$$

$$= \langle L_k f, L_k g\rangle + \frac{k}{2}\Big(1-\frac{k}{2}\Big)\langle f, g\rangle \tag{2.3.16}$$

for any $f, g \in \mathcal{B}_k(N,\nu)$. It follows that Δ_k is symmetric and that

$$\langle f, -\Delta_k f\rangle \geq \lambda_0(k)\langle f, f\rangle, \qquad \lambda_0(k) := \frac{|k|}{2}\Big(1-\frac{|k|}{2}\Big). \tag{2.3.17}$$

By Friedrichs' extension theorem (see e.g. [I3, Appendix A.1]) the operator Δ_k has a unique self-adjoint extension to $\mathcal{L}_k(N,\nu)$ (denoted also by by Δ_k).

From (2.3.17) we see that the spectrum of Δ_k is real and contained in $[\lambda_0(k), \infty)$. The holomorphic forms correspond to the bottom of the spectrum. To be precise, if $f_0 \in \mathcal{L}_k(N,\nu)$ has eigenvalue $\lambda_0(k)$ then the equations above show that

$$f_0(\tau) = \begin{cases} y^{\frac{k}{2}}\, F(\tau) & \text{if } k \geq 0, \\ y^{-\frac{k}{2}}\, \overline{F}(\tau) & \text{if } k < 0, \end{cases}$$

where $F : \mathbb{H} \to \mathbb{C}$ is a holomorphic cusp form of weight k. In particular, $\lambda_0(k)$ is not an eigenvalue when the space of cusp forms is trivial. Note also that if $a_0(n)$ are the coefficients of f_0, then

$$a_0(n) = 0 \quad \text{when} \quad \operatorname{sgn}(n_\nu) = -\operatorname{sgn}(k). \tag{2.3.18}$$

The spectrum of Δ_k on $\mathcal{L}_k(N,\nu)$ consists of an absolutely continuous spectrum of multiplicity equal to the number of singular cusps, and a discrete spectrum of finite multiplicity. The Eisenstein series, of which there is one for each singular cusp $\mathfrak{a}$, give rise to the continuous spectrum, which is bounded below by 1/4. When $(k,\nu) = (1/2, \chi)$ or $(-1/2, \overline{\chi})$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ there are no Eisenstein series since the only cusp is not singular.

Let $\mathcal{S}_k(N,\nu)$ denote the orthogonal complement in $\mathcal{L}_k(N,\nu)$ of the space generated by Eisenstein series. The spectrum of Δ_k on $\mathcal{S}_k(N,\nu)$ is countable and of finite multiplicity with no limit points except ∞. The exceptional eigenvalues are those which lie in $(\lambda_0(k), 1/4)$. Let $\lambda_1(N,\nu,k)$ denote the smallest eigenvalue larger than $\lambda_0(k)$ in the spectrum of Δ_k on $\mathcal{S}_k(N,\nu)$.

Selberg's eigenvalue conjecture states that

$$\lambda_1(N, \mathbf{1}, 0) \geq \frac{1}{4},$$

i.e., there are no exceptional eigenvalues. Selberg [Se] showed that

$$\lambda_1(N, \mathbf{1}, 0) \geq \frac{3}{16}.$$

The best progress toward this conjecture was made by Kim and Sarnak [Ki, Appendix 2], who proved that

$$\lambda_1(N, \mathbf{1}, 0) \geq \frac{1}{4} - \left(\frac{7}{64}\right)^2 = \frac{975}{4096},$$

as a consequence of Langlands functoriality for the symmetric fourth power of an automorphic representation on GL_2.

2.3.4 Maass cusp forms

The subspace $\mathcal{S}_k(N, \nu)$ consists of functions f whose zeroth Fourier coefficient at each singular cusp vanishes. Eigenfunctions of Δ_k in $\mathcal{S}_k(N, \nu)$ are called Maass cusp forms. Let $\{f_j\}$ be an orthonormal basis of $\mathcal{S}_k(N, \nu)$, and for each j let

$$\lambda_j = \frac{1}{4} + r_j^2$$

denote the Laplace eigenvalue and $\{a_j(n)\}$ the Fourier coefficients. Then we have the Fourier expansion

$$f_j(\tau) = \sum_{n_\nu \neq 0} a_j(n) W_{\frac{k}{2}\operatorname{sgn} n_\nu, ir_j}(4\pi|n_\nu|y)e(n_\nu x). \tag{2.3.19}$$

Suppose that f has spectral parameter r. Then by (2.3.16) we have

$$\|L_k f\|^2 = \left(r^2 + \frac{(k-1)^2}{4}\right)\|f\|^2. \tag{2.3.20}$$

Weyl's law describes the distribution of the spectral parameters r_j. Theorem 2.28 of [He] shows that

$$\sum_{0 \leq r_j \leq T} 1 - \frac{1}{4\pi}\int_{-T}^{T} \frac{\varphi'}{\varphi}\left(\frac{1}{2} + it\right) dt = \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H})}{4\pi} T^2 - \frac{K_0}{\pi} T \log T + O(T), \tag{2.3.21}$$

where $\varphi(s)$ and K_0 are the determinant (see [He, p. 298]) and dimension (see [He, p. 281]), respectively, of the scattering matrix $\Phi(s)$ whose entries are given in terms of constant terms of Eisenstein series. In particular we have

$$\sum_{0\leq r_j\leq T} 1 = \frac{1}{12}T^2 + O(T) \qquad \text{for } (k,\nu) = \Big(\frac{1}{2},\chi\Big). \tag{2.3.22}$$

In Chapter 5 we will be working mostly in the space $\mathcal{S}_{\frac{1}{2}}(1,\chi)$. Throughout that chapter we will denote an orthonormal basis for this space by $\{u_j\}$, and we will write the Fourier expansions as

$$u_j(\tau) = \sum_{n\neq 0} \rho_j(n) W_{\frac{\operatorname{sgn} n}{4},ir_j}(4\pi|\widetilde{n}|y)e(\widetilde{n}x), \qquad \widetilde{n} := n - \frac{23}{24}. \tag{2.3.23}$$

2.3.5 Hecke operators

We introduce Hecke operators on the spaces $\mathcal{S}_0(N,\mathbf{1})$ and $\mathcal{S}_{\frac{1}{2}}(1,\chi)$. For each prime $p\nmid N$ the Hecke operator T_p acts on a Maass form $f\in\mathcal{A}_0(N,\mathbf{1})$ as

$$T_pf(\tau) = p^{-\frac{1}{2}}\Big(\sum_{j \bmod p} f\left(\frac{\tau+j}{p}\right) + f(p\tau)\Big).$$

The T_p commute with each other and with the Laplacian Δ_0, so T_p acts on each $\mathcal{S}_0(N,\mathbf{1},r)$. The action of T_p on the Fourier expansion of $f\in\mathcal{S}_0(N,\mathbf{1},r)$ is given by

$$T_pf(\tau) = \sum_{n\neq 0}\Big(p^{\frac{1}{2}}a(pn) + p^{-\frac{1}{2}}a(n/p)\Big)W_{0,ir}(4\pi|n|y)e(nx). \tag{2.3.24}$$

Hecke operators on half-integral weight spaces have been defined for the theta multiplier (see e.g. [KS]). We define Hecke operators T_{p^2} for $p\geq 5$ on $\mathcal{A}_{1/2}(1,\chi)$ directly by

$$T_{p^2}f = \frac{1}{p}\left[\sum_{b \bmod p^2} e\left(\frac{-b}{24}\right) f\big|_{\frac{1}{2}}\begin{pmatrix}\frac{1}{p} & \frac{b}{p}\\ 0 & p\end{pmatrix} + e\left(\frac{p-1}{8}\right)\sum_{h=1}^{p-1} e\left(\frac{-hp}{24}\right)\left(\frac{h}{p}\right) f\big|_{\frac{1}{2}}\begin{pmatrix}1 & \frac{h}{p}\\ 0 & 1\end{pmatrix} + f\big|_{\frac{1}{2}}\begin{pmatrix}p & 0\\ 0 & \frac{1}{p}\end{pmatrix}\right].$$

One can show that

$$(T_{p^2}f)\big|_{\frac{1}{2}}\begin{pmatrix}1&1\\0&1\end{pmatrix}=e\left(\frac{1}{24}\right)f \qquad \text{and} \qquad (T_{p^2}f)\big|_{\frac{1}{2}}\begin{pmatrix}0&-1\\1&0\end{pmatrix}=\sqrt{-i}\,f \quad (2.3.25)$$

and that T_{p^2} commutes with $\Delta_{\frac{1}{2}}$, so T_{p^2} is an endomorphism of $\mathcal{S}_{\frac{1}{2}}(1,\chi,r)$. To describe the action of T_{p^2} on Fourier expansions, it is convenient to write $f\in\mathcal{S}_{\frac{1}{2}}(1,\chi,r)$ in the form

$$f(\tau)=\sum_{n\neq 0}a(n)W_{\frac{\operatorname{sgn}(n)}{4},ir}\left(\frac{\pi|n|y}{6}\right)e\left(\frac{nx}{24}\right).$$

Then a standard computation gives

$$T_{p^2}f(\tau)=\sum_{n\neq 0}b(n)W_{\frac{\operatorname{sgn}(n)}{4},ir}\left(\frac{\pi|n|y}{6}\right)e\left(\frac{nx}{24}\right), \qquad (2.3.26)$$

where

$$b(n)=p\,a(p^2n)+p^{-\frac{1}{2}}\left(\tfrac{12n}{p}\right)a(n)+p^{-1}a(n/p^2).$$

2.3.6 Further operators

The reflection operator

$$f(x+iy)\mapsto f(-x+iy)$$

defines an involution on $\mathcal{S}_0(N,\mathbf{1})$ which commutes with Δ_0 and the Hecke operators. We say that a Maass cusp form is even if it is fixed by reflection; such a form has Fourier expansion

$$f(\tau)=\sum_{n\neq 0}a(n)W_{0,ir}(4\pi|n|y)\cos(2\pi nx).$$

We also have the operator

$$f(\tau)\mapsto f(d\tau)$$

which in general raises the level and changes the multiplier. We will only need this operator in the case $d=24$ on $\mathcal{S}_{1/2}(1,\chi)$, where (2.1.7) shows that

$$f(\tau)\mapsto f(24\tau):\mathcal{S}_{\frac{1}{2}}(1,\chi,r)\to\mathcal{S}_{\frac{1}{2}}\Big(576,\left(\frac{12}{\bullet}\right)\nu_\theta,r\Big). \qquad (2.3.27)$$

2.3.7 Generalized Kloosterman sums

In Chapter 5, our main objects of study will be the generalized Kloosterman sums given by

$$S(m,n,c,\nu) := \sum_{\substack{0\le a,d<c \\ \gamma=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in\Gamma}} \overline{\nu}(\gamma) e\left(\frac{m_\nu a + n_\nu d}{c}\right) \tag{2.3.28}$$
$$= \sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma/\Gamma_\infty \\ \gamma=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)}} \overline{\nu}(\gamma) e\left(\frac{m_\nu a + n_\nu d}{c}\right)$$

(to see that these are equal, one checks that each summand is invariant under the substitutions $\gamma \to \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right)\gamma$ and $\gamma \to \gamma\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right)$ using axiom (iii) for multiplier systems and (2.3.7)). When $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $\nu = \mathbf{1}$ we recover the ordinary Kloosterman sums

$$S(m,n,c) = S(m,n,c,\mathbf{1}) = \sum_{\substack{d \bmod c \\ (d,c)=1}} e\left(\frac{m\overline{d} + nd}{c}\right),$$

where $d\overline{d} \equiv 1 \pmod{c}$. For the eta-multiplier, (2.1.3) gives

$$S(m,n,c,\chi) = \sqrt{i} \sum_{\substack{d \bmod c \\ (d,c)=1}} e^{\pi i s(d,c)} e\left(\frac{(m-1)\overline{d} + (n-1)d}{c}\right),$$

so the sums $A_c(n)$ appearing in Rademacher's formula are given by

$$A_c(n) = \sqrt{-i}\, S(1, 1-n, c, \chi). \tag{2.3.29}$$

We recall the bounds (1.2.5) and (1.2.13) from the introduction. The Weil-type bound for $S(m,n,c,\chi)$ given in Corollary 1.1.6 has the following useful consequence (the "trivial" bound):

$$\sum_{c\le X} \frac{|S(m,n,c,\chi)|}{c} \ll X^{\frac{1}{2}} \log X\, |mn|^{\varepsilon}, \tag{2.3.30}$$

which follows from a standard argument involving the mean value estimate for $\tau(c)$.

3 Periods of the j-function along infinite geodesics

This chapter contains the proof of Theorem 1.1.2, which has Theorem 1.1.1 as an immediate corollary. In Section 3.1 we recall some facts about binary quadratic forms, focusing on forms of square discriminant. In Section 3.2 we define mock modular forms and describe the functions $j_{m,Q}$ in terms of Poincaré series. The proof of Theorem 1.1.2 is in Section 3.3. We follow the proof given in [DIT] for nonsquare discriminants, modifying as needed when the discriminant is a square.

Notation. In this chapter we use the definitions given in Section 2.2. In particular, the slash operator is defined by

$$f\big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad-bc)^{\frac{k}{2}}(c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right)$$

and the Laplacian Δ_k is defined by

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+iky\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right).$$

3.1 Binary quadratic forms

In this section, we recall some basic facts about binary quadratic forms and the characters χ_D, and we give an explicit description of the classes $\Gamma\backslash\mathcal{Q}_d$ when $d>0$ is a square. Throughout, we assume that $d, D \equiv 0, 1 \pmod 4$.

Recall that the action of $\gamma = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \Gamma$ on $Q(x,y)$ is given by

$$(\gamma Q)(x,y) = Q(Dx-By, -Cx+Ay). \tag{3.1.1}$$

This action is compatible with the linear fractional action $\gamma\tau = \frac{A\tau+B}{C\tau+D}$ on the roots of $Q(\tau,1)$; if τ_Q is a root of Q, then $\gamma\tau_Q$ is a root of γQ.

Suppose that D is a fundamental discriminant. If $Q=[a,b,c]\in\mathcal{Q}_{dD}$, we define

$$\chi_D(Q) := \begin{cases} \left(\frac{D}{r}\right) & \text{if } (a,b,c,D) = 1 \text{ and } Q \text{ represents } r \text{ with } (r,D) = 1, \\ 0 & \text{if } (a,b,c,D) > 1. \end{cases} \tag{3.1.2}$$

The basic theory of these characters is presented nicely in [GKZ, Section 2]. It turns out that χ_D is well-defined on classes $\Gamma\backslash\mathcal{Q}_{dD}$ and that

$$\chi_D(-Q) = (\operatorname{sgn} D)\chi_D(Q).$$

If $Q = [a,b,c] \in \mathcal{Q}_d$ with $d > 0$ then the cycle S_Q is the curve in $\mathbb{H}$ defined by the equation

$$a|\tau|^2 + b\operatorname{Re}\tau + c = 0.$$

When $a = 0$, S_Q is the vertical line $\operatorname{Re}\tau = -c/b$ oriented upward. When $a \neq 0$, S_Q is a semicircle oriented counterclockwise if $a > 0$ and clockwise if $a < 0$. If $\gamma \in \Gamma$ then we have $\gamma S_Q = S_{\gamma Q}$. We define

$$d\tau_Q := \frac{\sqrt{d}\,d\tau}{Q(\tau,1)},$$

so that if $\tau' = \gamma\tau$ for some $\gamma \in \Gamma$, we have

$$d\tau'_{\gamma Q} = d\tau_Q. \tag{3.1.3}$$

When $d > 0$ is a square, we can describe a set of representatives for $\Gamma\backslash\mathcal{Q}_d$ explicitly, as the next lemma shows.

Lemma 3.1.1. *Suppose that $d = b^2$ for some $b \in \mathbb{N}$. Then the set*

$$\{[a,b,0] : 0 \leq a < b\}$$

is a complete set of representatives for $\Gamma\backslash\mathcal{Q}_d$.

Proof. Let $Q \in \mathcal{Q}_d$. We will show that

1. $Q \sim [a,b,0]$ for some a with $0 \leq a < b$, and
2. if $[a,b,0] \sim [a',b,0]$ then $a \equiv a' \pmod b$.

The roots of $Q(x,y)$ in $\mathbb{P}^1(\mathbb{Q})$ are of the form $-s/r$ for some integers r,s with $(r,s) = 1$. If $\gamma = \left(\begin{smallmatrix} r & s \\ * & * \end{smallmatrix}\right) \in \Gamma$ then $\gamma Q = [a, \varepsilon b, 0]$ for some $\varepsilon \in \{\pm 1\}$

and some $a \in \mathbb{Z}$. Since $\left(\begin{smallmatrix} 1 & 0 \\ k & 1 \end{smallmatrix}\right)[a, \varepsilon b, 0] = [a - \varepsilon k b, \varepsilon b, 0]$ we may assume that $0 \le a < b$. Suppose that $\varepsilon = -1$. Let $g = (a, b)$ and define $\overline{a}$ by the conditions $a\overline{a} \equiv g^2 \pmod{b}$ and $0 \le \overline{a} < b$. Then

$$\begin{pmatrix} a/g & -b/g \\ * & \overline{a}/g \end{pmatrix} [a, -b, 0] = [\overline{a}, b, 0],$$

and claim (i) follows.

Suppose that $[a, b, 0] \sim [a', b, 0]$. Then there exists $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$ with $A > 0$ such that

$$D(aD - bC) = a', \tag{3.1.4}$$

$$b(AD + BC) - 2aBD = b, \tag{3.1.5}$$

$$B(aB - Ab) = 0. \tag{3.1.6}$$

Let $g = (a, b)$. If $aB - Ab = 0$ then $A = a/g$ and $B = b/g$, so (3.1.5) implies that $AD - BC = -1$, a contradiction. So by (3.1.6) we have $B = 0$ which, together with (3.1.5), implies that $AD = 1$. Then (3.1.4) shows that $a' \equiv aD^2 \equiv a \pmod{b}$. This proves claim (ii). □

3.2 Poincaré series

Suppose that f is a mock modular form of weight k with multiplier ν_θ^{2k} on $\Gamma_0(4)$. We say that $f = \sum a(n)q^n$ satisfies the plus space condition if the coefficients $a(n)$ are supported on integers $n \gg -\infty$ with $(-1)^{k-1/2}n \equiv 0, 1 \bmod 4$.

In Section 2 of [DIT], the mock modular forms f_D are constructed explicitly using nonholomorphic Maass-Poincaré series. For $D > 0$ the form f_D is the holomorphic part of $D^{-1/2}h_D$, where h_D is defined in Proposition 1 of [DIT]. If

$$f_D(\tau) = \sum_{0 < d \equiv 0,1(4)} a(d, D)q^d$$

then by (2.15), (2.21), (2.29), and Lemma 5 of [DIT] we have[1]

$$a(d, D) = (dD)^{-\frac{1}{2}} \lim_{s \to \frac{3}{4}^+} \left(b(d, D, s) - \frac{b(d, 0, s)b(0, D, s)}{b(0, 0, s)} \right), \tag{3.2.1}$$

[1]Note that $b(d, D, s) = b_d(D, s)$ in the notation of [DIT].

where

$$b(d,D,s) = \sum_{c=1}^{\infty} K^+(d,D;4c)$$
$$\times \begin{cases} 2^{-\frac{3}{2}}\pi(dD)^{\frac{1}{4}}c^{-1}J_{2s-1}\left(\frac{\pi\sqrt{dD}}{c}\right) & \text{if } dD > 0, \\ 2^{-4s}\pi^{s+\frac{1}{4}}(d+D)^{s-\frac{1}{4}}c^{-2s} & \text{if } dD = 0 \text{ and } d+D \neq 0, \\ 2^{\frac{1}{2}-6s}\pi^{\frac{1}{2}}\Gamma(2s)c^{-2s} & \text{if } d = D = 0. \end{cases} \quad (3.2.2)$$

Here J_{2s-1} is the J-Bessel function and $K^+(d,D;4c)$ is the modified Kloosterman sum

$$K^+(d,D;4c) := (1-i)\sum_{\ell \bmod 4c}\left(\frac{4c}{\ell}\right)\varepsilon_\ell e\left(\frac{d\ell + D\bar{\ell}}{4c}\right) \times \begin{cases} 1 & \text{if } c \text{ is even,} \\ 2 & \text{otherwise,} \end{cases}$$

where $\bar{\ell}$ denotes the inverse of ℓ modulo $4c$. Equation (3.2.2) shows that $b(d,D,s) = b(D,d,s)$, so for $d, D > 0$ we have

$$a(d,D) = a(D,d). \quad (3.2.3)$$

To prove Theorem 1.1.2 we need to express $j_{m,Q}(\tau,s)$ in terms of certain modified Poincaré series $G_{m,Q}(\tau,s)$. Let $\phi : \mathbb{R}^+ \to \mathbb{C}$ be a smooth function satisfying $\phi(y) = O_\epsilon(y^{1+\epsilon})$ for any $\epsilon > 0$, and let $m \in \mathbb{Z}$. Define the Poincaré series associated to ϕ by

$$G_m(\tau,\phi) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} e(-m \operatorname{Re}\gamma\tau)\phi(\operatorname{Im}\gamma\tau). \quad (3.2.4)$$

As in [F] and [Ni], we make the specialization

$$\phi(y) = \phi_{m,s}(y) := \begin{cases} y^s & \text{if } m = 0, \\ 2\pi|m|^{\frac{1}{2}}y^{\frac{1}{2}}I_{s-\frac{1}{2}}(2\pi|m|y) & \text{if } m \neq 0, \end{cases} \quad (3.2.5)$$

where $I_{s-\frac{1}{2}}$ is the I-Bessel function and $\operatorname{Re} s > 1$ (to guarantee convergence). We write $G_m(\tau,s) := G_m(\tau,\phi_{m,s})$ and we define

$$j_m(\tau,s) := G_m(\tau,s) - \frac{2\pi^{s+\frac{1}{2}}m^{1-s}\sigma_{2s-1}(m)}{\Gamma(s+\frac{1}{2})\zeta(2s-1)}G_0(\tau,s). \quad (3.2.6)$$

As explained in Section 4 of [DIT] and Section 6.4 of [BFI], when $m > 0$ the function $G_m(\tau, s)$ has an analytic continuation to $\operatorname{Re} s > 3/4$, and when $m = 0$ the function $G_m(\tau, s)$ has a pole at $s = 1$ arising from its constant term. The factor multiplied by $G_0(\tau, s)$ in (3.2.6) is chosen to cancel the pole of $G_0(\tau, s)$ at $s = 1$ and to eliminate the constant term of $G_m(\tau, 1)$. Furthermore, we have $j_m(\tau, 1) = j_m(\tau)$.

Recall that for $d > 0$ a square and $Q \in \mathscr{Q}_d$, the functions $j_{m,Q}(\tau)$ are defined as

$$j_{m,Q}(\tau) := j_m(\tau) - 2 \sum_{\alpha \in \{\text{roots of } Q\}} \sinh(2\pi m \operatorname{Im} \gamma_\alpha \tau)\, e(m \operatorname{Re} \gamma_\alpha \tau). \tag{3.2.7}$$

Since $\phi_{m,1}(y) = 2\sinh(2\pi|m|y)$, the two terms subtracted from $j_m(\tau)$ in (3.2.7) are the terms in the Poincaré series (3.2.4) corresponding to γ_α for the roots α of Q. It turns out that these are the terms which cause the integral

$$\int_{C_Q} G_m(\tau, 1) \frac{d\tau}{Q(\tau, 1)}$$

to diverge. In analogy with (3.2.6) and (3.2.7), we define

$$j_{m,Q}(\tau, s) := G_{m,Q}(\tau, s) - \frac{2\pi^{s+\frac{1}{2}} m^{1-s} \sigma_{2s-1}(m)}{\Gamma(s + \frac{1}{2})\zeta(2s-1)} G_{0,Q}(\tau, s), \tag{3.2.8}$$

where $G_{m,Q}(\tau, s)$ is the modified Poincaré series

$$G_{m,Q}(\tau, s) := \sum_{\substack{\gamma \in \Gamma_\infty \backslash \Gamma \\ \gamma \neq \gamma_\alpha}} e(-m \operatorname{Re} \gamma\tau) \phi_{m,s}(\operatorname{Im} \gamma\tau).$$

Since the two terms subtracted from $G_0(\tau, s)$ are killed by the pole of $\zeta(2s-1)$, we conclude that

$$j_{m,Q}(\tau, 1) = j_{m,Q}(\tau). \tag{3.2.9}$$

Therefore, to compute the cycle integrals of the functions $j_{m,Q}(\tau)$, it is enough to compute the cycle integrals of the functions $G_{m,Q}(\tau, s)$.

3.3 Proof of Theorem 1.1.2

Throughout this section we assume that $dD > 0$ is a square. The main ingredient in the proof of Theorem 1.1.2 is the following proposition, which computes the traces of the functions $G_{m,Q}(\tau, s)$ in terms of the J-Bessel function and the exponential sum

$$\mathfrak{S}_m(d, D; 4c) := \sum_{\substack{b \bmod 4c \\ b^2 \equiv dD \bmod 4c}} \chi_D\left([c, b, \tfrac{b^2-dD}{4c}]\right) e\left(\frac{mb}{2c}\right).$$

See Proposition 4 of [DIT] for the analogous formula for the traces of the functions $G_m(\tau, s)$.

Proposition 3.3.1. *Let* $\operatorname{Re} s > 1$ *and* $m \geq 0$. *Suppose that* $dD > 0$ *is a square. Then*

$$\sum_{Q \in \Gamma \backslash \mathscr{Q}_{dD}} \frac{\chi_D(Q)}{B(s)} \int_{C_Q} G_{m,Q}(\tau, s)\, d\tau_Q$$
$$= \begin{cases} \frac{\pi}{\sqrt{2}} m^{\frac{1}{2}} (dD)^{\frac{1}{4}} \displaystyle\sum_{c=1}^{\infty} \frac{\mathfrak{S}_m(d, D; 4c)}{c^{\frac{1}{2}}} J_{s-\frac{1}{2}}\left(\frac{\pi m \sqrt{dD}}{c}\right) & \text{if } m > 0, \\ 2^{-s-1} (dD)^{\frac{s}{2}} \displaystyle\sum_{c=1}^{\infty} \frac{\mathfrak{S}_0(d, D; 4c)}{c^s} & \text{if } m = 0, \end{cases}$$

where $B(s) := 2^s \Gamma(\frac{s}{2})^2 / \Gamma(s)$.

Proof. Let $b = \sqrt{dD}$. By Lemma 3.1.1, a complete set of representatives for $\Gamma \backslash \mathscr{Q}_{dD}$ is given by

$$\{Q_a = [a, b, 0] : 0 \leq a < b\}.$$

Let $g = (a, b)$. Then the roots of $Q_a = ax^2 + bxy$ in $\mathbb{P}^1(\mathbb{Q})$ are 0 and $\beta := -\frac{b'}{a'}$, where $a' = a/g$ and $b' = b/g$. The corresponding matrices are

$$\gamma_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_\beta = \begin{pmatrix} * & * \\ a' & b' \end{pmatrix}.$$

Thus, replacing τ by $\gamma^{-1}\tau$ in the integral, we have

$$\sum_{Q\in\Gamma\backslash\mathcal{Q}_{dD}} \chi_D(Q) \int_{C_Q} G_{m,Q}(\tau,s)\, d\tau_Q$$
$$= \sum_{a \bmod b} \chi_D([a,b,0]) \sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma\\ \gamma\neq\gamma_0,\gamma_\beta}} \int_{C_{\gamma Q}} e(-m\operatorname{Re}\tau)\phi_{m,s}(\operatorname{Im}\tau)\, d\tau_{\gamma Q}.$$

The map $(\gamma, Q) \mapsto \gamma Q$ is a bijection

$$\Gamma_\infty\backslash\Gamma \times \Gamma\backslash\mathcal{Q}_{dD} \longleftrightarrow \Gamma_\infty\backslash\mathcal{Q}_{dD}$$

which sends $(\gamma_0, [a,b,0])$ to $[0,-b,a]$ and $(\gamma_\beta, [a,b,0])$ to $[0,b,g\,\overline{a'}]$, where $a'\overline{a'} \equiv 1 \bmod b'$ and $g = (a,b)$. Since $\left(\begin{smallmatrix}1 & k\\ 0 & 1\end{smallmatrix}\right)[0,b,c] = [0,b,c-kb]$, we conclude that

$$\sum_{Q\in\Gamma\backslash\mathcal{Q}_{dD}} \chi_D(Q) \int_{C_Q} G_{m,Q}(\tau,s)\, d\tau_Q$$
$$= \sum_{\substack{Q\in\Gamma_\infty\backslash\mathcal{Q}_{dD}\\ Q\neq[0,\pm b,*]}} \chi_D(Q) \int_{C_Q} e(-m\operatorname{Re}\tau)\phi_{m,s}(\operatorname{Im}\tau)\, d\tau_Q.$$

The remainder of the proof follows the proofs of Lemmas 7 and 8 and Proposition 4 of [DIT].

Since we have eliminated those terms in the sum with $a = 0$, we can parametrize each cycle C_Q with $Q = [a,b,c]$ by

$$\tau = \begin{cases} \operatorname{Re}\tau_Q + e^{i\theta}\operatorname{Im}\tau_Q & \text{if } a > 0,\\ \operatorname{Re}\tau_Q - e^{-i\theta}\operatorname{Im}\tau_Q & \text{if } a < 0, \end{cases} \qquad 0 \le \theta \le \pi$$

where

$$\tau_Q := -\frac{b}{2a} + i\frac{\sqrt{dD}}{2|a|}$$

is the apex of the semicircle. We then have

$$Q(\tau,1) = \frac{dD}{4a} \begin{cases} e^{2i\theta} - 1 & \text{if } a > 0,\\ e^{-2i\theta} - 1 & \text{if } a < 0, \end{cases}$$

which gives $d\tau_Q = d\theta/\sin\theta$. Hence for $a \neq 0$ we have

$$\int_{C_Q} e(-m\,\mathrm{Re}\,\tau)\phi_{m,s}(\mathrm{Im}\,\tau)\,d\tau_Q$$
$$= e\left(\frac{mb}{2a}\right)\int_0^\pi e\left(-\frac{m\sqrt{dD}}{2a}\cos\theta\right)\phi_{m,s}\left(\frac{\sqrt{dD}}{2|a|}\sin\theta\right)\frac{d\theta}{\sin\theta}. \quad (3.3.1)$$

Consider the sum of the terms corresponding to Q and $-Q$, where $Q = [a, b, c]$ and $a > 0$. Since $\chi_D(Q) = \chi_D(-Q)$ we find that

$$\chi_D(Q)\int_{C_Q} G_{m,Q}(\tau, s)\,d\tau_Q + \chi_D(-Q)\int_{C_{-Q}} G_{m,-Q}(\tau, s)\,d\tau_{-Q}$$
$$= 2\,\chi_D(Q)\,e\left(\frac{mb}{2a}\right)\int_0^\pi \cos\left(\frac{\pi m\sqrt{dD}}{a}\cos\theta\right)\phi_{m,s}\left(\frac{\sqrt{dD}}{2a}\sin\theta\right)\frac{d\theta}{\sin\theta}. \quad (3.3.2)$$

In what follows, we assume that $m > 0$ (the $m = 0$ case is similar). By (3.2.5) above and Lemma 9 of [DIT], the right-hand side of (3.3.2) equals

$$\pi\sqrt{\frac{2m}{a}}\,(dD)^{\frac14}B(s)\,\chi_D(Q)\,e\left(\frac{mb}{2a}\right)J_{s-\frac12}\left(\frac{\pi m\sqrt{dD}}{a}\right).$$

Therefore

$$\sum_{Q\in\Gamma\backslash\mathscr{Q}_{dD}} \chi_D(Q)\int_{C_Q} G_{m,Q}(\tau, s)\,d\tau_Q$$
$$= \pi\sqrt{2m}(dD)^{\frac14}B(s)\sum_{\substack{Q\in\Gamma_\infty\backslash\mathscr{Q}_{dD}\\ a>0}} \frac{\chi_D(Q)}{\sqrt{a}}e\left(\frac{mb}{2a}\right)J_{s-\frac12}\left(\frac{\pi m\sqrt{dD}}{a}\right).$$

Let $\mathscr{Q}^+_{dD} = \{[a, b, c] \in \mathscr{Q}_{dD} : a > 0\}$. Since $\left(\begin{smallmatrix}1 & k\\ 0 & 1\end{smallmatrix}\right)[a, b, c] = [a, b-2ka, *]$, we have a bijection

$$[a, b, c] \longleftrightarrow (a, b \bmod 2a)$$

between $\Gamma_\infty\backslash\mathscr{Q}_{dD}^+$ and $\{(a,b) : a\in\mathbb{N} \text{ and } 0\le b<2a\}$. Therefore,

$$\sum_{Q\in\Gamma\backslash\mathscr{Q}_{dD}} \chi_D(Q)\int_{C_Q} G_{m,Q}(\tau,s)\,d\tau_Q$$
$$= \pi\sqrt{2m}(dD)^{\frac14}B(s)\sum_{a=1}^{\infty} a^{-\frac12}J_{s-\frac12}\left(\frac{\pi m\sqrt{dD}}{a}\right)$$
$$\times \sum_{\substack{b(2a)\\ \frac{b^2-dD}{4a}\in\mathbb{Z}}} \chi\left([a,b,\tfrac{b^2-dD}{4a}]\right) e\left(\frac{mb}{2a}\right).$$

The latter sum is equal to $\frac12\mathfrak{S}_m(d,D,4a)$, so we conclude (after replacing a by c) that

$$\sum_{Q\in\Gamma\backslash\mathscr{Q}_{dD}} \frac{\chi_D(Q)}{B(s)}\int_{C_Q} G_{m,Q}(\tau,s)\,d\tau_Q$$
$$= \tfrac{\pi}{\sqrt2}m^{\frac12}(dD)^{\frac14}\sum_{c=1}^{\infty}\frac{\mathfrak{S}_m(d,D;4c)}{c^{\frac12}}J_{s-\frac12}\left(\frac{\pi m\sqrt{dD}}{c}\right).$$

This completes the proof. □

We now complete the proof of Theorem 1.1.2, following the proof of Theorem 3 in [DIT]. Let

$$T_m(s) := \sum_{Q\in\Gamma\backslash\mathscr{Q}_{dD}} \frac{\chi_D(Q)}{B(s)}\int_{C_Q} G_{m,Q}(\tau,s)d\tau_Q.$$

Recall that $d\tau_Q=\sqrt{dD}\,d\tau/Q(\tau,1)$. By (3.2.8) and (3.2.3), to prove Theorem 1.1.2 we need to show that

$$\sum_{n|m}\left(\frac{D}{n}\right)(m/n)\,a\left(d,\frac{m^2D}{n^2}\right)$$
$$= (dD)^{-\frac12}\lim_{s\to1}\left(T_m(s)-\frac{2\pi^{s+\frac12}m^{1-s}\sigma_{2s-1}(m)}{\Gamma(s+\frac12)\zeta(2s-1)}T_0(s)\right). \quad (3.3.3)$$

By Proposition 3 of [DIT] we have

$$\mathfrak{S}_m(d,D;4c)=\tfrac12\sum_{n|(m,c)}\left(\tfrac{D}{n}\right)\sqrt{\tfrac{n}{c}}\,K^+\left(d,\tfrac{m^2D}{n^2};\tfrac{4c}{n}\right),$$

which, together with Proposition 3.3.1, gives

$$T_m(s) = \tfrac{\pi}{2\sqrt{2}} m^{\frac{1}{2}} (dD)^{\frac{1}{4}} \sum_{n|m} \left(\tfrac{D}{n}\right) n^{-\frac{1}{2}} \times \sum_{c=1}^{\infty} c^{-1} K^{+}\left(d, \tfrac{m^2}{n^2} D; 4c\right) J_{s-\frac{1}{2}}\left(\tfrac{\pi m \sqrt{dD}}{nc}\right) \quad (3.3.4)$$

if $m > 0$, and

$$T_0(s) = 2^{-s-2} (dD)^{\frac{s}{2}} \sum_{n=1}^{\infty} \left(\tfrac{D}{n}\right) n^{-s} \sum_{c=1}^{\infty} c^{-s-\frac{1}{2}} K^{+}(d, 0; 4c). \quad (3.3.5)$$

Comparing (3.3.4) and (3.3.5) with (3.2.2), we see that

$$T_m(s) = \begin{cases} \sum_{n|m} \left(\tfrac{D}{n}\right) b(d, \tfrac{m^2}{n^2} D, \tfrac{s}{2} + \tfrac{1}{4}) & \text{if } m > 0, \\ \pi^{-\frac{s+1}{2}} 2^{s-1} D^{\frac{s}{2}} L_D(s)\, b(d, 0, \tfrac{s}{2} + \tfrac{1}{4}) & \text{if } m = 0, \end{cases} \quad (3.3.6)$$

where $L_D(s) = \sum_{n>0} \left(\frac{D}{n}\right) n^{-s}$ is the Dirichlet L-function. By (3.2.1) and (3.3.6), the left-hand side of (3.3.3) equals

$$(dD)^{-\frac{1}{2}} \lim_{s \to 1} \left(T_m(s) - \frac{2^{1-s} \pi^{\frac{s+1}{2}} D^{-\frac{s}{2}}}{L_D(s) b(0, 0, \frac{s}{2} + \frac{1}{4})} T_0(s) \sum_{n|m} \left(\tfrac{D}{n}\right) b(0, \tfrac{m^2 D}{n^2}, \tfrac{s}{2} + \tfrac{1}{4}) \right).$$

It remains to show that

$$b(0, 0, \tfrac{s}{2} + \tfrac{1}{4})^{-1} \sum_{n|m} \left(\tfrac{D}{n}\right) b(0, \tfrac{m^2 D}{n^2}, \tfrac{s}{2} + \tfrac{1}{4}) = \frac{2^s D^{\frac{s}{2}} m^{1-s} \sigma_{2s-1}(m) \pi^{\frac{s}{2}} L_D(s)}{\Gamma(s + \frac{1}{2}) \zeta(2s - 1)},$$

which follows from Lemma 4 of [DIT] since $b(d, D, s) = b_d(D, s)$ in the notation of that paper.

4 Harmonic Maass forms of weight $5/2$

In this chapter, we prove Theorem 1.1.4, which has Theorem 1.1.3 as an immediate corollary. Our proof of Theorem 1.1.4 follows the method of Duke, Imamoḡlu, and Tóth [DIT] in the case when mn is not a square, and the author [A1] in the case when mn is a square. We construct the functions $P_v(\tau, s)$ and $P_{v,Q}(\tau, s)$ as Poincaré series and then evaluate the traces of these Poincaré series directly. We match these evaluations to the formulas given in [AA1] for the coefficients $p(m, n)$.

In Section 4.1 we review some facts about weak Maass forms and construct the functions $P_v(\tau, s)$. In Section 4.2 we discuss binary quadratic forms and establish some facts which we will need for the proof of the main theorem. We construct the functions $P_{v,Q}(\tau, s)$ at the end of Section 4.2. In Section 4.3 we prove a proposition which is equivalent to Theorem 1.1.5 and is a crucial ingredient in the proof of Theorem 1.1.4. The proof of Theorem 1.1.4 is in Section 4.4.

Notation. In this chapter we use the definitions given in Section 2.2. In particular, the slash operator is defined by

$$f\big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad - bc)^{\frac{k}{2}} (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right)$$

and the Laplacian Δ_k is defined by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

4.1 Poincaré series and $P_v(\tau, s)$

In this section we construct the weak Maass forms $P_v(\tau, s)$ in terms of Poincaré series. We first show how weak Maass forms can be constructed using Poincaré series associated with Whittaker functions. We then discuss the Atkin-Lehner involutions W_d which are used to build the functions $P_v(\tau, s)$.

4.1.1 Weak Maass forms and Poincaré series

Define

$$\Gamma_6 := \Gamma_0(6)/\{\pm 1\}.$$

We follow Section 2.6 of [BrO] in constructing Poincaré series for Γ_6 attached to special values of the M-Whittaker function $M_{\mu,\nu}(y)$ (see Section 13.14 of [DL] for the definition and relevant properties). Let v be a positive integer, and for $s \in \mathbb{C}$ and $y > 0$, define

$$\mathcal{M}_{s,k}(y) := y^{-k/2} M_{-k/2, s-1/2}(y) \tag{4.1.1}$$

and

$$\phi_v(\tau, s, k) := \mathcal{M}_{s,k}(4\pi v y) e(-vx).$$

Then

$$\phi_v(\tau, s, k) \ll y^{\mathrm{Re}(s) - k/2} \text{ as } y \to 0. \tag{4.1.2}$$

Letting $\Gamma_\infty := \{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\} \subseteq \Gamma_6$ denote the stabilizer of ∞, we define the Poincaré series

$$\mathbb{F}_v(\tau, s, k) := \frac{1}{\Gamma(2s)} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_6} (\phi_v\big|_k \gamma)(\tau, s, k). \tag{4.1.3}$$

On compact subsets of $\mathbb{H}$, we have (by (4.1.2)) the bound

$$|\mathbb{F}_v(\tau, s, k)| \ll y^{\mathrm{Re}(s) - k/2} \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_6} |c\tau + d|^{-2\,\mathrm{Re}(s)},$$

so $\mathbb{F}_v(\tau, s, k)$ converges absolutely uniformly on compact subsets of $\mathbb{H}$ for $\mathrm{Re}(s) > 1$. A computation involving [DL, 13.14.1] shows that

$$\Delta_k\, \phi_v(\tau, s, k) = (s - k/2)(1 - k/2 - s)\phi_v(\tau, s, k).$$

Since $\mathbb{F}_v$ clearly satisfies $\mathbb{F}_v\big|_k \gamma = \mathbb{F}_v$ for all $\gamma \in \Gamma_6$, we see that, for fixed s with $\mathrm{Re}(s) > 1$, the function $\mathbb{F}_v(\tau, s, k)$ is a weak Maass form of weight k and Laplace eigenvalue $(s - k/2)(1 - k/2 - s)$.

We are primarily interested in the case when k is negative. In this case the special value $\mathbb{F}_v(\tau, 1 - k/2, k)$ is a harmonic Maass form. Its principal part at ∞ is given by $q^{-v} + c_0$ for some $c_0 \in \mathbb{C}$, while its principal parts at the other cusps are constant. Thus, $\xi_k \mathbb{F}_v(\tau, 1 - k/2, k)$ is a cusp form of weight

$2 - k$ on Γ_6.

4.1.2 Atkin-Lehner involutions

We recall some basic facts about Atkin-Lehner involutions (see, for example, Section IX.7 of [Kna] or Section 2.4 of [O]). Suppose that N is a positive squarefree integer and that $d \mid N$. Let $W_d = W_d^N$ denote any matrix with determinant d of the form

$$W_d = \begin{pmatrix} d\alpha & \beta \\ N\gamma & d\delta \end{pmatrix}$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$. The relation

$$W_d \, \Gamma_0(N) \, W_d^{-1} = \Gamma_0(N) \tag{4.1.4}$$

shows that the map $f \mapsto f\big|_k W_d$ (called the Atkin-Lehner involution W_d) is independent of the choices of $\alpha, \beta, \gamma, \delta$ and defines an involution on the space of weight k forms on $\Gamma_0(N)$. If d and d' are divisors of N, then

$$f\big|_k W_d\big|_k W_{d'} = f\big|_k W_{d*d'},$$

where $d*d' = dd'/(d,d')^2$. When $d = N$ it is convenient to take $W_N = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$. Finally, the Atkin-Lehner involutions act transitively on the cusps of $\Gamma_0(N)$; that is, for each cusp $\mathfrak{a} \in \Gamma_0(N)\backslash\mathbb{P}^1(\mathbb{Q})$, there exists a unique $d \mid N$ such that $W_d \, \infty = \mathfrak{a}$.

Recall that $F_v(\tau) = q^{-v} + O(1)$ is a weakly holomorphic modular form of weight -2 on $\Gamma_0(6)$ with eigenvalues 1 and -1 under W_6 and W_2, respectively. We claim that

$$F_v(\tau) = \sum_{d|6} \mu(d) \left(\mathbb{F}_v\big|_{-2} W_d\right)(\tau, 2, -2). \tag{4.1.5}$$

To prove this, let $\widetilde{F}_v(\tau)$ denote the right-hand side of (4.1.5). Since the function $\xi_{-2}\mathbb{F}_v(\tau, 2, -2)$ lies in the one-dimensional space of cusp forms of weight 4 on $\Gamma_0(6)$, it must be proportional to

$$g(\tau) := \left(\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)\right)^2.$$

Since ξ_k commutes with the slash operator and $g(\tau)$ is invariant under W_d

for each $d \mid 6$, we have, for some $\alpha \in \mathbb{C}$, the relation

$$\xi_{-2}\widetilde{F}_v(\tau) = \alpha \sum_{d|6} \mu(d) g(\tau) = 0.$$

Thus $F_v(\tau) - \widetilde{F}_v(\tau)$ is holomorphic on $\mathbb{H}$ and vanishes at every cusp. Hence $F_v(\tau) = \widetilde{F}_v(\tau)$.

4.1.3 The functions $P_v(\tau)$, $P_v(\tau, s)$

To construct the functions $P_v(\tau)$ and $P_v(\tau, s)$, we require the Maass raising operator

$$R_k := 2i\frac{\partial}{\partial\tau} + \frac{k}{y} \tag{4.1.6}$$

which raises the weight of a weak Maass form by 2. For each $v \geq 1$, we define

$$P_v(\tau) := \frac{1}{4\pi v} R_{-2} F_v(\tau).$$

Then $P_v(\tau)$ is a weak Maass form of weight 0 and Laplace eigenvalue -2. By (4.1.5) and Proposition 2.2 of [BrO] this is equivalent to defining

$$\begin{aligned} P_v(\tau) :=& \sum_{d|6} \mu(d) \mathbb{F}_v(W_d\,\tau, 2, 0) \\ =& \frac{1}{6} \sum_{d|6} \mu(d) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_6} \phi_v\left(\gamma W_d\,\tau, 2, 0\right). \end{aligned}$$

Similarly, we define $P_v(\tau, s)$ as

$$\begin{aligned} P_v(\tau, s) :=& C(s) \sum_{d|6} \mu(d) \mathbb{F}_v(W_d\,\tau, s, 0) \\ =& \frac{C(s)}{\Gamma(2s)} \sum_{d|6} \mu(d) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_6} \phi_v\left(\gamma W_d\,\tau, s, 0\right), \end{aligned} \tag{4.1.7}$$

where

$$C(s) := \frac{2^s}{\pi} \Gamma\left(\frac{s+1}{2}\right)^2. \tag{4.1.8}$$

We have chosen the non-standard normalizing factor $C(s)$ so that later results are cleaner to state. Note that $C(2) = 1$, so

$$P_v(\tau, 2) = P_v(\tau). \tag{4.1.9}$$

In the next section we will define the dampened functions $P_{v,Q}(\tau, s)$.

4.2 Binary quadratic forms and $P_{v,Q}(\tau, s)$

In this section we recall some basic facts about binary quadratic forms and the genus characters χ_m. A good reference for this material is Section I of [GKZ]. Throughout this section, we assume that $m, n \equiv 1 \pmod{24}$ and that m is squarefree. The latter condition ensures that m is a fundamental discriminant.

Suppose that $r \in \{1, 5, 7, 11\}$. We recall that

$$\mathcal{Q}_{n,6}^{(r)} := \big\{ax^2 + bxy + cy^2 : b^2 - 4ac = n, \\ 6 \mid a,\ b \equiv r \pmod{12},\ \text{and } a > 0 \text{ if } n < 0\big\}.$$

Let Γ_6^* denote the group generated by $\Gamma_6 = \Gamma_0(6)/\{\pm 1\}$ and the Atkin-Lehner involutions W_d for $d \mid 6$. Matrices $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_6^*$ act on such forms on the left by

$$\gamma Q(x, y) := \frac{1}{\det \gamma} Q(Dx - By, -Cx + Ay).$$

It is easy to check that this action is compatible with the action $\gamma\tau := \frac{A\tau+B}{C\tau+D}$ on the roots of Q: for all $\gamma \in \Gamma^*$, we have

$$\gamma\, \tau_Q = \tau_{\gamma Q}. \tag{4.2.1}$$

The set $\Gamma \backslash \mathcal{Q}_{n,6}^{(r)}$ forms a finite group under Gaussian composition which is isomorphic to the narrow class group of $\mathbb{Q}(\sqrt{n})/\mathbb{Q}$ when n is a fundamental discriminant. Let $\mathcal{Q}_{n,6}$ denote the union

$$\mathcal{Q}_{n,6} := \bigcup_{r \in \{1,5,7,11\}} \mathcal{Q}_{n,6}^{(r)}.$$

For $d \mid 6$, the Atkin-Lehner involution $W_d = \left(\begin{smallmatrix} d\alpha & \beta \\ 6\gamma & d\delta \end{smallmatrix}\right)$ acts on quadratic forms by

$$W_d\,Q(x,y) := \frac{1}{d}\,Q(d\delta x - \beta y, -6\gamma x + d\alpha y). \tag{4.2.2}$$

A computation involving (4.2.2) and the relation $d\alpha\delta - \frac{6}{d}\beta\gamma = 1$ shows that

$$W_d\,[6a,b,c] = \left[6*, b\left(1 + \tfrac{12}{d}\beta\gamma\right) + 12*, *\right]. \tag{4.2.3}$$

It is convenient to choose $W_2 = \left(\begin{smallmatrix} 2 & -1 \\ 6 & -2 \end{smallmatrix}\right)$ and $W_3 = \left(\begin{smallmatrix} 3 & 1 \\ 6 & 3 \end{smallmatrix}\right)$. Then (4.2.3) shows that

$$W_d : \mathcal{Q}_{n,6}^{(r)} \longleftrightarrow \mathcal{Q}_{n,6}^{(r')} \tag{4.2.4}$$

is a bijection, where

$$r' \equiv r \times \begin{cases} 1 & \text{if } d = 1, \\ 7 & \text{if } d = 2, \\ 5 & \text{if } d = 3, \\ 11 & \text{if } d = 6, \end{cases} \pmod{12}. \tag{4.2.5}$$

Moreover, we have

$$\mathcal{Q}_{n,6} = \bigcup_{d|6} W_d\,\mathcal{Q}_{n,6}^{(r)} \tag{4.2.6}$$

for any $r \in \{1,5,7,11\}$.

We turn now to the extended genus character χ_m. For $Q \in \mathcal{Q}_{mn,6}$, define

$$\chi_m(Q) := \begin{cases} \left(\frac{m}{r}\right) & \text{if } (a,b,c,m) = 1 \text{ and } Q \text{ represents } r \text{ with } (r,m) = 1, \\ 0 & \text{if } (a,b,c,m) > 1. \end{cases} \tag{4.2.7}$$

The following lemma lists some properties of χ_m.

Lemma 4.2.1. *Suppose that* $m, n \equiv 1 \pmod{24}$ *and that* m *is squarefree.*

P1) The map $\chi_m : \Gamma_6\backslash\mathcal{Q}_{mn,6} \to \{-1,0,1\}$ *is well-defined; i.e.* $\chi_m(\gamma Q) = \chi_m(Q)$ *for all* $\gamma \in \Gamma_6$.

P2) If $(a,a') = 1$ *then*

$$\chi_m([6aa',b,c]) = \chi_m([6a,b,a'c])\chi_m([6a',b,ac]).$$

P3) For each $d \mid 6$ *we have*

$$\chi_m(Q) = \chi_m(W_d\, Q).$$

P4) Suppose that $[6a, b, c] \in \mathcal{Q}_{mn,6}$*, and let* $g := \pm(a, m)$*, where the sign is chosen so that* $g \equiv 1 \pmod 4$*. Then*

$$\chi_m([6a, b, c]) = \left(\frac{m/g}{6a}\right)\left(\frac{g}{c}\right).$$

P5) We have $\chi_m(-Q) = \mathrm{sgn}(m)\chi_m(Q)$*.*

Proof. Property P3 for $d = 1, 6$ and P1, P2, and P4 are special cases of Proposition 1 of [GKZ]. Property P5 follows easily from P4. A generalization of P3 is stated without proof in [GKZ], so we provide a proof here for our special case.

We want to show that P3 holds for $d = 2, 3$. Suppose that

$$Q = [6a, b, c] \text{ with } (a, b, c, m) = 1.$$

Choosing $W_2 = \left(\begin{smallmatrix} 2 & -1 \\ 6 & -2 \end{smallmatrix}\right)$ and $W_3 = \left(\begin{smallmatrix} 3 & 1 \\ 6 & 3 \end{smallmatrix}\right)$, we find that

$$W_2\,[6a, b, c] = [6(2a + b + 3c), *, 3a + b + 2c],$$
$$W_3\,[6a, b, c] = [6(3a - b + 2c), *, 2a - b + 3c].$$

We will use P4. For W_2, we want to show that

$$\left(\frac{m/g}{6(2a + b + 3c)}\right)\left(\frac{g}{3a + b + 2c}\right) = \left(\frac{m/g}{6a}\right)\left(\frac{g}{c}\right). \tag{4.2.8}$$

Since g divides $mn + 4ac = b^2$ and g is squarefree, we see that $g \mid b$, so

$$\left(\frac{g}{3a + b + 2c}\right) = \left(\frac{g}{c}\right)\left(\frac{g}{2}\right). \tag{4.2.9}$$

If $a = 0$ then $g = m$, and (4.2.8) follows from (4.2.9) and the fact that $m \equiv 1 \pmod 8$. If $a \neq 0$, then the relation

$$8a(2a + b + 3c) = (4a + b)^2 - m$$

shows that
$$\left(\frac{m/g}{6(2a+b+3c)}\right)\left(\frac{m/g}{6a}\right)=\left(\frac{m/g}{2}\right).$$
Together with (4.2.9), this completes the proof for W_2. The proof for W_3 is similar. $\square$

The remainder of this section follows Sections 3 and 4 of [DIT] and Section 3 of [A1]. Let Γ_Q denote the stabilizer of Q in $\Gamma_6 = \Gamma_0(6)/\{\pm 1\}$. When the discriminant of Q is negative or a positive square, the group Γ_Q is trivial. However, when the discriminant $n > 0$ is not a square, the group Γ_Q is infinite cyclic. If $Q = [a, b, c] \in \mathcal{Q}_{n,6}$ with $(a, b, c) = 1$, we have $\Gamma_Q = \langle g_Q \rangle$, where
$$g_Q := \begin{pmatrix} \frac{t+bu}{2} & cu \\ -au & \frac{t-bu}{2} \end{pmatrix}$$
and t, u are the smallest positive integral solutions to Pell's equation $t^2 - nu^2 = 4$. When $(a, b, c) = \delta > 1$, we have $\Gamma_Q = \langle g_{Q/\delta} \rangle$.

For $Q = [a, b, c] \in \mathcal{Q}_{n,6}$ with $n > 0$, let S_Q denote the geodesic in $\mathbb{H}$ connecting the two roots of $Q(\tau, 1)$. Explicitly, S_Q is the curve in $\mathbb{H}$ defined by
$$a|\tau|^2 + b\,\mathrm{Re}(\tau) + c = 0.$$
When $a \neq 0$, S_Q is a semicircle, which we orient counter-clockwise if $a > 0$ and clockwise if $a < 0$. When $a = 0$, S_Q is the vertical line $\mathrm{Re}(\tau) = -c/b$, which we orient upward. If $\gamma \in \Gamma_6$ then we have
$$\gamma S_Q = S_{\gamma Q} \tag{4.2.10}$$
Fix any $z \in S_Q$ and define the cycle C_Q as the directed arc on S_Q from z to $g_Q z$. We define
$$d\tau_Q := \frac{\sqrt{n}\,d\tau}{Q(\tau, 1)}, \tag{4.2.11}$$
so that if $\tau' = \gamma\tau$ for some $\gamma \in \Gamma_6^*$, we have
$$d\tau'_{\gamma Q} = d\tau_Q. \tag{4.2.12}$$

Suppose that Q has positive non-square discriminant and that f is a Γ_6-invariant function that is continuous on S_Q. A straightforward generalization of Lemma 6 of [DIT] shows that the integral

$$\int_{C_Q} f(\tau) d\tau_Q$$

is a well-defined (i.e. independent of the choice of $z \in S_Q$) invariant of the equivalence class of Q.

We now define the functions $P_{v,Q}(\tau, s)$. Let $Q = [a, b, c]$ be a binary quadratic form with square discriminant. Then the equation $Q(x, y) = 0$ has two inequivalent solutions $[r_1 : s_1]$ and $[r_2 : s_2]$ in $\mathbb{P}^1(\mathbb{Q})$, which we write as fractions $\mathfrak{a}_i := r_i/s_i$, with $(r_i, s_i) = 1$ and possibly $s_i = 0$. For each i, there is a unique d_i such that

$$W_{d_i}\, \mathfrak{a}_i \sim_{\Gamma_6} \infty.$$

Thus, up to translation, there is a unique $\gamma_i \in \Gamma_6$ such that

$$\gamma_i W_{d_i}\, \mathfrak{a}_i = \infty.$$

The function $P_{v,Q}(\tau, s)$ is defined by deleting the two terms of $P_v(\tau, s)$ in (4.1.7) corresponding to the pairs (γ_i, W_{d_i}); that is,

$$P_{v,Q}(\tau, s) := \frac{C(s)}{\Gamma(2s)} \sum_{d|6} \mu(d) \sum_{\substack{\gamma \in \Gamma_\infty \backslash \Gamma_6 \\ \gamma W_d\, \mathfrak{a}_i \neq \infty}} \phi_v(\gamma W_d\, \tau, s, 0). \tag{4.2.13}$$

Suppose that $\sigma \in \Gamma$. Then by (4.2.1), the roots of σQ are $\sigma \mathfrak{a}_1$ and $\sigma \mathfrak{a}_2$, and we find (using (4.1.4)) that

$$P_{v,\sigma Q}(\sigma\tau, s) = P_{v,Q}(\tau, s).$$

Together with (4.2.10) and (4.2.12), this shows that the integral

$$\int_{C_Q} P_{v,Q}(\tau, s) \frac{d\tau}{Q(\tau, 1)}$$

is invariant under $Q \mapsto \sigma Q$.

4.3 Kloosterman sums and the proof of Theorem 1.1.5

In this section we prove an identity (Proposition 4.3.2 below) connecting the Kloosterman sum (1.1.21) with the twisted quadratic Weyl sum (1.1.24). This is an essential ingredient in the proof of Theorem 1.1.4, and is equivalent to the evaluation of the Kloosterman sum in Theorem 1.1.5.

Throughout this section, v is a positive integer coprime to 6 and $m, n \equiv 1 \pmod{24}$ with m squarefree. We will use the notation

$$a' := \frac{a-1}{24}$$

whenever $a \equiv 1 \pmod{24}$. The Kloosterman sum is defined as

$$K(a,b;c) := \sum_{d(c)^*} e^{\pi i s(d,c)} e\left(\frac{\overline{d}a + db}{c}\right),$$

where $d(c)^*$ indicates that the sum is taken over residue classes coprime to c, and $\overline{d}$ denotes the inverse of d modulo c. The factor $e^{\pi i s(d,c)}$ makes the Kloosterman sum very difficult to evaluate. The following lemma shows that $e^{\pi i s(d,c)}$ is related to the Gauss-type sums

$$H_{d,c}(\delta) := \frac{1}{2} \sum_{j(2c)} e\left(\frac{d(6j+\delta)^2}{24c}\right) \tag{4.3.1}$$

which were introduced by Fischer in [Fi].

Lemma 4.3.1. *Suppose that $(v,6) = (c,d) = 1$ and define*

$$\alpha := 1 - \overline{d}c - \overline{d}v,$$
$$\beta := 1 - \overline{d}c + \overline{d}v,$$

with $\overline{d}$ chosen such that

$$d\overline{d} \equiv \begin{cases} 1 \pmod{c} & \text{if } c \text{ is odd,} \\ 1 \pmod{2c} & \text{if } c \text{ is even.} \end{cases}$$

Then we have

$$\sqrt{3c}\left(\frac{12}{v}\right) e\left(\frac{\overline{d}(v^2-1)}{24c}\right) e^{\pi i\, s(d,c)}$$
$$= e\left(\frac{2v+d\alpha^2}{24c}\right) H_{-d,c}(\alpha) + e\left(\frac{-2v+d\beta^2}{24c}\right) H_{-d,c}(\beta). \quad (4.3.2)$$

Proof. Define

$$f(v) := \left(\frac{12}{v}\right) e\left(\frac{2v+d\alpha^2-\overline{d}(v^2-1)}{24c}\right) H_{-d,c}(\alpha).$$

We will prove (4.3.2) by showing that $f(v)+f(-v) = \sqrt{3c}\, e^{\pi i\, s(d,c)}$.

We first show that for fixed d, c, the function $f(v)$ depends only on $v \bmod 6$. By (3.8) of [Wh] we find that $H_{-d,c}(\alpha)$ depends only on $\alpha \bmod 6$. Define $\varepsilon \in \{-1, 1\}$ by $v \equiv \varepsilon \pmod 6$. Then $\left(\frac{12}{v}\right) = e\left(\frac{v-\varepsilon}{12}\right)$, and we have

$$\left(\frac{12}{v}\right) e\left(\frac{2v+d\alpha^2-\overline{d}(v^2-1)}{24c}\right)$$
$$= e\left(v^2\frac{\overline{d}(d\overline{d}-1)/c}{24} - v\frac{(d\overline{d}-1)/c - d\overline{d}^2 - 1}{12} - \frac{\varepsilon}{12} + \frac{d(\overline{d}c-1)^2+\overline{d}}{24c}\right).$$

This depends only on $v \bmod 6$ since $v^2 \equiv 1 \pmod{24}$ and, by definition,

$$\frac{d\overline{d}-1}{c} - d\overline{d}^2 - 1 \in 2\mathbb{Z}.$$

Now $f(v)+f(-v) = f(\varepsilon)+f(-\varepsilon)$ is independent of v, so to prove (4.3.2) it suffices to show that $f(1)+f(-1) = \sqrt{3c}\, e^{\pi i\, s(d,c)}$. This is proved in Section 4 of [Wh]. $\square$

The quadratic Weyl sum $\mathfrak{S}_v(m, n; 24c)$ is defined as

$$\mathfrak{S}_v(m,n;24c) := \sum_{\substack{b(24c)\\ b^2\equiv mn(24c)}} \chi_{12}(b)\chi_m\left([6c, b, \tfrac{b^2-mn}{24c}]\right) e\left(\frac{bv}{12c}\right),$$

where χ_m is defined in (4.2.7). We clearly have $\mathfrak{S}_v(m,n;24c) = \overline{\mathfrak{S}_v(m,n;24c)}$, so the exponential $e\left(\frac{bv}{12c}\right)$ may be replaced by $\cos\left(\frac{bv\pi}{6c}\right)$. When $m = 1$, we obtain a simpler expression for $\mathfrak{S}_1(1, n; 24c)$ as follows. The summands of $\mathrm{Re}(\mathfrak{S}_v(1, n; 24c))$ are invariant under both $b \mapsto b + 12c$ and $b \mapsto -b$, so we may sum over those b modulo $12c$ for which $b \equiv 1 \pmod 6$, and multiply the sum by 4. Writing $b = 6\ell + 1$, we obtain (cf. formula (1.1.23))

$$\mathfrak{S}_1(1, n; 24c) = 4 \sum_{\substack{\ell \bmod 2c \\ (3\ell^2+\ell)/2 \equiv n'(c)}} (-1)^\ell \cos\left(\frac{(6\ell+1)v\pi}{6c}\right). \tag{4.3.3}$$

The following proposition gives an expression for $\mathfrak{S}_v(m, n; 24c)$ in terms of Kloosterman sums. Its proof occupies most of the remainder of the section. Theorem 1.1.5 follows from (4.3.4) by Möbius inversion in two variables.

Proposition 4.3.2. *Suppose that $m, n \equiv 1 \pmod{24}$ and that m is square-free. Suppose that $c, v > 0$ and that $(v, 6) = 1$. Then*

$$\mathfrak{S}_v(m, n; 24c) = 4\sqrt{3} \sum_{u \mid (v,c)} \left(\frac{12}{v/u}\right)\left(\frac{m}{u}\right)\sqrt{\frac{u}{c}}\, K\left(\left(\tfrac{v^2}{u^2}m\right)', n'; \frac{c}{u}\right). \tag{4.3.4}$$

Remark. Proposition 4.3.2 resembles Proposition 3 of [DIT], which is proved using a slight modification of Kohnen's argument in [Ko, Proposition 5]. Using an elegant idea of Tóth [To], Duke [D2] greatly simplified Kohnen's proof for the case $m = D = 1$ (in the notation of [DIT]). Jenkins [J] later extended this argument to the case of general m. However, Kohnen's argument remains the only proof of the general case.

Although special cases of (4.3.4) are amenable to the methods of Duke and Jenkins, we prove Proposition 4.3.2 in full generality by adapting Kohnen's argument. The proof is quite technical.

In the proof of Proposition 4.3.2 we will encounter the quadratic Gauss sum

$$G(a, b, c) := \sum_{x(c)} e\left(\frac{ax^2 + bx}{c}\right), \qquad c > 0. \tag{4.3.5}$$

For any $d \mid (a, c)$, we see by replacing x by $x + c/d$ that

$$G(a, b, c) = e\left(\tfrac{b}{d}\right) G(a, b, c). \tag{4.3.6}$$

This implies that $G(a, b, c) = 0$ unless $d \mid b$. In that case,

$$G(a, b, c) = d \cdot G\left(\frac{a}{d}, \frac{b}{d}, \frac{c}{d}\right). \tag{4.3.7}$$

If $(a,c)=1$, we have the well-known evaluations (see Theorems 1.5.1, 1.5.2, and 1.5.4 of [BEW])

$$G(a,0,c)=\begin{cases}0 & \text{if } 2\,||\,c,\\ (1+i)\varepsilon_a^{-1}\sqrt{c}\left(\frac{c}{a}\right) & \text{if } 4\mid c,\\ \varepsilon_c\sqrt{c}\left(\frac{a}{c}\right) & \text{if } c \text{ is odd,}\end{cases} \tag{4.3.8}$$

where

$$\varepsilon_a:=\begin{cases}1 & \text{if } a\equiv 1 \pmod 4,\\ i & \text{if } a\equiv 3 \pmod 4.\end{cases}$$

If $4\mid c$ and $(a,c)=1$ then, by replacing x by $x+c/2$, we find that $G(a,b,c)=0$ if b is odd. If b is even and $4\mid c$, or if c is odd, then by completing the square and using (4.3.8), we find that

$$G(a,b,c)=\begin{cases}e\left(\dfrac{-\overline{a}\,b^2}{4c}\right)(1+i)\varepsilon_a^{-1}\sqrt{c}\left(\dfrac{c}{a}\right) & \text{if } b \text{ is even and } 4\mid c,\\ e\left(\dfrac{-\overline{4a}\,b^2}{c}\right)\varepsilon_c\sqrt{c}\left(\dfrac{a}{c}\right) & \text{if } c \text{ is odd.}\end{cases} \tag{4.3.9}$$

Finally, these Gauss sums satisfy the multiplicative property

$$G(a,b,qr)=G(ar,b,q)\,G(aq,b,r),\qquad (q,r)=1 \tag{4.3.10}$$

which is a straightforward generalization of [BEW, Lemma 1.2.5].

We will need an explicit formula for $\chi_m([6c,b,\frac{b^2-mn}{24c}])$, which follows from P4 of Lemma 4.2.1 (see also Proposition 6 of [Ko]). For each odd prime p, let

$$p^*:=(-1)^{\frac{p-1}{2}}p$$

so that $(\frac{a}{p})=(\frac{p^*}{a})$. If m is squarefree, then

$$\chi_m([6c,b,\tfrac{b^2-mn}{24c}])=\prod_{\substack{p^\lambda||c\\ p\nmid m}}\left(\frac{m}{p^\lambda}\right)\prod_{\substack{p^\lambda||c\\ p|m}}\left(\frac{m/p^*}{p^\lambda}\right)\left(\frac{p^*}{(b^2-mn)/p^\lambda}\right). \tag{4.3.11}$$

Proof of Proposition 4.3.2. Both sides of (4.3.4) are periodic in v with period $12c$, so it suffices to show that their Fourier transforms are equal. For each

$h \in \mathbb{Z}$ we will show that

$$\frac{1}{12c}\sum_{v(12c)} e\left(\frac{-hv}{12c}\right)\mathfrak{S}_v(m,n;24c)$$
$$= \frac{4\sqrt{3}}{12c}\sum_{v(12c)} e\left(\frac{-hv}{12c}\right)\sum_{u|(c,v)}\left(\frac{12}{v/u}\right)\left(\frac{m}{u}\right)\sqrt{\frac{u}{c}}\,K\left(\left(\tfrac{v^2}{u^2}m\right)',n';\frac{c}{u}\right). \tag{4.3.12}$$

Let $L(h)$ and $R(h)$ denote the left- and right-hand sides of (4.3.12), respectively. Then we have

$$L(h) = \sum_{b^2\equiv mn(24c)} \chi_{12}(b)\chi_m\left([6c,b,\tfrac{b^2-mn}{24c}]\right)\times\frac{1}{12c}\sum_{v(12c)} e\left(\frac{(b-h)v}{12c}\right)$$
$$= \begin{cases} 2\,\chi_{12}(h)\,\chi_m\left([6c,h,\tfrac{h^2-mn}{24c}]\right) & \text{if } h^2\equiv mn \pmod{24c},\\ 0 & \text{otherwise.}\end{cases} \tag{4.3.13}$$

For the right-hand side, we have

$$R(h) = \frac{1}{\sqrt{3}\,c}\sum_{u|c}\left(\frac{m}{u}\right)\sqrt{\frac{u}{c}}\sum_{v(12c/u)}\left(\frac{12}{v}\right)e\left(\frac{-hv}{12c/u}\right)$$
$$\times\sum_{d(c/u)^*} e^{\pi i\, s(d,c/u)}e\left(\frac{\overline{d}(v^2m)'+dn'}{c/u}\right)$$
$$= \frac{1}{\sqrt{3}\,c}\sum_{u|c}\left(\frac{m}{c/u}\right)\frac{1}{\sqrt{u}}\sum_{d(u)^*} e\left(\frac{dn'}{u}\right)$$
$$\times\sum_{v(12u)} e^{\pi i\, s(d,u)}e\left(\frac{-\overline{d}}{24u}\right)\left(\frac{12}{v}\right)e\left(\frac{\overline{d}mv^2-2hv}{24u}\right).$$

Using Lemma 4.3.1, (4.3.1), and the fact that $u+v$ is even, we find that

$$e^{\pi i\, s(d,u)}e\left(\frac{-\overline{d}}{24u}\right)\left(\frac{12}{v}\right) = \frac{1}{2\sqrt{3u}}\,e\left(\frac{-\overline{d}v^2}{24u}\right)$$
$$\times\sum_{j(2u)} e\left(\frac{-d(3j^2+j)/2}{u}+\frac{j}{2}\right)\left(e\left(\frac{v(6j+1)}{12u}\right)+e\left(\frac{-v(6j+1)}{12u}\right)\right).$$

Thus we obtain

$$R(h) = \frac{1}{6c} \sum_{u|c} \left(\frac{m}{c/u}\right) u^{-1} \sum_{d(u)^*} e\left(\frac{dn'}{u}\right) \sum_{j(2u)} e\left(-\frac{d(3j^2+j)/2}{u} + \frac{j}{2}\right)$$
$$\times \left(G(\overline{d}(m-1)/2, 6j+1-h, 12u) + G(\overline{d}(m-1)/2, -6j-1-h, 12u)\right), \tag{4.3.14}$$

where $G(a,b,c)$ is the quadratic Gauss sum defined in (4.3.5). Since $12 \mid (m-1)/2$, we see by (4.3.7) that

$$G(d(m-1)/2, \pm(6j+1)-h, 12u)$$
$$= \begin{cases} 12\, G\left(\overline{d}m', \frac{\pm(6j+1)-h}{12}, u\right) & \text{if } h \equiv \pm(6j+1) \pmod{12}, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $R(h) = 0$ unless $h \equiv \pm 1 \pmod 6$.

For the remainder of the proof, we assume that $h \equiv 1 \pmod 6$ (the other case is analogous). Then in (4.3.14) the second Gauss sum is zero, and we take only those j for which $j \equiv \frac{h-1}{6} \pmod 2$. We write $j = 2k + \frac{h-1}{6}$; then

$$\sum_{j(2u)} e\left(-\frac{d(3j^2+j)/2}{u} + \frac{j}{2}\right) G(\overline{d}(m-1)/2, 6j+1-h, 12u)$$
$$= 12e\left(\frac{h-1}{12} - \frac{d(h^2-1)/24}{u}\right) \sum_{k(u)} e\left(-\frac{dk(6k+h)}{u}\right) G(\overline{d}m', k, u).$$

Since $e(\frac{h-1}{12}) = \chi_{12}(h)$ we have

$$R(h) = \frac{2}{c}\, \chi_{12}(h) \sum_{u|c} \left(\frac{m}{c/u}\right) u^{-1} F_h(u), \tag{4.3.15}$$

where

$$F_h(u) := \sum_{d(u)^*} e\left(\frac{d}{u}\left(n' - \frac{h^2-1}{24}\right)\right) \sum_{k(u)} e\left(\frac{-dk(6k+h)}{u}\right) G(\overline{d}m', k, u).$$

We show that $F_h(u)$ is multiplicative as a function of u. To prove this, suppose that $(q,r) = 1$, and choose $\overline{r}$ and $\overline{q}$ such that $r\overline{r} + q\overline{q} = 1$. Let $\alpha := n' - (h^2-1)/24$, and write $d = r\overline{r}x + q\overline{q}y$ and $k = k_1 r + k_2 q$. Using

(4.3.5) and (4.3.10) we find that

$$F_h(qr) = \sum_{x(q)^*} e\left(\frac{\overline{r}x\alpha}{q}\right) \sum_{k_1(q)} e\left(\frac{-k_1x(6k_1r+h)}{q}\right) G(r\overline{x}m', k_1r, q)$$
$$\times \sum_{y(r)^*} e\left(\frac{\overline{q}y\alpha}{r}\right) \sum_{k_2(r)} e\left(\frac{-k_2y(6k_2q+h)}{r}\right) G(q\overline{y}m', k_2q, r).$$

Replacing k_1, k_2, x, and y by $k_1\overline{r}$, $k_2\overline{q}$, xr, and yq, respectively, we conclude that

$$F_h(qr) = F_h(q)F_h(r).$$

Clearly $\frac{1}{c}\sum_{u|c}\left(\frac{m}{c/u}\right)u^{-1}F_h(u)$ is multiplicative as a function of c. Thus, by (4.3.11), (4.3.13), and (4.3.15), to show that $L(h) = R(h)$ it suffices to show that for each prime power $p^\lambda \mid\mid c$ we have

$$p^{-\lambda}\sum_{j=0}^{\lambda}\left(\frac{m}{p^{\lambda-j}}\right)p^{-j}F_h(p^j) = \begin{cases} \left(\frac{m}{p^\lambda}\right) & \text{if } p \nmid m, \\ \left(\frac{m/p^*}{p^\lambda}\right)\left(\frac{p^*}{(h^2-mn)/p^\lambda}\right) & \text{if } p \mid m, \end{cases} \tag{4.3.16}$$

when $h^2 \equiv mn \pmod{24p^\lambda}$, and 0 otherwise.

Suppose first that p is an odd prime. Set

$$p^\mu := (m', p^j).$$

Then $G(\overline{d}m', k, p^j) = 0$ unless $p^\mu \mid k$. In the latter case, using (4.3.7) and (4.3.9), we find that

$$G(\overline{d}m', k, p^j) = \varepsilon_{p^{j-\mu}}p^{\frac{j+\mu}{2}}\left(\frac{\overline{d}m'/p^\mu}{p^{j-\mu}}\right)e\left(\frac{-d\overline{(4m'/p^\mu)}(k/p^\mu)^2}{p^{j-\mu}}\right).$$

Writing $k = p^\mu\ell$, we find that

$$F_h(p^j) = \varepsilon_{p^{j-\mu}}\left(\frac{m'/p^\mu}{p^{j-\mu}}\right)p^{\frac{j+\mu}{2}}\sum_{d(p^j)^*}\left(\frac{d}{p^{j-\mu}}\right)e\left(\frac{d}{p^j}\left(n' - \frac{h^2-1}{24}\right)\right)$$
$$\times \sum_{\ell(p^{j-\mu})} e\left(\frac{-d(6p^\mu + \overline{(4m'/p^\mu)})\ell^2 - dh\ell}{p^{j-\mu}}\right). \tag{4.3.17}$$

Since

$$\overline{(4m'/p^\mu)}m = \overline{(4m'/p^\mu)}(24p^\mu m'/p^\mu + 1) \equiv 6p^\mu + \overline{(4m'/p^\mu)} \pmod{p^{j-\mu}},$$

the inner sum is equal to

$$G(-dm\overline{(4m'/p^\mu)}, -dh, p^{j-\mu}).$$

We first consider the case where $p \nmid m$, and we choose $\overline{m}$ such that $\overline{m}m \equiv 1 \pmod{24p^\lambda}$. Using (4.3.9) again, we find that

$$F_h(p^j) = p^j \left(\frac{-m}{p^{j-\mu}}\right) \varepsilon_{p^{j-\mu}}^2 \sum_{d(p^j)^*} e\left(\frac{d}{p^j}\left(n' - \frac{h^2-1}{24} + \overline{m}m'h^2\right)\right).$$

Note that

$$n' - \frac{h^2-1}{24} + \overline{m}m'h^2 \equiv \frac{n - \overline{m}h^2}{24} \pmod{p^\lambda}.$$

Since $\varepsilon_{p^{j-\mu}}^2 = \left(\frac{-1}{p^{j-\mu}}\right)$ and $m \equiv 1 \pmod{p^\mu}$ we conclude that the quantity in (4.3.16) is

$$\begin{aligned} p^{-\lambda}\sum_{j=0}^{\lambda}\left(\frac{m}{p^{\lambda-j}}\right)p^{-j}F_h(p^j) &= p^{-\lambda}\left(\frac{m}{p^\lambda}\right)\sum_{j=0}^{\lambda}\sum_{d(p^j)^*} e\left(\frac{dp^{\lambda-j}}{p^\lambda}\left(\frac{n-\overline{m}h^2}{24}\right)\right) \\ &= p^{-\lambda}\left(\frac{m}{p^\lambda}\right)\sum_{d(p^\lambda)} e\left(\frac{d}{p^\lambda}\left(\frac{n-\overline{m}h^2}{24}\right)\right) \\ &= \begin{cases} \left(\frac{m}{p^\lambda}\right) & \text{if } p^\lambda \mid \dfrac{n-\overline{m}h^2}{24}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The condition $p^\lambda \mid (n - \overline{m}h^2)/24$ is equivalent to $h^2 \equiv mn \pmod{24p^\lambda}$, so (4.3.16) is true in the case where p is an odd prime not dividing m.

We turn now to the case where $p \mid m$. Then $p \nmid m'$, so $\mu = 0$ in (4.3.17), and since m is squarefree, $(m/p, p) = 1$. Furthermore, all of the terms in the sum on the left-hand side of (4.3.16) vanish except for the term $j = \lambda$. From

(4.3.17) we have

$$F_h(p^\lambda) = \varepsilon_{p^\lambda}\left(\frac{m'}{p^\lambda}\right)p^{\lambda/2} \times \sum_{d(p^\lambda)^*}\left(\frac{d}{p^\lambda}\right)e\left(\frac{d}{p^\lambda}\left(n'-\frac{h^2-1}{24}\right)\right)G(-dm\overline{(4m')},-dh,p^\lambda),$$

which is zero unless $p \mid h$. Assume that $p \mid h$; then, using (4.3.9), we obtain

$$F_h(p^\lambda) = p^{\lambda+\frac{1}{2}}\varepsilon_p\left(\frac{m'}{p}\right)\left(\frac{-m/p}{p^{\lambda-1}}\right) \times \sum_{d(p^\lambda)^*}\left(\frac{d}{p}\right)e\left(\frac{d}{p^\lambda}\left(n'-\frac{h^2-1}{24}+\frac{\overline{(m/p)}m'h^2}{p}\right)\right).$$

Set

$$\alpha := n' - \frac{h^2-1}{24} + \frac{\overline{(m/p)}m'h^2}{p}.$$

Replacing d by $d+p$, we see that the sum

$$\sum_{d(p^\lambda)^*}\left(\frac{d}{p}\right)e\left(\frac{d\alpha}{p^\lambda}\right)$$

is zero unless $p^{\lambda-1} \mid \alpha$. Assume that $p^{\lambda-1} \mid \alpha$. Then

$$\sum_{d(p^\lambda)^*}\left(\frac{d}{p}\right)e\left(\frac{d\alpha}{p^\lambda}\right) = p^{\lambda-1}\sum_{d(p)}\left(\frac{d}{p}\right)e\left(\frac{d\alpha/p^{\lambda-1}}{p}\right) = \varepsilon_p\, p^{\lambda-\frac{1}{2}}\left(\frac{\alpha/p^{\lambda-1}}{p}\right),$$

where the last equality uses Theorem 1.1.5 of [BEW] and (4.3.8). We have $p \neq 3$ since $m \equiv 1 \pmod{24}$, so $\left(\frac{m'}{p}\right) = \left(\frac{-24}{p}\right)$. Therefore

$$F_h(p^\lambda) = p^{2\lambda}\varepsilon_p^2\left(\frac{-m/p}{p^{\lambda-1}}\right)\left(\frac{-24\alpha/p^{\lambda-1}}{p}\right).$$

We have

$$\frac{24\alpha}{p^{\lambda-1}} = \frac{p(n-h^2)+\overline{(m/p)}24m'h^2}{p^\lambda}$$

which, together with the fact that $\overline{(m/p)}m \equiv p \pmod{p^{\lambda+1}}$, yields

$$\frac{24\alpha}{p^{\lambda-1}} \equiv \frac{\overline{(m/p)}\,[mn - mh^2 + 24m'h^2]}{p^\lambda} \equiv \frac{\overline{(m/p)}\,[mn - h^2]}{p^\lambda} \pmod{p}.$$

Therefore

$$F_h(p^\lambda) = p^{2\lambda}\left(\frac{-m/p}{p^\lambda}\right)\left(\frac{(h^2-mn)/p^\lambda}{p}\right) = p^{2\lambda}\left(\frac{m/p^*}{p^\lambda}\right)\left(\frac{p^*}{(h^2-mn)/p^\lambda}\right),$$

under the assumption that $p^{\lambda-1} \mid \alpha$. This assumption is equivalent to $h^2 \equiv mn \pmod{24p^\lambda}$ and implies that $p \mid h$, which justifies our previous assumption. Thus we conclude that

$$F_h(p^\lambda) = \begin{cases} p^{2\lambda}\left(\frac{m/p^*}{p^\lambda}\right)\left(\frac{p^*}{(h^2-mn)/p^\lambda}\right) & \text{if } h^2 \equiv mn \pmod{24p^\lambda}, \\ 0 & \text{otherwise,} \end{cases} \tag{4.3.18}$$

which verifies (4.3.16) in the case where p is an odd prime dividing m.

Now suppose that $p = 2$. Since $2 \nmid m$ and $\left(\frac{m}{2}\right) = 1$, we want to show that

$$\sum_{j=0}^{\lambda} 2^{-j} F_h(2^j) = \begin{cases} 2^\lambda & \text{if } h^2 \equiv mn \pmod{24 \cdot 2^\lambda}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3.19}$$

We recall the definition

$$F_h(u) := \sum_{d(u)^*} e\left(\frac{d}{u}\left(n' - \frac{h^2-1}{24}\right)\right) \sum_{k(u)} e\left(-\frac{dk(6k+h)}{u}\right) G(\overline{d}m', k, u).$$

Define μ by

$$2^\mu = (m', 2^j).$$

Then

$$G(\overline{d}m', k, 2^j) = \begin{cases} 2^\mu G(\overline{d}m'/2^\mu, k/2^\mu, 2^{j-\mu}) & \text{if } 2^\mu \mid k, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\beta := n' - \frac{h^2-1}{24} + h^2\,\overline{m}\,m', \text{ with } \overline{m}\,m \equiv 1 \pmod{24 \cdot 2^\lambda}.$$

We claim that

$$F_h(2^j) = 2^j \sum_{d(2^j)^*} e\left(\frac{d\beta}{2^j}\right). \tag{4.3.20}$$

If $\mu = j$ then $G(\overline{d}m'/2^\mu, k/2^\mu, 2^{j-\mu}) = 1$ and $2^j \mid m'$, so

$$F_h(2^j) = 2^j \sum_{d(2^j)^*} e\left(\frac{d\beta}{2^j}\right).$$

If $\mu = j-1$ then

$$G(\overline{d}m'/2^\mu, k/2^\mu, 2^{j-\mu}) = \begin{cases} 2 & \text{if } k/2^\mu \text{ is odd,} \\ 0 & \text{if } k/2^\mu \text{ is even.} \end{cases}$$

Since $2^{j-1} \mid m'$ and $\overline{m}$ is odd, we have $\beta \equiv n' - (h^2-1)/24 - 2^{j-1}h \pmod{2^j}$, which yields

$$\begin{aligned} F_h(2^j) &= 2^j \sum_{d(2^j)^*} e\left(\frac{d}{2^j}\left(n' - \frac{h^2-1}{24}\right) - \frac{d\cdot 2^{j-1}(6\cdot 2^{j-1}+h)}{2^j}\right) \\ &= 2^j \sum_{d(2^j)^*} e\left(\frac{d\beta}{2^j}\right). \end{aligned}$$

If $\mu \le j-2$ then by (4.3.9) we have

$$G(\overline{d}m'/2^\mu, k/2^\mu, 2^{j-\mu}) = \\ \begin{cases} (1+i)\,\varepsilon^{-1}_{\overline{d}m'/2^\mu}\, 2^{\frac{j-\mu}{2}} \left(\frac{2^{j-\mu}}{\overline{d}m'/2^\mu}\right) e\left(-\frac{d(\overline{m'/2^\mu})(k/2^\mu)^2}{2^{j-\mu+2}}\right) & \text{if } k/2^\mu \text{ is even,} \\ 0 & \text{if } k/2^\mu \text{ is odd.} \end{cases}$$

Writing $k = 2^{\mu+1}\ell$, we have

$$\begin{aligned} F_h(2^j) = (1+i)\,2^{\frac{j+\mu}{2}} \sum_{d(2^j)^*} e\left(\frac{d}{2^j}\left(n' - \frac{h^2-1}{24}\right)\right) \varepsilon^{-1}_{\overline{d}m'/2^\mu} \left(\frac{2^{j-\mu}}{\overline{d}m'/2^\mu}\right) \\ \times \sum_{\ell(2^{j-\mu-1})} e\left(-\frac{d\cdot 2^{\mu+1}\ell(6\cdot 2^{\mu+1}\ell+h)}{2^j} - \frac{d(\overline{m'/2^\mu})\ell^2}{2^{j-\mu}}\right). \end{aligned}$$

Since $(\overline{m'/2^\mu}) + 24 \cdot 2^\mu \equiv (\overline{m'/2^\mu})m \pmod{2^{j-\mu}}$, the inner sum equals

$$\frac{1}{2}G(-dm(\overline{m'/2^\mu}), -2dh, 2^{j-\mu})$$
$$= (1+i)\, 2^{\frac{j-\mu}{2}-1}\, \varepsilon^{-1}_{-dm(\overline{m'/2^\mu})} \left(\frac{2^{j-\mu}}{-dm(\overline{m'/2^\mu})}\right) e\left(\frac{dh^2\, \overline{m}\, m'}{2^j}\right).$$

Since

$$\varepsilon^{-1}_{dm'/2^\mu}\varepsilon^{-1}_{-dm(\overline{m'/2^\mu})} = -i,$$

we have

$$F_h(2^j) = 2^j \sum_{d(2^j)^*} e\left(\frac{d\beta}{2^j}\right).$$

We conclude in every case that

$$\sum_{j=0}^{\lambda} 2^{-j}F_h(2^j) = \sum_{j=0}^{\lambda}\sum_{d\,(2^j)^*} e\left(\frac{d\beta}{2^j}\right)$$
$$= \sum_{d(2^\lambda)} e\left(\frac{d\beta}{2^\lambda}\right) = \begin{cases} 2^\lambda & \text{if } h^2 \equiv mn \pmod{24\cdot 2^\lambda}, \\ 0 & \text{otherwise,} \end{cases}$$

which completes the proof. □

We now prove the Weil-type bound of Corollary 1.1.6, namely

$$K(M', n'; c) \ll \tau((v,c))\tau(c)(mn,c)^{\frac{1}{2}}\sqrt{c}. \tag{4.3.21}$$

Proof of Corollary 1.1.6. Estimating the right-hand side of (1.1.25) trivially, we find that

$$|K(M', n'; c)| \le \frac{4}{\sqrt{3}}\,\tau\big((v,c)\big)c^{\frac{1}{2}}R\big(mn, 24c\big), \tag{4.3.22}$$

where

$$R(y,\ell) = \#\{x \bmod \ell : x^2 \equiv y \bmod \ell\}.$$

The function $R(y,\ell)$ is multiplicative in ℓ. If p is prime and $p \nmid y$, then

$$R(y, p^e) \le \begin{cases} 2 & \text{if } p \text{ is odd}, \\ 4 & \text{if } p = 2. \end{cases} \tag{4.3.23}$$

If $y = p^d y'$ with $p \nmid y'$ then $R(y, p^e) \leq p^{d/2} R(y', p^{e-d})$. It follows that

$$R(y, \ell) \leq 2 \cdot 2^{\omega(\ell)} (y, \ell)^{\frac{1}{2}}, \tag{4.3.24}$$

where $\omega(\ell)$ is the number of primes dividing ℓ. By (4.3.22), (4.3.23), (4.3.24), and the fact that $2^{\omega(\ell)} \leq \tau(\ell)$, we obtain (4.3.21). □

4.4 Proof of Theorem 1.1.4

We begin by recording exact formulas for the coefficients $p(m, n)$ in terms of Kloosterman sums and the I- and J-Bessel functions $I_\alpha(x)$ and $J_\alpha(x)$. The following formulas are found in Proposition 11 of [AA1]. Let $\widetilde{h}_m$ denote the functions in that paper; then our functions h_m described in (1.1.18) and (1.1.17) are normalized as

$$h_m = \begin{cases} -\frac{3m^{3/2}}{4\sqrt{\pi}} \widetilde{h}_m & \text{if } m > 0, \\ |m|^{3/2} \widetilde{h}_m & \text{if } m < 0. \end{cases} \tag{4.4.1}$$

Suppose that $m, n \equiv 1 \pmod{24}$ are not both negative. By Propositions 8 and 11 of [AA1] we have

$$p(m, n) = \begin{cases} \dfrac{2\pi}{|mn|^{\frac{1}{4}}} \displaystyle\sum_{c>0} \frac{K(m', n'; c)}{c} I_{\frac{3}{2}} \left(\frac{\pi\sqrt{|mn|}}{6c} \right) & \text{if } mn < 0, \\ \dfrac{4}{(mn)^{\frac{1}{4}}} \displaystyle\sum_{c>0} \frac{K(m', n'; c)}{c} \left[\frac{\partial}{\partial s} J_{s-\frac{1}{2}} \left(\frac{\pi\sqrt{mn}}{6c} \right) \bigg|_{s=2} \right] & \text{if } mn > 0. \end{cases} \tag{4.4.2}$$

To prove Theorem 1.1.4 we will show that the traces $\mathrm{Tr}_v(m, n)$ can also be expressed as infinite series involving Kloosterman sums and Bessel functions. This is essentially accomplished in the following proposition. To simplify the statement of the proposition for square and non-square positive discriminants, we set $P_{v,Q}(\tau, s) := P_v(\tau, s)$ whenever Q has positive non-square discriminant. The functions $P_v(\tau)$, $P_v(\tau, s)$, and $P_{v,Q}(\tau, s)$ are defined in (4.1.9), (4.1.7), and (4.2.13), respectively.

Proposition 4.4.1. *Suppose that* $m, n \equiv 1 \pmod{24}$ *and that* m *is square-*

free. If $mn<0$ then

$$\frac{1}{|mn|^{\frac{1}{2}}}\sum_{Q\in\Gamma\backslash\mathcal{Q}^{(1)}_{mn,6}}\chi_m(Q)P_v(\tau_Q)$$
$$=\frac{2\pi}{|mn|^{\frac{1}{4}}}\sum_{d|v}\sqrt{d}\left(\frac{12}{d}\right)\left(\frac{m}{v/d}\right)\sum_{c>0}\frac{K((d^2m)',n';c)}{c}I_{\frac{3}{2}}\left(\frac{\pi\sqrt{|d^2mn|}}{6c}\right). \tag{4.4.3}$$

If $\mathrm{Re}(s)>1$ *and* $m,n>0$ *then*

$$\frac{1}{2\pi}\sum_{Q\in\Gamma\backslash\mathcal{Q}^{(1)}_{mn,6}}\chi_m(Q)\int_{C_Q}P_{v,Q}(\tau,s)\frac{d\tau}{Q(\tau,1)}$$
$$=\frac{4}{(mn)^{\frac{1}{4}}}\sum_{d|v}\sqrt{d}\left(\frac{12}{d}\right)\left(\frac{m}{v/d}\right)\sum_{c>0}\frac{K((d^2m)',n';c)}{c}J_{s-\frac{1}{2}}\left(\frac{\pi\sqrt{d^2mn}}{6c}\right). \tag{4.4.4}$$

Before proving Proposition 4.4.1, we remark that when $s=2$, the right-hand side of (4.4.4) is often identically zero. This follows from equation (3.15) of [AA1], which states that

$$\sum_{c>0}\frac{K(m',n';c)}{c}J_{3/2}\left(\frac{\pi\sqrt{mn}}{6c}\right)=\begin{cases}0 & \text{if } m\neq n,\\ \frac{1}{2\pi} & \text{if } m=n.\end{cases}$$

The only situation in which the right-hand side of (4.4.4) does not vanish is when $n=mt^2$ for some integer t and $v=t\ell$, for some integer ℓ with $(\ell,m)=1$. In that case (4.4.4) becomes

$$\sum_{Q\in\mathcal{Q}^{(1)}_{(mt)^2,6}}\chi_m(Q)\int_{C_Q}P_{v,Q}(\tau,2)\frac{d\tau}{Q(\tau,1)}=\frac{4}{\sqrt{m}}\left(\frac{12}{t}\right)\left(\frac{m}{\ell}\right).$$

Proof of Proposition 4.4.1. Suppose that $mn<0$, and let $L_v^-(m,n)$ denote the left-hand side of (4.4.3). Using the definition of $P_v(\tau)=P_v(\tau,2)$ in (4.1.7), we find that

$$L_v^-(m,n)=\tfrac{1}{6}|mn|^{-\frac{1}{2}}\sum_{d|6}\sum_{Q\in\Gamma_6\backslash\mathcal{Q}^{(1)}_{mn,6}}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_6}\mu(d)\,\chi_m(Q)\,\phi_v\left(\gamma W_d\,\tau_Q,2,0\right).$$

Using P1 and P3 of Lemma 4.2.1 and equation (4.2.1), this becomes

$$L_v^-(m,n) = \tfrac{1}{6}|mn|^{-\frac{1}{2}} \sum_{d|6} \sum_{Q\in\Gamma_6\backslash\mathcal{Q}_{mn,6}^{(1)}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_6} \mu(d)\,\chi_m(\gamma W_d\,Q)\,\phi_v\left(\tau_{\gamma W_d\,Q},2,0\right).$$

By (4.2.4) and (4.2.6) the map $(\gamma, d, Q) \mapsto \gamma W_d\,Q$ gives a bijection

$$\Gamma_\infty\backslash\Gamma_6 \times \{1,2,3,6\} \times \Gamma_6\backslash\mathcal{Q}_{mn,6}^{(1)} \longleftrightarrow \Gamma_\infty\backslash\mathcal{Q}_{mn,6}. \tag{4.4.5}$$

If $Q \in \mathcal{Q}_{mn,6}^{(1)}$ and $Q' = W_dQ = [a,b,c]$ then $\mu(d) = \left(\frac{12}{b}\right)$ by (4.2.4) and (4.2.5). With $\mathcal{M}_{s,k}(y)$ as in (4.1.1), we have

$$L_v^-(m,n) = \tfrac{1}{6}|mn|^{-1/2} \sum_{\substack{Q\in\Gamma_\infty\backslash\mathcal{Q}_{mn,6}\\ Q=[a,b,c]}} \left(\tfrac{12}{b}\right)\chi_m(Q)\mathcal{M}_{2,0}(4\pi v\operatorname{Im}\tau_Q)e(-v\operatorname{Re}\tau_Q).$$

Since $\mathcal{Q}_{mn,6}$ contains only positive definite forms (those with $a>0$), we have

$$\tau_Q = -\frac{b}{2a} + i\frac{\sqrt{|mn|}}{2a}.$$

By [DL, (13.18.8)], we have

$$\mathcal{M}_{2,0}(4\pi vy) = M_{0,3/2}(4\pi vy) = 12\pi\sqrt{vy}\,I_{3/2}(2\pi vy),$$

which gives

$$L_v^-(m,n) = \frac{\pi\sqrt{2v}}{|mn|^{\frac{1}{4}}} \sum_{\substack{Q\in\Gamma_\infty\backslash\mathcal{Q}_{mn,6}\\ Q=[a,b,c]}} \left(\frac{12}{b}\right)\frac{\chi_m(Q)}{\sqrt{a}} I_{\frac{3}{2}}\left(\frac{\pi v\sqrt{|mn|}}{a}\right) e\left(\frac{bv}{2a}\right). \tag{4.4.6}$$

Now suppose that m and n are both positive, and let $L_v^+(m,n)$ denote the left-hand side of (4.4.4). As in (4.2.11), we write

$$d\tau_Q = \frac{\sqrt{mn}\,d\tau}{Q(\tau,1)}.$$

If mn is not a square, then by (4.1.7) we have

$$L_v^+(m,n) = \frac{C(s)}{2\pi\Gamma(2s)\sqrt{mn}} \sum_{d|6} \mu(d) \sum_{Q\in\Gamma_6\backslash\mathcal{Q}_{mn}^{(1)}} \chi_m(Q) \times \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_6} \int_{C_Q} \phi_v(\gamma W_d\,\tau, s, 0)\, d\tau_Q. \quad (4.4.7)$$

For each Q, let $\Gamma_Q \subseteq \Gamma_6$ denote the stabilizer of Q. We rewrite the sum over $\Gamma_\infty\backslash\Gamma_6$ as a sum over $\gamma \in \Gamma_\infty\backslash\Gamma_6/\Gamma_Q$ and a sum over $g \in \Gamma_Q$. Since $S_Q = \cup_{g\in\Gamma_Q} C_Q$, the inner sum in (4.4.7) becomes

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_6/\Gamma_Q} \int_{S_Q} \phi_v(\gamma W_d\,\tau, s, 0)\, d\tau_Q.$$

In each integral, we replace τ by $W_d^{-1}\gamma^{-1}\tau$. Using (4.2.12) and P1 and P3 of Lemma 4.2.1, we obtain

$$L_v^+(m,n) = \frac{C(s)}{2\pi\Gamma(2s)\sqrt{mn}} \sum_{d|6} \sum_{Q\in\Gamma_6\backslash\mathcal{Q}_{mn,6}^{(1)}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_6/\Gamma_Q} \mu(d)\chi_m(\gamma W_d\,Q) \times \int_{S_{\gamma W_d\,Q}} \phi_v(\tau, s, 0)\, d\tau_{\gamma W_d\,Q}. \quad (4.4.8)$$

As in (4.4.5), we have the bijection

$$\Gamma_\infty\backslash\Gamma_6/\Gamma_Q \times \{1,2,3,6\} \times \Gamma_6\backslash\mathcal{Q}_{mn,6}^{(1)} \longleftrightarrow \Gamma_\infty\backslash\mathcal{Q}_{mn,6}$$

given by $(\gamma, d, Q) \mapsto \gamma W_d\,Q$. Thus

$$L_v^+(m,n) = \frac{C(s)}{2\pi\Gamma(2s)\sqrt{mn}} \times \sum_{\substack{Q\in\Gamma_\infty\backslash\mathcal{Q}_{mn,6}\\ Q=[a,b,c]}} \left(\frac{12}{b}\right) \chi_m(Q) \int_{S_Q} \mathcal{M}_{s,0}(4\pi v \operatorname{Im}\tau) e(-v\operatorname{Re}\tau)\, d\tau_Q. \quad (4.4.9)$$

In order to treat the square case together with the non-square case, we will show that, with the added condition $a \neq 0$ in the sum, (4.4.9) holds when mn is a square. Of course, $a \neq 0$ is implied in (4.4.9) when mn is not a square.

Suppose that $mn > 0$ is a square. For each $Q \in \mathcal{Q}_{mn,6}^{(1)}$ define $\mathfrak{a}_{i,Q} :=$

$r_{i,Q}/s_{i,Q}$ as in (4.2.13). The stabilizer Γ_Q is trivial for all $Q \in \mathcal{Q}_{mn,6}$ and, using (4.2.13), we have (4.4.8) with the added condition $\gamma W_d\, \mathfrak{a}_{i,Q} \neq \infty$ on the third sum; that is,

$$L_v^+(m,n) = \frac{C(s)}{2\pi\Gamma(2s)\sqrt{mn}} \sum_{d|6} \sum_{Q\in\Gamma\backslash\mathcal{Q}_{mn,6}^{(1)}} \sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma_6 \\ \gamma W_d\,\mathfrak{a}_{i,Q}\neq\infty}} \mu(d)\chi_m(\gamma W_d\, Q) \times \int_{S_{\gamma W_d\, Q}} \phi_v(\tau,s,0)\, d\tau_{\gamma W_d\, Q}. \quad (4.4.10)$$

The quadratic forms Q having $\infty = [1,0]$ as a root are of the form $Q = [0, \pm b, *]$, where $b = \sqrt{mn}$. Thus the condition $\gamma W_d\, \mathfrak{a}_{i,Q} \neq \infty$ is equivalent to $\gamma W_d\, Q \neq [0, \pm b, *]$. Applying the bijection (4.4.5), which holds in this case since Γ_Q is trivial, we obtain (4.4.9) with the restriction $a \neq 0$ on the sum.

Treating the square and non-square case together, we assume that m and n are arbitrary positive integers satisfying $m, n \equiv 1 \pmod{24}$. Suppose that $Q = [a,b,c]$. The apex of the semicircle S_Q is

$$-\frac{b}{2a} + i\frac{\sqrt{mn}}{2|a|},$$

so we parametrize S_Q by

$$\begin{aligned} \tau &= -\frac{b}{2a} + \frac{\sqrt{mn}}{2a} e^{i\,\mathrm{sgn}(a)\theta} \\ &= -\frac{b}{2a} + \frac{\sqrt{mn}}{2a}\cos\theta + i\frac{\sqrt{mn}}{2|a|}\sin\theta, \qquad 0 \le \theta \le \pi. \end{aligned} \quad (4.4.11)$$

Then we have

$$Q(\tau,1) = \frac{mn}{4a}(e^{2i\,\mathrm{sgn}(a)\theta} - 1),$$

which gives

$$d\tau_Q = \frac{\sqrt{mn}\, d\tau}{Q(\tau,1)} = \frac{d\theta}{\sin\theta}. \quad (4.4.12)$$

Combining (4.4.9), (4.4.11), and (4.4.12), we obtain

$$L_v^+(m,n) = \frac{C(s)}{2\pi\Gamma(2s)\sqrt{mn}} \sum_{\substack{Q\in\Gamma_\infty\backslash\mathcal{Q}_{mn,6} \\ Q=[a,b,c],\, a\neq 0}} R_Q(m,n),$$

where

$$R_Q(m,n) :=$$
$$\left(\frac{12}{b}\right)\chi_m(Q)e\left(\frac{bv}{2a}\right)\int_0^\pi \mathcal{M}_{s,0}\left(\frac{2\pi v\sqrt{mn}}{|a|}\sin\theta\right)e\left(-\frac{v\sqrt{mn}}{2a}\cos\theta\right)\frac{d\theta}{\sin\theta}.$$

For each $Q=[a,b,c]\in\Gamma_\infty\backslash\mathcal{Q}_{mn,6}$ with $a>0$, we have (using P5 of Lemma 4.2.1)

$$R_Q(m,n)+R_{-Q}(m,n) = 2\left(\frac{12}{b}\right)\chi_m(Q)e\left(\frac{bv}{2a}\right)$$
$$\times\int_0^\pi \mathcal{M}_{s,0}\left(\frac{2\pi v\sqrt{mn}}{a}\sin\theta\right)\cos\left(\frac{\pi v\sqrt{mn}}{a}\cos\theta\right)\frac{d\theta}{\sin\theta}.$$

By [DL, (13.18.8)] we have

$$\mathcal{M}_{s,0}(y) = M_{0,s-1/2}(y) = 2^{2s-1}\,\Gamma(s+1/2)\,\sqrt{y}\,I_{s-1/2}(y/2),$$

hence

$$L_v^+(m,n) = \frac{2^{2s-1/2}C(s)\Gamma(s+1/2)}{\sqrt{\pi}\,\Gamma(2s)(mn)^{1/4}}\sum_{Q\in\Gamma_\infty\backslash\mathcal{Q}^+_{mn,6}}\left(\frac{12}{b}\right)\chi_m(Q)\sqrt{\frac{v}{a}}\,e\left(\frac{bv}{2a}\right)$$
$$\times\int_0^\pi I_{s-1/2}\left(\frac{\pi v\sqrt{mn}}{a}\sin\theta\right)\cos\left(\frac{\pi v\sqrt{mn}}{a}\cos\theta\right)\frac{d\theta}{\sqrt{\sin\theta}}, \quad (4.4.13)$$

where $\mathcal{Q}^+_{mn,6}$ consists of those $Q=[a,b,c]$ with $a>0$. Lemma 9 of [DIT] asserts that for $\operatorname{Re}s>0$ we have

$$\int_0^\pi \cos(t\cos\theta)I_{s-1/2}(t\sin\theta)\frac{d\theta}{\sqrt{\sin\theta}} = 2^{s-1}\frac{\Gamma(s/2)^2}{\Gamma(s)}J_{s-1/2}(t).$$

Since

$$\frac{2^{3s-3/2}C(s)\Gamma(s+1/2)\Gamma(s/2)^2}{\sqrt{\pi}\,\Gamma(2s)\,\Gamma(s)} = 2\sqrt{2},$$

we obtain

$$L_v^+(m,n) = \frac{2\sqrt{2v}}{(mn)^{\frac14}}\sum_{Q\in\Gamma_\infty\backslash\mathcal{Q}^+_{mn,6}}\left(\frac{12}{b}\right)\frac{\chi_m(Q)}{\sqrt{a}}\,e\left(\frac{bv}{2a}\right)J_{s-\frac12}\left(\frac{\pi v\sqrt{mn}}{a}\right). \quad (4.4.14)$$

Combining (4.4.6) and (4.4.14), we have

$$L_v^{\pm}(m,n) = \frac{\sqrt{2v}}{|mn|^{\frac{1}{4}}} \sum_{\substack{Q\in\Gamma_\infty\backslash\mathcal{Q}^+_{mn,6} \\ Q=[a,b,c]}} \left(\frac{12}{b}\right) \frac{\chi_m(Q)}{\sqrt{a}} e\left(\frac{bv}{2a}\right) \varphi^{\pm}\left(\frac{\pi v\sqrt{|mn|}}{a}\right), \tag{4.4.15}$$

where

$$\varphi^-(x) = \pi I_{3/2}(x),$$
$$\varphi^+(x) = 2J_{s-1/2}(x).$$

Since $\left(\begin{smallmatrix}1 & k\\ 0 & 1\end{smallmatrix}\right)[a,b,c] = [a, b-2ka, *]$, we have a bijection

$$\Gamma_\infty\backslash\mathcal{Q}^+_{mn,6} \longleftrightarrow \{(a,b) : a>0,\ 6\mid a,\ 0\le b<2a\},$$

which gives

$$L_v^{\pm}(m,n) = \frac{\sqrt{2v}}{|mn|^{\frac{1}{4}}} \sum_{\substack{a>0\\ 6|a}} a^{-1/2}\varphi^{\pm}\left(\frac{\pi v\sqrt{|mn|}}{a}\right)$$
$$\times \sum_{\substack{b \bmod 2a \\ \frac{b^2-mn}{4a}\in\mathbb{Z}}} \left(\frac{12}{b}\right)\chi_m\left([a,b,\tfrac{b^2-mn}{4a}]\right) e\left(\frac{bv}{2a}\right). \tag{4.4.16}$$

We write $a = 6c$ and find that the inner sum in (4.4.16) equals $\frac{1}{2}\mathfrak{S}_v(m,n;24c)$ (see (1.1.24)), so

$$L_v^{\pm}(m,n) = \frac{\sqrt{v}}{2\sqrt{3}\,|mn|^{\frac{1}{4}}} \sum_{c>0} \frac{\mathfrak{S}_v(m,n;24c)}{\sqrt{c}} \varphi^{\pm}\left(\frac{\pi v\sqrt{|mn|}}{6c}\right).$$

Applying Proposition 4.3.2, we obtain

$$L_v^{\pm}(m,n) = \frac{2\sqrt{v}}{|mn|^{\frac{1}{4}}} \sum_{c>0} c^{-1/2}\varphi^{\pm}\left(\frac{\pi v\sqrt{|mn|}}{6c}\right)$$
$$\times \sum_{u|(v,c)} \left(\frac{12}{v/u}\right)\left(\frac{m}{u}\right)\sqrt{\frac{u}{c}}K\left(\left(\tfrac{v^2}{u^2}m\right)', u'; \frac{c}{u}\right).$$

We replace c by cu and switch the order of summation to obtain

$$L_v^{\pm}(m,n) = \frac{2}{|mn|^{\frac{1}{4}}} \sum_{u|v} \left(\frac{12}{v/u}\right)\left(\frac{m}{u}\right)\sqrt{\frac{v}{u}} \times \sum_{c>0} \frac{1}{c} K\left(\left(\tfrac{v^2}{u^2}m\right)', u'; c\right) \varphi^{\pm}\left(\frac{\pi v/u\sqrt{|mn|}}{6c}\right).$$

Finally, letting $d = v/u$ we conclude that

$$L_v^{\pm}(m,n) = \frac{2}{|mn|^{\frac{1}{4}}} \sum_{d|v} \sqrt{d}\left(\frac{12}{d}\right)\left(\frac{m}{v/d}\right) \sum_{c>0} \frac{K((d^2m)', n'; c)}{c} \varphi^{\pm}\left(\frac{\pi\sqrt{|d^2mn|}}{6c}\right),$$

from which Proposition 4.4.1 follows. $\square$

Theorem 1.1.4 now follows easily from Proposition 4.4.1.

Proof of Theorem 1.1.4. As above, we let $P_{v,Q}(\tau,s) := P_v(\tau,s)$ when Q has positive non-square discriminant. Then the definition of the traces in (1.1.15) becomes

$$\mathrm{Tr}_v(m,n) = \begin{cases} |mn|^{-1/2} \displaystyle\sum_{Q\in\Gamma\backslash\mathcal{Q}^{(1)}_{mn,6}} \chi_m(Q) P_v(\tau_Q) & \text{if } mn<0, \\ \dfrac{1}{2\pi} \displaystyle\sum_{Q\in\Gamma\backslash\mathcal{Q}^{(1)}_{mn,6}} \chi_m(Q) \int_{C_Q} \left[\frac{\partial}{\partial s} P_{v,Q}(\tau,s)\Big|_{s=2}\right] \frac{d\tau}{Q(\tau,1)} & \text{if } mn>0. \end{cases}$$

When $mn<0$, (1.1.20) follows immediately from (4.4.2) and (4.4.3). When $m,n>0$, we take the derivative of each side with respect to s, then set $s=2$. Comparing the resulting equation with (4.4.2) gives (1.1.20). $\square$

5 Kloosterman sums and Maass cusp forms

In this chapter we prove the theorems contained in Section 1.2 of the introduction. The material in this chapter is joint with Scott Ahlgren and appears in [AA2].

We begin by proving a general version of Theorem 1.2.5 in Section 5.1. The basic tool relating Kloosterman sums and Maass cusp forms is a version of Kuznetsov's trace formula, which we prove in Section 5.2. In Section 5.3 we introduce theta lifts which give a Shimura-type correspondence from Maass cusp forms of half-integral weight with the eta multiplier to Maass cusp forms of integral weight on $\Gamma_0(6)$. In Section 5.4 we obtain estimates for the K-Bessel transform $\check{\phi}(r)$ appearing in the Kuznetsov trace formula which, together with Theorem 1.2.5, suffice to prove Theorem 1.2.3 (Section 5.5). In Section 5.6 we prove Theorem 5.6.1, which gives an estimate for sums of coefficients of Maass cusp forms of half-integral weight. In Section 5.7 we use Theorem 5.6.1 and a modification of the argument of [ST] to prove Theorems 1.2.4, 1.2.1 and 1.2.2.

Notation. In this chapter we follow the definitions given in Section 2.3.2. In particular, the slash operator is defined as

$$f\big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{c\tau+d}{|c\tau+d|}\right)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right)$$

and the Laplacian is given by

$$\Delta_k = y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) - iky\frac{\partial}{\partial x}.$$

Furthermore, throughout this chapter, ϵ denotes an arbitrarily small positive number whose value is allowed to change with each occurence. Implied constants in any equation which contains ϵ are allowed to depend on ϵ. For all other parameters, we will use a subscript (e.g. $\ll_{a,b,c}$) to signify dependencies in the implied constant.

5.1 A mean value estimate for coefficients of Maass cusp forms

In this section we prove a general version of Theorem 1.2.5 which applies to the Fourier coefficients of weight $\pm 1/2$ Maass cusp forms with multiplier ν for $\Gamma_0(N)$, where ν satisfies the following assumptions:

1. There exists $\beta = \beta_\nu \in (1/2, 1)$ such that
$$\sum_{c>0} \frac{|S(n,n,c,\nu)|}{c^{1+\beta}} \ll_\nu n^{\epsilon}. \tag{5.1.1}$$

2. None of the cusps of $\Gamma_0(N)$ is singular for ν.

By (2.3.30) the multipliers χ and $\overline{\chi}$ on $\mathrm{SL}_2(\mathbb{Z})$ satisfy these assumptions with $\beta = 1/2 + \epsilon$.

Fix an orthonormal basis of cusp forms $\{f_j\}$ for $\mathcal{S}_k(\Gamma_0(N), \nu)$. For each j, let $a_j(n)$ denote the n-th Fourier coefficient of f_j and let r_j denote the spectral parameter.

Theorem 5.1.1. *Suppose that $k = \pm 1/2$ and that ν satisfies conditions (1) and (2) above. Then for all $x \geq 2$ and $n \geq 1$ we have*

$$n_\nu \sum_{0<r_j\leq x} \frac{|a_j(n)|^2}{\operatorname{ch} \pi r_j} = \begin{cases} \dfrac{x^{\frac{3}{2}}}{3\pi^2} + O_\nu\left(x^{\frac{1}{2}} \log x + n^{\beta+\epsilon}\right) & \text{if } k = 1/2, \\ \dfrac{x^{\frac{5}{2}}}{5\pi^2} + O_\nu\left(x^{\frac{3}{2}} \log x + n^{\beta+\epsilon} x^{\frac{1}{2}}\right) & \text{if } k = -1/2. \end{cases} \tag{5.1.2}$$

The $n > 0$ case of Theorem 1.2.5 follows from a direct application of Theorem 5.1.1 when $(k, \nu) = (1/2, \chi)$. For $n < 0$, we apply Theorem 5.1.1 to the case $(k, \nu) = (-1/2, \overline{\chi})$ and use the relation (2.3.13).

5.1.1 An auxiliary version of the Kuznetsov trace formula

We begin with an auxiliary version of Kuznetsov's formula ([Ku, §5]) which is Lemma 3 of [P2] with $m = n$. By assumption (2) there are no Eisenstein series for the multiplier system ν. While Proskurin assumes that $k > 0$ throughout his paper, this lemma is still valid for $k<0$ by the same proof. We include the term $\Gamma(2\sigma-1)$ which is omitted on the right-hand side of [P2, Lemma 3].

Lemma 5.1.2. *Suppose that $k=\pm 1/2$ and that ν satisfies conditions (1) and (2) above. For $n>0$, $t\in\mathbb{R}$, and $\sigma>1$ we have*

$$-\frac{1}{i^{k+1}}\sum_{c>0}\frac{S(n,n,c,\nu)}{c^{2\sigma}}\int_L K_{it}\left(\frac{4\pi n_\nu}{c}q\right)\left(q+\frac{1}{q}\right)^{2\sigma-2}q^{k-1}\,dq=\frac{\Gamma(2\sigma-1)}{4(2\pi n_\nu)^{2\sigma-1}}$$
$$-\frac{2^{1-2\sigma}\pi^{2-2\sigma}n_\nu^{2-2\sigma}}{\Gamma(\sigma-\frac{k}{2}+\frac{it}{2})\Gamma(\sigma-\frac{k}{2}-\frac{it}{2})}\sum_{r_j}|a_j(n)|^2\Lambda\left(\sigma+\frac{it}{2},\sigma-\frac{it}{2},r_j\right),\quad (5.1.3)$$

where L is the contour $|q|=1$, $\mathrm{Re}(q)>0$, from $-i$ to i and

$$\Lambda(s_1,s_2,r)=\Gamma\left(s_1-\frac{1}{2}-ir\right)\Gamma\left(s_1-\frac{1}{2}+ir\right)\Gamma\left(s_2-\frac{1}{2}-ir\right)\Gamma\left(s_2-\frac{1}{2}+ir\right).$$

We justify substituting $\sigma=1$ in this lemma. Suppose that $t\in\mathbb{R}$ and write $a=4\pi n_\nu/c$ and $q=e^{i\theta}$ with $-\pi/2<\theta<\pi/2$. By [DL, (10.27.4), (10.25.2), and (5.6.7)] we have, for c sufficiently large,

$$K_{it}\big(ae^{i\theta}\big)\ll e^{-\frac{\pi|t|}{2}}\big(e^{\theta t}+e^{-\theta t}\big).$$

So $K_{it}(aq)\ll 1$ on L and the integral over L is bounded uniformly in a. By this and the assumption (5.1.1), the left-hand side of (5.1.3) is absolutely uniformly convergent for $\sigma\in[1,2]$. In this range we have

$$\left|\Gamma\left(\sigma-\frac{1}{2}+iy\right)\right|=\frac{|\Gamma(\sigma+\frac{1}{2}+iy)|}{|\sigma-\frac{1}{2}+iy|}\le 2\left|\Gamma\left(\sigma+\frac{1}{2}+iy\right)\right|.$$

Using this together with the fact that $\Lambda\left(\sigma+\frac{it}{2},\sigma-\frac{it}{2},r_j\right)$ is positive, we see that the convergence of the right hand side of (5.1.3) for $\sigma>1$ implies the uniform convergence of the right hand side for $\sigma\in[1,2]$. We may therefore take $\sigma=1$.

To evaluate the integral on the left-hand side of (5.1.3) we require the following lemma.

Lemma 5.1.3. *Suppose that $a > 0$. Then*

$$\int_L K_{it}(aq)q^{k-1}\,dq = i\sqrt{2}\,a^{-k} \\ \times \begin{cases} \displaystyle\int_0^a u^{-\frac{1}{2}} \int_0^\infty \cos(t\xi)\big(\cos(u\operatorname{ch}\xi) - \sin(u\operatorname{ch}\xi)\big)\,d\xi\,du & \text{if } k = \frac{1}{2}, \\ \displaystyle\int_a^\infty u^{-\frac{3}{2}} \int_0^\infty \cos(t\xi)\big(\cos(u\operatorname{ch}\xi) + \sin(u\operatorname{ch}\xi)\big)\,d\xi\,du & \text{if } k = -\frac{1}{2}. \end{cases} \tag{5.1.4}$$

Proof. The case $k = 1/2$ follows from equation (44) in [P2] (correcting a typo in the upper limit of integration) and the integral representation [DL, (10.9.8)].

For $k = -1/2$, we write

$$\int_L K_{it}(aq)q^{k-1}\,dq = -\left(\int_{-iR}^{-i} + \int_i^{iR} + \int_{L_R}\right) K_{it}(aq)q^{k-1}\,dq,$$

where the first two integrals are along the imaginary axis, and L_R is the semicircle $|q| = R$, $\operatorname{Re}(q) > 0$, traversed clockwise. By [DL, (10.40.10)], the integral over L_R approaches zero as $R \to \infty$. For the first and second integrals we change variables to $q = -iu/a$ and $q = iu/a$, respectively, to obtain

$$\int_L K_{it}(aq)q^{k-1}\,dq = -\sqrt{a}\int_a^\infty \left(\sqrt{-i}K_{it}(iu) - \sqrt{i}K_{it}(-iu)\right)\frac{du}{u^{3/2}}.$$

The lemma follows from writing $K_{it}(\pm iu)$ as a linear combination of $J_{it}(u)$ and $J_{-it}(u)$ using [DL, (10.27.9), (10.27.3)] and applying the integral representation [DL, (10.9.8)]. □

Starting with (5.1.3) with $\sigma = 1$ and using the fact that

$$\Lambda(1 + it, 1 - it, r) = \frac{\pi^2}{\operatorname{ch}(\pi r + \pi t)\operatorname{ch}(\pi r - \pi t)}, \tag{5.1.5}$$

we replace t by $2t$, multiply by

$$\frac{4n_\nu}{\pi^2}\,g(t), \qquad \text{where } g(t) := \left|\Gamma\left(1 - \frac{k}{2} + it\right)\right|^2 \operatorname{sh}\pi t,$$

and integrate on t from 0 to x. Applying Lemma 5.1.3 to the result, we obtain

$$\begin{aligned} n_\nu \sum_{r_j} \frac{|a_j(n)|^2}{\operatorname{ch} \pi r_j} h_x(r_j) \\ = \frac{1}{2\pi^3} \int_0^x g(t)\, dt + \frac{(4n_\nu)^{1-k}\sqrt{2}}{i^k \pi^{2+k}} \sum_{c>0} \frac{S(n,n,c,\nu)}{c^{2-k}} I\left(x, \frac{4\pi n_\nu}{c}\right), \end{aligned} \tag{5.1.6}$$

where

$$h_x(r) := \int_0^x \frac{2 \operatorname{sh} \pi t \operatorname{ch} \pi r}{\operatorname{ch}(\pi r + \pi t) \operatorname{ch}(\pi r - \pi t)}\, dt \tag{5.1.7}$$

and

$$I(x,a) := \begin{cases} \displaystyle\int_0^x g(t) \int_0^a u^{-\frac{1}{2}} \int_0^\infty \cos(2t\xi) & \\ \qquad \times \left(\cos(u \operatorname{ch} \xi) - \sin(u \operatorname{ch} \xi)\right) d\xi\, du\, dt & \text{if } k = \frac{1}{2}, \\ \displaystyle\int_0^x g(t) \int_a^\infty u^{-\frac{3}{2}} \int_0^\infty \cos(2t\xi) & \\ \qquad \times \left(\cos(u \operatorname{ch} \xi) + \sin(u \operatorname{ch} \xi)\right) d\xi\, du\, dt & \text{if } k = -\frac{1}{2}. \end{cases} \tag{5.1.8}$$

5.1.2 The function $g(t)$ and the main term

A computation involving [DL, (5.11.3), (5.11.11)] shows that for fixed x and $y \geq 1$ we have

$$\Gamma(x+iy)\Gamma(x-iy) = 2\pi y^{2x-1} e^{-\pi y} \left(1 + O\left(\frac{1}{y}\right)\right),$$

from which it follows that

$$g(t) = \pi t^{1-k} + O(t^{-k}) \qquad \text{as } t \to \infty. \tag{5.1.9}$$

We have

$$g(t) = \pi \Gamma\left(1 - \tfrac{k}{2}\right)^2 t + O(t^2) \qquad \text{as } t \to 0. \tag{5.1.10}$$

This gives the main term

$$\frac{1}{2\pi^3} \int_0^x g(t)\, dt = \begin{cases} \dfrac{x^{\frac{3}{2}}}{3\pi^2} + O\big(x^{\frac{1}{2}}\big) & \text{if } k = \frac{1}{2}, \\ \dfrac{x^{\frac{5}{2}}}{5\pi^2} + O\big(x^{\frac{3}{2}}\big) & \text{if } k = -\frac{1}{2}. \end{cases} \tag{5.1.11}$$

5.1.3 Preparing to estimate the error term

The error term in (5.1.2) is obtained by uniformly estimating the integral $I(x,a)$ in the ranges $a \le 1$ and $a \ge 1$. The analysis is similar in spirit to [Ku, Section 5], but the nature of the function $g(t)$ introduces substantial difficulties. In this section we collect a number of facts which we will require for these estimates.

Given an interval $[a,b]$, let $\operatorname{var} f$ denote the total variation of f on $[a,b]$ which, for f differentiable, is given by

$$\operatorname{var} f = \int_a^b |f'(x)|\, dx. \tag{5.1.12}$$

We begin by collecting some facts about the functions $g'(t)$, $(g(t)/t)'$, and $t(g(t)/t)'$.

Lemma 5.1.4. *Let $k = \pm 1/2$. If $x \ge 1$ then on the interval $[0,x]$ we have*

$$\operatorname{var}\,(g(t)/t)' \ll 1 + x^{-1-k}, \tag{5.1.13}$$

$$\operatorname{var}\, t(g(t)/t)' \ll 1 + x^{-k}. \tag{5.1.14}$$

For $t > 0$ we have

$$g'(t) > 0, \qquad \operatorname{sgn}\,(g(t)/t)' = -\operatorname{sgn} k. \tag{5.1.15}$$

Proof. Let $\sigma = 1 - k/2$. Taking the logarithmic derivative of $g(t)/t$, we find that

$$\left(\frac{g(t)}{t}\right)' = g(t)h(t), \tag{5.1.16}$$

where

$$h(t) := \frac{1}{t}\left[i\big(\psi(\sigma+it) - \psi(\sigma - it)\big) - \frac{1}{t} + \pi \coth \pi t\right], \qquad \psi := \frac{\Gamma'}{\Gamma}.$$

We have

$$g'(t) = g(t)\left[i\big(\psi(\sigma+it) - \psi(\sigma-it)\big) + \pi \coth \pi t\right],$$

and

$$h'(t) = -\frac{h(t)}{t} - \frac{1}{t}\left[\psi'(\sigma+it)+\psi'(\sigma-it) - \tfrac{1}{t^2} + \pi^2(\operatorname{csch}\pi t)^2\right].$$

At $t=0$ we have the Taylor series

$$g(t) = \pi\Gamma(\sigma)^2 t + \pi\Gamma(\sigma)^2\Big(\frac{\pi^2}{6} - \psi'(\sigma)\Big)t^3 + \dots, \tag{5.1.17}$$

$$h(t) = \frac{\pi^2}{3} - 2\psi'(\sigma) + \Big(\frac{1}{3}\psi'''(\sigma) - \frac{\pi^4}{45}\Big)t^2 + \dots. \tag{5.1.18}$$

Thus we have the estimates

$$g(t) \ll t, \quad g'(t) \ll 1, \quad h(t) \ll 1, \quad \text{and} \quad h'(t) \ll t \quad \text{as } t \to 0. \tag{5.1.19}$$

For large t we use [DL, (5.11.2) and (5.15.8)] to obtain the asymptotic expansions

$$i\big(\psi(\sigma+it) - \psi(\sigma-it)\big) = -2\arctan\left(\frac{t}{\sigma}\right) - \frac{t}{\sigma^2+t^2} + O\Big(\frac{1}{t^2}\Big), \tag{5.1.20}$$

$$\psi'(\sigma+it) + \psi'(\sigma-it) = \frac{2\sigma}{\sigma^2+t^2} + \frac{\sigma^2-t^2}{(\sigma^2+t^2)^2} + O\Big(\frac{1}{t^3}\Big). \tag{5.1.21}$$

By (5.1.9), (5.1.20), and (5.1.21) we have the estimates

$$g(t) \ll t^{1-k}, \quad g'(t) \ll t^{-k}, \quad h(t) \ll \frac{1}{t^2}, \quad \text{and} \quad h'(t) \ll \frac{1}{t^3} \quad \text{as } t \to \infty. \tag{5.1.22}$$

From (5.1.22) and (5.1.12) it follows that

$$\operatorname{var}\,(g(t)/t)' = \int_0^x |g(t)h'(t) + g'(t)h(t)|\; dt \ll 1 + x^{-1-k}$$

and

$$\operatorname{var}\; t(g(t)/t)' = \int_0^x |t\, g(t)h'(t) + t\, g'(t)h(t) + g(t)h(t)|\; dt \ll 1 + x^{-k}.$$

Using [DL, (4.36.3), (5.7.6)] we have

$$h(t) = 2\sum_{n=1}^{\infty}\left(\frac{1}{t^2+n^2} - \frac{1}{t^2+(n-\frac{k}{2})^2}\right).$$

The second claim in (5.1.15) follows from (5.1.16). The first is simpler. $\square$

Suppose that $f \geq 0$ and ϕ are of bounded variation on $[a, b]$ with no common points of discontinuity. By Corollary 2 of [Kno], there exists $[\alpha, \beta] \subseteq [a, b]$ such that

$$\int_a^b f\, d\phi = (\inf f + \operatorname{var} f) \int_\alpha^\beta d\phi, \tag{5.1.23}$$

where the integrals are of Riemann-Stieltjes type. We obtain the following by taking $\phi(x) = \int_0^x G(t)\, dt$.

Lemma 5.1.5. *Suppose that $F \geq 0$ and G are of bounded variation on $[a, b]$ and that G is continuous. Then*

$$\left| \int_a^b F(x)G(x)\, dx \right| \leq (\inf F + \operatorname{var} F) \sup_{[\alpha,\beta]\subseteq[a,b]} \left| \int_\alpha^\beta G(x)\, dx \right|. \tag{5.1.24}$$

We will frequently make use of the following well-known estimates for oscillatory integrals (see, for instance, [T, Chapter IV, p. 61]).

Lemma 5.1.6 (First derivative estimate)**.** *Suppose that F and G are real-valued functions on $[a, b]$ with F differentiable, such that $G(x)/F'(x)$ is monotonic. If $|F'(x)/G(x)| \geq m > 0$ then*

$$\int_a^b G(x)e(F(x))\, dx \ll \frac{1}{m}.$$

Lemma 5.1.7 (Second derivative estimate)**.** *Suppose that F and G are real-valued functions on $[a, b]$ with F twice differentiable, such that $G(x)/F'(x)$ is monotonic. If $|G(x)| \leq M$ and $|F''(x)| \geq m > 0$ then*

$$\int_a^b G(x)e(F(x))\, dx \ll \frac{M}{\sqrt{m}}.$$

5.1.4 The first error estimate

We estimate the error arising from $I(x, a)$ (recall (5.1.8)) when $k = 1/2$.

Proposition 5.1.8. *Suppose that $k = 1/2$. For $a > 0$ we have*

$$I(x, a) \ll \begin{cases} \sqrt{a}\left(\log(1/a) + 1\right) & \text{if } a \leq 1, \\ 1 & \text{if } a \geq 1. \end{cases} \tag{5.1.25}$$

Proof. We write $I(x,a)$ as a difference of integrals, one involving $\cos(u\,\mathrm{ch}\,\xi)$, the other involving $\sin(u\,\mathrm{ch}\,\xi)$. We will estimate the first integral, the second being similar. We have

$$\int_0^\infty \cos(2t\xi)\cos(u\,\mathrm{ch}\,\xi)\,d\xi = \int_0^T \cos(2t\xi)\cos(u\,\mathrm{ch}\,\xi)\,d\xi + R_T(u,t),$$

where, by the second derivatve estimate,

$$R_T(u,t) \ll u^{-\frac{1}{2}}e^{-\frac{T}{2}}.$$

For $0<\varepsilon<a$ we have, using (5.1.9),

$$\int_0^x g(t)\int_\varepsilon^a u^{-\frac{1}{2}}R_T(u,t)\,du\,dt \ll e^{-\frac{T}{2}}x^{\frac{3}{2}}\big(|\log a| + |\log\varepsilon|\big) \to 0 \qquad \text{as } T\to\infty.$$

So our goal is to estimate

$$\lim_{\varepsilon\to 0^+}\lim_{T\to\infty} I(x,\varepsilon,a,T),$$

where

$$I(x,\varepsilon,a,T) = \int_0^x g(t)\int_\varepsilon^a u^{-\frac{1}{2}}\int_0^T \cos(2t\xi)\cos(u\,\mathrm{ch}\,\xi)\,d\xi\,du\,dt.$$

Integrating the innermost integral by parts, we find that

$$\begin{aligned} I(x,\varepsilon,a,T) &= \frac{1}{2}\int_0^x \frac{g(t)}{t}\sin(2tT)\,dt\int_\varepsilon^a u^{-\frac{1}{2}}\cos(u\,\mathrm{ch}\,T)\,du \\ &\quad + \frac{1}{2}\int_0^T \mathrm{sh}\,\xi\int_0^x \frac{g(t)}{t}\sin(2t\xi)\,dt\int_\varepsilon^a \sqrt{u}\,\sin(u\,\mathrm{ch}\,\xi)\,du\,d\xi. \qquad (5.1.26)\end{aligned}$$

The first derivative estimate (recalling (5.1.10) and (5.1.15)) gives

$$\int_0^x \frac{g(t)}{t}\sin(2tT)\,dt \ll \frac{1}{T} \qquad \text{and} \qquad \int_\varepsilon^a u^{-\frac{1}{2}}\cos(u\,\mathrm{ch}\,T)\,du \ll \frac{1}{\sqrt{\varepsilon}\,\mathrm{ch}\,T}, \qquad (5.1.27)$$

so the first term on the right-hand side of (5.1.26) approaches zero as $T\to\infty$.

Turning to the second term, set

$$G_x(\xi) := \int_0^x \frac{g(t)}{t}\sin(2t\xi)\,dt.$$

Integrating by parts gives

$$G_x(\xi) = \frac{\pi\Gamma(\frac{3}{4})^2}{2\xi} - \frac{g(x)}{x}\frac{\cos(2x\xi)}{2\xi} + \frac{1}{2\xi}\int_0^x \left(\frac{g(t)}{t}\right)' \cos(2t\xi)\,dt. \tag{5.1.28}$$

Applying Lemma 5.1.5 and (5.1.13) to the integral in (5.1.28) we find that

$$\frac{1}{\xi}\int_0^x \left(\frac{g(t)}{t}\right)' \cos(2t\xi)\,dt \ll \frac{1}{\xi} \sup_{0\le\alpha<\beta\le x}\left|\int_\alpha^\beta \cos(2t\xi)\,dt\right| \ll \frac{1}{\xi^2}.$$

On the other hand, (5.1.15) gives

$$\left|\frac{1}{\xi}\int_0^x \left(\frac{g(t)}{t}\right)' \cos(2t\xi)\,dt\right| \le \frac{1}{\xi}\int_0^x -\left(\frac{g(t)}{t}\right)' dt \ll \frac{1}{\xi}. \tag{5.1.29}$$

It follows that

$$G_x(\xi) = \frac{\pi\Gamma(\frac{3}{4})^2}{2\xi} - \frac{g(x)}{x}\frac{\cos(2x\xi)}{2\xi} + O\left(\min(\xi^{-1},\xi^{-2})\right). \tag{5.1.30}$$

The integral

$$\int_\varepsilon^a \sqrt{u}\,\sin(u\,\mathrm{ch}\,\xi)\,du$$

evaluates to $H(a,\xi) - H(\varepsilon,\xi)$, where

$$H(u,\xi) := -\frac{\sqrt{u}\,\cos(u\,\mathrm{ch}\,\xi)}{\mathrm{ch}\,\xi} + \frac{\sqrt{\pi/2}}{(\mathrm{ch}\,\xi)^{\frac{3}{2}}}C\left(\sqrt{\frac{2u\,\mathrm{ch}\,\xi}{\pi}}\right), \tag{5.1.31}$$

and $C(x)$ denotes the Fresnel integral

$$C(x) := \int_0^x \cos\left(\frac{\pi}{2}t^2\right)\,dt.$$

For $u > 0$, write

$$J(x,u) := \int_0^\infty \mathrm{sh}\,\xi\, G_x(\xi) H(u,\xi)\,d\xi. \tag{5.1.32}$$

To show that $J(x,u)$ converges, suppose that $B \ge A \ge T$. By (5.1.30), (5.1.31), and the bound $C(x) \le 1$ we have

$$\begin{aligned}
&\int_A^B \operatorname{sh}\xi\, G_x(\xi) H(u,\xi)\, d\xi \\
&\quad = -\sqrt{u}\int_A^B \operatorname{th}\xi \cos(u\operatorname{ch}\xi)\left(\frac{\pi\Gamma(\frac{3}{4})^2}{2\xi} - \frac{g(x)}{x}\frac{\cos(2x\xi)}{2\xi} + O\Big(\frac{1}{\xi^2}\Big)\right) d\xi \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad + O\left(\int_A^B \frac{\operatorname{sh}\xi}{\xi(\operatorname{ch}\xi)^{\frac{3}{2}}}\, d\xi\right) \\
&\quad \ll u^{-\frac{1}{2}} e^{-A/2} + u^{\frac{1}{2}} A^{-1} + e^{-A/2},
\end{aligned}$$

where we have used the first derivative estimate in the last inequality. We have

$$\lim_{\varepsilon\to 0^+}\lim_{T\to\infty} I(x,\varepsilon,a,T) = \frac{1}{2}J(x,a) - \frac{1}{2}\lim_{\varepsilon\to 0^+} J(x,\varepsilon), \tag{5.1.33}$$

so it remains only to estimate $J(x,u)$ in the ranges $u \le 1$ and $u \ge 1$.

First, suppose that $u \le 1$. We estimate each of the six terms obtained by multiplying (5.1.30) and (5.1.31) in (5.1.32). Starting with the terms involving $\cos(u\operatorname{ch}\xi)$, we have

$$\begin{aligned}
\sqrt{u}\int_0^\infty \frac{\operatorname{th}\xi}{\xi}\cos(u\operatorname{ch}\xi)\, d\xi &= \sqrt{u}\left(\int_0^{\log\frac{1}{u}} + \int_{\log\frac{1}{u}}^\infty\right)\frac{\operatorname{th}\xi}{\xi}\cos(u\operatorname{ch}\xi)\, d\xi \\
&\ll \sqrt{u}\,\big(\log(1/u)+1\big)
\end{aligned}$$

by applying the second derivative estimate to the second integral. By the same method,

$$\sqrt{u}\,\frac{g(x)}{x}\int_0^\infty \frac{\operatorname{th}\xi}{\xi}\cos(2x\xi)\cos(u\operatorname{ch}\xi)\, d\xi \ll \sqrt{u}\,\big(\log(1/u)+1\big),$$

since $g(x)/x \ll 1$. For the third term we have

$$\begin{aligned}
&\sqrt{u}\int_0^\infty \operatorname{th}\xi\cos(u\operatorname{ch}\xi)\min(\xi^{-1},\xi^{-2})\, d\xi \\
&\qquad\qquad\qquad \ll \sqrt{u}\int_0^1 \frac{\operatorname{th}\xi}{\xi}\, d\xi + \sqrt{u}\int_1^\infty \frac{\operatorname{th}\xi}{\xi^2}\, d\xi \ll \sqrt{u}.
\end{aligned}$$

For the terms involving the Fresnel integral, we use the trivial estimate

$C(x) \le \min(x,1)$. For the first of these terms we have

$$\int_0^\infty \frac{\operatorname{sh}\xi}{\xi(\operatorname{ch}\xi)^{\frac{3}{2}}} C\left(\sqrt{\frac{2u\operatorname{ch}\xi}{\pi}}\right) d\xi$$
$$\ll \sqrt{u}\int_0^{\log\frac{1}{u}} \frac{\operatorname{th}\xi}{\xi}\,d\xi + \int_{\log\frac{1}{u}}^\infty \frac{d\xi}{\sqrt{\operatorname{ch}\xi}} \ll \sqrt{u}\,\big(\log(1/u)+1\big).$$

By the same method, the remaining two terms are $\ll \sqrt{u}\,(\log(1/u)+1)$. It follows that

$$J(x,u) \ll \sqrt{u}\,\big(\log(1/u)+1\big) \qquad \text{for } u \le 1. \tag{5.1.34}$$

For $u \ge 1$ we show that $J(x,u) \ll 1$. Write

$$J(x,u) = J_1(x,u) + J_2(x,u) := \left(\int_0^{1/\sqrt{u}} + \int_{1/\sqrt{u}}^\infty\right) \operatorname{sh}\xi\, G_x(\xi) H(u,\xi)\,d\xi.$$

Since $G_x(\xi) \ll \xi^{-1}$ and $H(u,\xi) \ll \sqrt{u}/\operatorname{ch}\xi$, we have

$$J_1(x,u) \ll \sqrt{u}\int_0^{1/\sqrt{u}} \frac{\operatorname{th}\xi}{\xi}\,d\xi \ll 1.$$

We break $J_2(x,u)$ into six terms, using (5.1.28) in place of (5.1.30). We start with the terms involving $\pi\Gamma(3/4)^2/2\xi$. By the first derivative estimate we have

$$\sqrt{u}\int_{1/\sqrt{u}}^\infty \frac{\operatorname{th}\xi}{\xi}\cos(u\operatorname{ch}\xi)\,d\xi \ll 1.$$

Estimating the Fresnel integral trivially, we have

$$\int_{1/\sqrt{u}}^\infty \frac{\operatorname{sh}\xi}{\xi(\operatorname{ch}\xi)^{\frac{3}{2}}} C\left(\sqrt{\frac{2u\operatorname{ch}\xi}{\pi}}\right) d\xi \ll \int_0^\infty \frac{d\xi}{\sqrt{\operatorname{ch}\xi}} \ll 1.$$

The two terms involving $\cos(2x\xi)/\xi$ are treated similarly, and are $\ll 1$. The term involving the integral in (5.1.28) and the Fresnel integral can be estimated trivially using (5.1.29); it is also $\ll 1$.

The final term,

$$\sqrt{u}\int_{1/\sqrt{u}}^\infty \frac{\operatorname{sh}\xi\cos(u\operatorname{ch}\xi)}{\xi\operatorname{ch}\xi}\int_0^x \left(\frac{g(t)}{t}\right)'\cos(2t\xi)\,dt\,d\xi,$$

is more delicate. We write the integral as

$$\sqrt{u}\int_{1/\sqrt{u}}^{\infty}\left\{\frac{1}{\xi\operatorname{ch}\xi}\int_0^x\left(\frac{g(t)}{t}\right)'\cos(2t\xi)\,dt\right\}\cdot\left\{\operatorname{sh}\xi\cos(u\operatorname{ch}\xi)d\xi\right\} = \sqrt{u}\int_{1/\sqrt{u}}^{\infty}U\,dV$$

and integrate by parts. By (5.1.29) we have

$$\sqrt{u}\,UV\Big|_{1/\sqrt{u}}^{\infty} \ll \sqrt{u}\cdot\frac{\sqrt{u}}{\operatorname{ch}(1/\sqrt{u})}\cdot\frac{1}{u} \ll 1.$$

Write $U' = R + S$, where

$$R = \left(\frac{1}{\xi\operatorname{ch}\xi}\right)'\int_0^x\left(\frac{g(t)}{t}\right)'\cos(2t\xi)\,dt, \quad S = \frac{2}{\xi\operatorname{ch}\xi}\int_0^x t\left(\frac{g(t)}{t}\right)'\sin(2t\xi)\,dt.$$

By (5.1.29) we have

$$R \ll \frac{1}{\xi\operatorname{ch}\xi}\left(\frac{1}{\xi}+\operatorname{th}\xi\right) \ll \begin{cases}(\xi^2\operatorname{ch}\xi)^{-1} & \text{if } \xi\le 1,\\ (\xi\operatorname{ch}\xi)^{-1} & \text{if } \xi\ge 1.\end{cases}$$

Applying Lemma 5.1.5 and the estimate (5.1.14) we find that

$$S \ll \frac{1}{\xi^2\operatorname{ch}\xi}.$$

Thus we have

$$\sqrt{u}\int_{1/\sqrt{u}}^{\infty}V\,dU \ll \frac{1}{\sqrt{u}}\int_{1/\sqrt{u}}^{1}\frac{d\xi}{\xi^2}+\frac{1}{\sqrt{u}}\int_1^{\infty}\frac{d\xi}{\xi\operatorname{ch}\xi} \ll 1.$$

We conclude that

$$J(x,u) \ll 1 \qquad \text{for } u \ge 1. \tag{5.1.35}$$

The proposition follows from (5.1.33), (5.1.34), and (5.1.35). □

5.1.5 The second error estimate

In the case when $k = -1/2$ we must keep track of the dependence on x.

Proposition 5.1.9. *Suppose that $k = -1/2$. For $a > 0$ we have*

$$I(x,a) \ll \sqrt{x} \times \begin{cases} a^{-\frac{1}{2}}\big(\log(1/a)+1\big) & \textit{if } a \le 1, \\ a^{-1} & \textit{if } a \ge 1. \end{cases} \tag{5.1.36}$$

Proof. As before, we estimate the term involving $\cos(u \operatorname{ch} \xi)$ and we write

$$\begin{aligned} &\int_0^\infty \cos(2t\xi)\cos(u \operatorname{ch} \xi)\, d\xi \\ &= \frac{1}{2t}\sin(2tT)\cos(u \operatorname{ch} T) + \frac{u}{2t}\int_0^T \operatorname{sh} \xi \sin(2t\xi)\sin(u \operatorname{ch} \xi)\, d\xi + R_T(u,t), \end{aligned} \tag{5.1.37}$$

with $R_T(u,t) \ll u^{-1/2}e^{-T/2}$. Recalling (5.1.9) and (5.1.10), we have

$$\int_0^x g(t)\int_a^\infty u^{-\frac{3}{2}} R_T(u,t)\, du\, dt \ll a^{-1}x^{\frac{5}{2}}e^{-T/2} \to 0 \qquad \text{as } T \to \infty$$

and, by the first derivative estimate,

$$\int_0^x \frac{g(t)}{t}\sin(2tT)\, dt \int_a^\infty u^{-\frac{3}{2}}\cos(u \operatorname{ch} T)\, du \ll a^{-\frac{3}{2}}x^{\frac{3}{2}}e^{-T} \to 0 \qquad \text{as } T \to \infty.$$

Our goal is to estimate

$$J(x,a) := \int_0^\infty \operatorname{sh} \xi\, G_x(\xi) H(a,\xi)\, d\xi, \tag{5.1.38}$$

where

$$G_x(\xi) := \int_0^x \frac{g(t)}{t}\sin(2t\xi)\, dt \tag{5.1.39}$$

and

$$H(a,\xi) := \int_a^\infty u^{-\frac{1}{2}}\sin(u \operatorname{ch} \xi)\, du. \tag{5.1.40}$$

The integral defining $J(x,a)$ converges by an argument as in the proof of Proposition 5.1.8.

Together with (5.1.9) and (5.1.15), the first derivative estimate gives

$$G_x(\xi) \ll \frac{\sqrt{x}}{\xi}. \tag{5.1.41}$$

For a better estimate, we integrate by parts in (5.1.39) to get

$$G_x(\xi) = \frac{\pi\Gamma(\frac{5}{4})^2}{2\xi} - \frac{g(x)}{x}\frac{\cos(2x\xi)}{2\xi} + \frac{1}{2\xi}\int_0^x \left(\frac{g(t)}{t}\right)' \cos(2t\xi)\, dt. \qquad (5.1.42)$$

We estimate the third term in two ways. Taking absolute values and using (5.1.15), it is $\ll \sqrt{x}/\xi$; by (5.1.13) and Lemma 5.1.5 it is $\ll 1/\xi^2$. Thus

$$\frac{1}{\xi}\int_0^x \left(\frac{g(t)}{t}\right)' \cos(2t\xi)\, dt \ll \min\left(\frac{\sqrt{x}}{\xi}, \frac{1}{\xi^2}\right). \qquad (5.1.43)$$

We also need two estimates for $H(a,\xi)$. The first derivative estimate applied to (5.1.40) gives

$$H(a,\xi) \ll \frac{1}{\sqrt{a}\ \mathrm{ch}\,\xi}, \qquad (5.1.44)$$

while integrating (5.1.40) by parts and applying the first derivative estimate gives

$$H(a,\xi) = \frac{\cos(a\,\mathrm{ch}\,\xi)}{\sqrt{a}\ \mathrm{ch}\,\xi} + O\left(\frac{1}{a^{\frac{3}{2}}(\mathrm{ch}\,\xi)^2}\right). \qquad (5.1.45)$$

First suppose that $a \le 1$. Write

$$J(x,a) = J_1(x,a) + J_2(x,a) = \left(\int_0^{1/a} + \int_{1/a}^{\infty}\right) \mathrm{sh}\,\xi\, G_x(\xi) H(a,\xi)\, d\xi.$$

By (5.1.41) and (5.1.44) we have

$$J_1(x,a) \ll \sqrt{\frac{x}{a}}\left(\int_0^1 + \int_1^{1/a}\right)\frac{\mathrm{th}\,\xi}{\xi}\, d\xi \ll \sqrt{\frac{x}{a}}\big(1+\log(1/a)\big).$$

By (5.1.45), (5.1.41), (5.1.42), and the second bound in (5.1.43) we have

$$\begin{aligned} J_2(x,a) &= \frac{1}{\sqrt{a}}\int_{1/a}^{\infty} \mathrm{th}\,\xi\, G_x(\xi)\cos(a\,\mathrm{ch}\,\xi)\, d\xi + O\left(\frac{\sqrt{x}}{a^{\frac{3}{2}}}\int_{1/a}^{\infty}\frac{\mathrm{sh}\,\xi}{\xi(\mathrm{ch}\,\xi)^2}\, d\xi\right) \\ &\ll \frac{1}{\sqrt{a}}\int_{1/a}^{\infty}\frac{\mathrm{th}\,\xi}{\xi}\cos(a\,\mathrm{ch}\,\xi)\, d\xi \\ &\qquad + \frac{g(x)}{x\sqrt{a}}\int_{1/a}^{\infty}\frac{\mathrm{th}\,\xi}{\xi}\cos(2x\xi)\cos(a\,\mathrm{ch}\,\xi)\, d\xi + \sqrt{x/a}. \end{aligned}$$

Applying the first derivative estimate to each integral above, we find that

$J_2(x,a) \ll \sqrt{x/a}$, from which it follows that

$$J(x,a) \ll \sqrt{\frac{x}{a}}\bigl(\log(1/a)+1\bigr) \qquad \text{for } a \le 1. \tag{5.1.46}$$

Now suppose that $a \ge 1$. Using (5.1.41), we find that the contribution to $J(x,a)$ from the second term in (5.1.45) is

$$a^{-\frac{3}{2}} \int_0^\infty \frac{\operatorname{sh}\xi}{(\operatorname{ch}\xi)^2} G_x(\xi)\, d\xi \ll a^{-\frac{3}{2}}\sqrt{x}.$$

We break the integral involving $\cos(a\operatorname{ch}\xi)$ at $1/\sqrt{a}$. By (5.1.41) the contribution from the initial segment is $\ll a^{-1}\sqrt{x}$. It remains to estimate

$$\frac{1}{\sqrt{a}} \int_{1/\sqrt{a}}^\infty \operatorname{th}\xi\, G_x(\xi) \cos(a\operatorname{ch}\xi)\, d\xi,$$

which we break into three terms using (5.1.42). For the first two terms the first derivative estimate gives

$$\frac{1}{\sqrt{a}} \int_{1/\sqrt{a}}^\infty \frac{\operatorname{th}\xi}{\xi} \cos(a\operatorname{ch}\xi)\, d\xi \ll \frac{1}{a},$$
$$\frac{g(x)}{x\sqrt{a}} \int_{1/\sqrt{a}}^\infty \frac{\operatorname{th}\xi}{\xi} \cos(2x\xi)\cos(a\operatorname{ch}\xi)\, d\xi \ll \frac{\sqrt{x}}{a}.$$

The remaining term,

$$\frac{1}{\sqrt{a}} \int_{1/\sqrt{a}}^\infty \frac{\operatorname{th}\xi}{\xi} \cos(a\operatorname{ch}\xi) \int_0^x \left(\frac{g(t)}{t}\right)' \sin(2t\xi)\, dt\, d\xi,$$

requires more care, but we follow the estimate for the corresponding term in the proof of Proposition 5.1.8 using Lemma 5.1.5 and (5.1.14). The details are similar, and the contribution is $\ll \frac{\sqrt{x}}{a}$, which, together with (5.1.46), completes the proof. □

5.1.6 Proof of Theorem 5.1.1

We give the proof for the case $k = 1/2$, as the other case is analogous. By Proposition 5.1.8 we have

$$I(x,a) \ll \min(1, a^{1/2}\log(1/a)) \ll_\delta a^\delta \qquad \text{for any } 0 \le \delta < 1/2.$$

So by (5.1.6) and (5.1.11) we have

$$n_\nu \sum_{r_j} \frac{|a_j(n)|^2}{\operatorname{ch}\pi r_j} h_x(r_j) = \frac{x^{\frac{3}{2}}}{3\pi^2} + O_\delta\left(x^{\frac{1}{2}} + n^{\frac{1}{2}+\delta} \sum_{c>0} \frac{|S(n,n,c,\nu)|}{c^{3/2+\delta}}\right).$$

Setting $\delta = \beta - 1/2$, with β as in (5.1.1), this becomes

$$n_\nu \sum_{r_j} \frac{|a_j(n)|^2}{\operatorname{ch}\pi r_j} h_x(r_j) = \frac{x^{\frac{3}{2}}}{3\pi^2} + O\left(x^{\frac{1}{2}} + n^{\beta+\epsilon}\right). \tag{5.1.47}$$

The function $h_x(r)$ (recall (5.1.7)) is a smooth approximation to the characteristic function of $[0, x]$. We recall some properties from [Ku, (5.4)–(5.7)] for $x \geq 1$:

$$h_x(r) = \frac{2}{\pi}\arctan(e^{\pi x - \pi r}) + O(e^{-\pi r}) \qquad \text{for } r \geq 1, \tag{5.1.48}$$

$$h_x(r) = 1 + O(x^{-3} + e^{-\pi r}) \qquad \text{for } 1 \leq r \leq x - \log x, \tag{5.1.49}$$

$$h_x(r) \ll e^{-\pi(r-x)} \qquad \text{for } r \geq x + \log x, \tag{5.1.50}$$

$$0 < h_x(r) < 1 \qquad \text{for } r \geq 0. \tag{5.1.51}$$

For $x \geq 1$ and $0 \leq r \leq x$ we bound $h_x(r)$ uniformly from below as follows. If $r \geq 1$ we have

$$h_x(r) = 4\operatorname{ch}\pi r \int_0^x \frac{\operatorname{sh}\pi t}{\operatorname{ch}2\pi r + \operatorname{ch}2\pi t}\,dt \geq \frac{2\operatorname{ch}\pi r}{\operatorname{ch}2\pi r}\int_0^r \operatorname{sh}\pi t\,dt > \frac{1}{4},$$

while if $r \leq 1$ we have

$$h_x(r) \geq 4\operatorname{ch}\pi r \int_0^1 \frac{\operatorname{sh}\pi t}{\operatorname{ch}2\pi r + \operatorname{ch}2\pi t}\,dt \geq 4\int_0^1 \frac{\operatorname{sh}\pi t}{\operatorname{ch}2\pi + \operatorname{ch}2\pi t}\,dt > \frac{3}{100}.$$

Similarly, we have $h_x(r) > 2/5$ for $r \in i[0, 1/4]$.

Set

$$A(x) := n_\nu \sum_{\substack{r_j \leq x \\ \text{or } r_j \in i\mathbb{R}}} \frac{|a_j(n)|^2}{\operatorname{ch}\pi r_j}. \tag{5.1.52}$$

Then

$$A(x) \ll n_\nu \sum_{\substack{r_j \leq x \\ \text{or } r_j \in i\mathbb{R}}} \frac{|a_j(n)|^2}{\operatorname{ch}\pi r_j} h_x(r_j) \ll n_\nu \sum_{r_j} \frac{|a_j(n)|^2}{\operatorname{ch}\pi r_j} h_x(r_j)$$

which, together with (5.1.47), gives

$$A(x) \ll x^{\frac{3}{2}} + n^{\beta+\epsilon}. \tag{5.1.53}$$

Let $A^*(x)$ denote the sum (5.1.52) restricted to $1 \le r_j \le x$. By (5.1.53) with $x = 1$ we have $A(x) = A^*(x) + O(n^{\beta+\epsilon})$, so in what follows we work with $A^*(x)$.

Assume that $x \ge 2$ and set $X = x + 2\log x$ so that $X - \log X \ge x$. Then by (5.1.49) there exist $c_1, c_2 > 0$ such that

$$n_\nu \sum_{1\le r_j} \frac{|a_j(n)|^2}{\operatorname{ch} \pi r_j} h_X(r_j) \ge A^*(x)\left(1 - \frac{c_1}{x^3}\right) - c_2\, n_\nu \sum_{1\le r_j\le x} \frac{|a_j(n)|^2}{\operatorname{ch} \pi r_j} e^{-\pi r_j}.$$

On the other hand, (5.1.47) gives

$$n_\nu \sum_{1\le r_j} \frac{|a_j(n)|^2}{\operatorname{ch} \pi r_j} h_X(r_j) = \frac{x^{\frac{3}{2}}}{3\pi^2} + O\left(x^{\frac{1}{2}} \log x + n^{\beta+\epsilon}\right).$$

By (5.1.53) we have

$$\begin{aligned} n_\nu \sum_{1\le r_j\le x} \frac{|a_j(n)|^2}{\operatorname{ch} \pi r_j} e^{-\pi r_j} &= n_\nu \sum_{\ell=1}^{\lfloor x\rfloor} \sum_{\ell\le r_j\le \ell+1} \frac{|a_j(n)|^2}{\operatorname{ch} \pi r_j} e^{-\pi r_j} \\ &\ll \sum_{\ell=1}^{\infty} e^{-\pi\ell}\big(\ell^{\frac{3}{2}} + n^{\beta+\epsilon}\big) \ll n^{\beta+\epsilon}, \end{aligned} \tag{5.1.54}$$

so that

$$A^*(x) \le \frac{x^{\frac{3}{2}}}{3\pi^2} + O\left(x^{\frac{1}{2}} \log x + n^{\beta+\epsilon}\right). \tag{5.1.55}$$

Similarly, replacing x by $x - \log x$ in (5.1.47) and using (5.1.50), (5.1.51) gives

$$A^*(x) \ge \frac{x^{\frac{3}{2}}}{3\pi^2} + O\left(x^{\frac{1}{2}} \log x + n^{\beta+\epsilon}\right) + O\Big(n_\nu \sum_{r_j>x} \frac{|a_j(n)|^2}{\operatorname{ch} \pi r_j} e^{-\pi(r_j - x + \log x)}\Big).$$

Arguing as in (5.1.54), we see that the error term is $\ll x^{-\frac{3}{2}} + n^{\beta+\epsilon} x^{-3}$, from which we obtain

$$A^*(x) \ge \frac{x^{\frac{3}{2}}}{3\pi^2} + O\left(x^{\frac{1}{2}} \log x + n^{\beta+\epsilon}\right). \tag{5.1.56}$$

Equations (5.1.55) and (5.1.56) together give Theorem 5.1.1. □

5.2 The Kuznetsov trace formula

We require a variant of Kuznetsov's formula [Ku, Theorem 1], which relates weighted sums of the Kloosterman sums $S(m,n,c,\chi)$ on one side to spectral data on the other side (in this case, weighted sums of coefficients of Maass cusp forms). In [P2], Proskurin generalized Kuznetsov's theorem to any weight k and multiplier system ν, but only for the case $m,n>0$. Here we treat the mixed sign case when $k=1/2$ and $\nu=\chi$; the proof follows the same lines. Blomer [Bl] has recorded this formula for twists of the theta-multiplier by a Dirichlet character; we provide a sketch of the proof in the present case since there are some details which are unique to this situation.

Suppose that $\phi:[0,\infty)\to\mathbb{C}$ is four times continuously differentiable and satisfies

$$\phi(0)=\phi'(0)=0, \quad \phi^{(j)}(x)\ll_\epsilon x^{-2-\varepsilon} \quad (j=0,\ldots,4) \quad \text{as } x\to\infty \tag{5.2.1}$$

for some $\varepsilon>0$. Define

$$\check{\phi}(r):=\operatorname{ch}\pi r\int_0^\infty K_{2ir}(u)\phi(u)\frac{du}{u}. \tag{5.2.2}$$

We state the analogue of the main result of [P2] in this case.

Theorem 5.2.1. *Suppose that ϕ satisfies the conditions* (5.2.1). *As in* (2.3.23) *let $\rho_j(n)$ denote the coefficients of an orthornomal basis $\{u_j\}$ for $\mathcal{S}_{1/2}(1,\chi)$ with spectral parameters r_j. If $m>0$ and $n<0$ then*

$$\sum_{c>0}\frac{S(m,n,c,\chi)}{c}\phi\left(\frac{4\pi\sqrt{\widetilde{m}|\widetilde{n}|}}{c}\right)=8\sqrt{i}\sqrt{\widetilde{m}|\widetilde{n}|}\sum_{r_j}\frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch}\pi r_j}\check{\phi}(r_j).$$

The proof involves evaluating the inner product of Poincaré series in two ways. Let $\tau=x+iy\in\mathbb{H}$ and $s=\sigma+it\in\mathbb{C}$. Let $k\in\mathbb{R}$ and let ν be a multiplier system for Γ in weight k. For $m>0$ the Poincaré series $\mathcal{U}_m(\tau,s,k,\nu)$ is defined by

$$\mathcal{U}_m(\tau,s,k,\nu):=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\overline{\nu(\gamma)}\,j(\gamma,\tau)^{-k}\operatorname{Im}(\gamma\tau)^s e(m_\nu\gamma\tau), \qquad \sigma>1. \tag{5.2.3}$$

This function satisfies the transformation law

$$\mathcal{U}_m(\cdot,s,k,\nu)\big|_k\gamma = \nu(\gamma)\,\mathcal{U}_m(\cdot,s,k,\nu) \qquad \text{for all } \gamma\in\Gamma$$

and has Fourier expansion (see equation (15) of [P2])

$$\mathcal{U}_m(\tau,s,k,\nu) = y^s e(m_\nu\tau) + y^s \sum_{\ell\in\mathbb{Z}}\sum_{c>0} \frac{S(m,\ell,c,\nu)}{c^{2s}} B(c,m_\nu,\ell_\nu,y,s,k)e(\ell_\nu x), \tag{5.2.4}$$

where

$$B(c,m_\nu,\ell_\nu,y,s,k) = y\int_{-\infty}^{\infty} e\left(\frac{-m_\nu}{c^2y(u+i)}\right)\left(\frac{u+i}{|u+i|}\right)^{-k} e(-\ell_\nu yu)\frac{du}{y^{2s}(u^2+1)^s}. \tag{5.2.5}$$

In the next two lemmas we will obtain expressions for the inner product

$$\left\langle \mathcal{U}_m\left(\cdot,s_1,\tfrac12,\chi\right),\ \overline{\mathcal{U}_{1-n}\left(\cdot,s_2,-\tfrac12,\overline{\chi}\right)}\right\rangle.$$

The first lemma is stated in such a way that symmetry may be exploited in its proof.

Lemma 5.2.2. *Suppose that (k,ν) is one of the pairs $(1/2,\chi)$ or $(-1/2,\overline{\chi})$. Suppose that $m>0$, $n<0$. Then for $\operatorname{Re} s_1>1$, $\operatorname{Re} s_2>1$ we have*

$$\begin{aligned}\left\langle \mathcal{U}_m\left(\cdot,s_1,k,\nu\right),\ \overline{\mathcal{U}_{1-n}\left(\cdot,s_2,-k,\overline{\nu}\right)}\right\rangle &= i^{-k}2^{3-s_1-s_2}\pi\left(\frac{m_\nu}{|n_\nu|}\right)^{\frac{s_2-s_1}{2}}\frac{\Gamma(s_1+s_2-1)}{\Gamma(s_1-k/2)\Gamma(s_2+k/2)}\\ &\quad\times\sum_{c>0}\frac{S(m,n,c,\nu)}{c^{s_1+s_2}}K_{s_1-s_2}\left(\frac{4\pi\sqrt{m_\nu|n_\nu|}}{c}\right).\end{aligned} \tag{5.2.6}$$

Proof of Lemma 5.2.2. We will prove (5.2.6) under the assumption $\operatorname{Re} s_2>\operatorname{Re} s_1$. Recall the definitions (2.3.28), (2.1.1), and (2.3.7), and recall that

$$(1-n)_{\overline{\nu}} = -n_\nu. \tag{5.2.7}$$

Replacing γ by $-\gamma^{-1}$ in (2.3.28) and using $\nu(-I)=e^{-\pi ik}$, we find that

$$S(m,n,c,\nu) = e^{\pi ik}\,S(1-n,1-m,c,\overline{\nu}).$$

The case $\operatorname{Re} s_1 > \operatorname{Re} s_2$ then follows since both sides of the putative identity are invariant under the changes $s_1 \leftrightarrow s_2$, $k \leftrightarrow -k$, $\nu \leftrightarrow \overline{\nu}$, $m \leftrightarrow 1-n$ (see [DL, (10.27.3)]). The case $\operatorname{Re} s_1 = \operatorname{Re} s_2$ follows by continuity.

Let $I_{m,n}(s_1, s_2)$ denote the inner product on the left-hand side of (5.2.6). Unfolding using (5.2.4) and (5.2.3), we find that

$$\begin{aligned} I_{m,n}(s_1, s_2) &= \int_{\Gamma_\infty \backslash \mathbb{H}} \mathcal{U}_m(\tau, s_1, k, \nu)(\operatorname{Im} \tau)^{s_2} e(-n_\nu)\, d\mu \\ &= \int_0^\infty y^{s_1+s_2-2} \sum_{c>0} \frac{S(m,n,c,\nu)}{c^{2s_1}} B\left(c, m_\nu, n_\nu, y, s_1, k\right) e^{2\pi n_\nu y}\, dy. \end{aligned} \tag{5.2.8}$$

By [P2, (17)] we have

$$B(c, m_\nu, n_\nu, y, s_1, k) \ll_{s_1} y^{1-2\operatorname{Re} s_1} e^{-\pi |n_\nu| y}.$$

Therefore the integral in (5.2.8) is majorized by the convergent (since $\operatorname{Re} s_2 > \operatorname{Re} s_1$) integral

$$\int_0^\infty y^{\operatorname{Re} s_2 - \operatorname{Re} s_1 - 1} e^{3\pi n_\nu y}\, dy.$$

Interchanging the integral and sum in (5.2.8) and using (5.2.5), we find that

$$\begin{aligned} I_{m,n}(s_1, s_2) = \sum_{c>0} \frac{S(m,n,c,\nu)}{c^{2s_1}} &\int_{-\infty}^\infty \left(\frac{u+i}{|u+i|}\right)^{-k} (u^2+1)^{-s_1} \\ &\times \left(\int_0^\infty y^{s_2-s_1-1} e\left(\frac{-m_\nu}{c^2 y(u+i)} - n_\nu y u\right) e^{2\pi n_\nu y}\, dy\right) du. \end{aligned}$$

Set $w = 2\pi n_\nu (iu-1) y$ and $a = 4\pi\sqrt{m_\nu |n_\nu|}/c$. Then the inner integral equals

$$(2\pi n_\nu (iu-1))^{s_1-s_2} \int_0^{(1-iu)\infty} \exp\left(-w - \frac{a^2}{4w}\right) \frac{dw}{w^{s_1-s_2+1}}.$$

We may shift the path of integration to the positive real axis since

$$\int_T^{(1-iu)T} \exp\left(-w - \frac{a^2}{4w}\right) \frac{dw}{w^{s_1+s_2+1}} \ll T^{-\operatorname{Re}(s_1-s_2)} e^{-T} \qquad \text{as } T \to \infty.$$

Using the integral representation [DL, (10.32.10)] for the K-Bessel function, we find that the inner integral is equal to

$$2\left(\frac{m_\nu}{|n_\nu|}\right)^{\frac{s_2-s_1}{2}}(1-iu)^{s_1-s_2}c^{s_1-s_2}K_{s_1-s_2}\left(\frac{4\pi\sqrt{m_\nu|n_\nu|}}{c}\right).$$

It follows that

$$I_{m,n}(s_1,s_2)=2\left(\frac{m_\nu}{|n_\nu|}\right)^{\frac{s_2-s_1}{2}}\sum_{c>0}\frac{S(m,n,c,\nu)}{c^{s_1+s_2}}K_{s_1-s_2}\left(\frac{4\pi\sqrt{m_\nu|n_\nu|}}{c}\right)$$
$$\times\int_{-\infty}^{\infty}\left(\frac{u+i}{|u+i|}\right)^{-k}(u^2+1)^{-s_1}(1-iu)^{s_1-s_2}\,du.$$

Lemma 5.2.2 follows after using [DL, (5.12.8)] to evaluate the integral. □

Recall the definition

$$\Lambda(s_1,s_2,r)=\Gamma(s_1-\tfrac{1}{2}-ir)\Gamma(s_1-\tfrac{1}{2}+ir)\Gamma(s_2-\tfrac{1}{2}-ir)\Gamma(s_2-\tfrac{1}{2}+ir).\quad(5.2.9)$$

Lemma 5.2.3. *Suppose that* $\operatorname{Re}s_1>1,\operatorname{Re}s_2>1$. *As in* (2.3.23) *let* $\rho_j(n)$ *and* r_j *denote the coefficients and spectral parameters of an orthornomal basis* $\{u_j\}$ *for* $\mathcal{S}_{1/2}(1,\chi)$. *If* $m>0$ *and* $n<0$, *then*

$$\left\langle\mathcal{U}_m\left(\cdot,s_1,\tfrac{1}{2},\chi\right),\overline{\mathcal{U}_{1-n}\left(\cdot,s_2,-\tfrac{1}{2},\overline{\chi}\right)}\right\rangle$$
$$=\frac{(4\pi m_\chi)^{1-s_1}(4\pi|n_\chi|)^{1-s_2}}{\Gamma(s_1-1/4)\Gamma(s_2+1/4)}\sum_{r_j}\overline{\rho_j(m)}\rho_j(n)\Lambda(s_1,s_2,r_j).\quad(5.2.10)$$

Proof of Lemma 5.2.3. In this situation there are no Eisenstein series. So for any $f_1,f_2\in\mathcal{L}_{\frac{1}{2}}(1,\chi)$ we have the Parseval identity [P2, (27)]

$$\langle f_1,f_2\rangle=\sum_{r_j}\langle f_1,u_j\rangle\overline{\langle f_2,u_j\rangle}.\quad(5.2.11)$$

By [P2, (32)] we have

$$\left\langle\mathcal{U}_m\left(z,s_1,\tfrac{1}{2},\chi\right),u_j\right\rangle=\overline{\rho_j(m)}(4\pi m_\chi)^{1-s_1}\frac{\Gamma(s_1-\frac{1}{2}-ir_j)\Gamma(s_1-\frac{1}{2}+ir_j)}{\Gamma(s_1-\frac{1}{4})}.\quad(5.2.12)$$

Unfolding as in the last lemma using the definition (5.2.3) for $\mathcal{U}_{1-n}$ and the

Fourier expansion (2.3.23) for u_j gives

$$\overline{\left\langle \mathcal{U}_{1-n}\left(\cdot, s_2, -\tfrac{1}{2}, \overline{\chi}\right), u_j \right\rangle} = \left\langle u_j, \overline{\mathcal{U}_{1-n}\left(\cdot, s_2, -\tfrac{1}{2}, \overline{\chi}\right)} \right\rangle$$
$$= \rho_j(n) \int_0^\infty y^{s_2-2} W_{-\frac{1}{4}, ir_j}(4\pi |n_\chi| y) e^{2\pi n_\chi y}\, dy.$$

Using [DL, (13.23.4) and (16.2.5)] we find that the latter expression equals

$$\rho_j(n)(4\pi|n_\chi|)^{1-s_2} \frac{\Gamma(s_2 - \frac{1}{2} - ir_j)\Gamma(s_2 - \frac{1}{2} + ir_j)}{\Gamma(s_2 + \frac{1}{4})}.$$

The lemma follows from this together with (5.2.11) and (5.2.12). □

We are now ready to prove Theorem 5.2.1. Recall the notation $\widetilde{n} = n_\chi$.

Proof of Theorem 5.2.1. Equating the right-hand sides of Lemmas 5.2.2 (for $k = 1/2$) and 5.2.3 and setting

$$s_1 = \sigma + \frac{it}{2}, \qquad s_2 = \sigma - \frac{it}{2}$$

we obtain (for $\sigma > 1$)

$$i^{-\frac{1}{2}} 2^{3-2\sigma} \pi \Gamma(2\sigma - 1) \sum_{c>0} \frac{S(m,n,c,\chi)}{c^{2\sigma}} K_{it}\left(\frac{4\pi\sqrt{\widetilde{m}|\widetilde{n}|}}{c}\right)$$
$$= (4\pi\widetilde{m})^{1-\sigma}(4\pi|\widetilde{n}|)^{1-\sigma} \sum_{r_j} \overline{\rho_j(m)}\rho_j(n) \Lambda(\sigma + \tfrac{it}{2}, \sigma - \tfrac{it}{2}, r_j). \quad (5.2.13)$$

We justify the substitution of $\sigma = 1$ in (5.2.13). By [DL, (10.45.7)] we have

$$K_{it}(x) \ll (t \operatorname{sh} \pi t)^{-1/2} \quad \text{as } x \to 0.$$

Using (2.3.30) we see that the left side of (5.2.13) converges absolutely uniformly for $\sigma \in [1, 2]$. For the right-hand side we use the inequality $|\rho_j(m)\rho_j(n)| \leq |\rho_j(m)|^2 + |\rho_j(n)|^2$. The argument which follows Lemma 5.1.2 shows that the right side converges absolutely uniformly for $\sigma \in [1, 2]$.

Using (5.1.5), we find that

$$i^{-\frac{1}{2}} \sum_{c>0} \frac{S(m,n,c,\chi)}{c^2} K_{it}\left(\frac{4\pi\sqrt{\widetilde{m}|\widetilde{n}|}}{c}\right) = \frac{\pi}{2} \sum_{r_j} \frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch}\pi(\frac{t}{2} - r_j)\operatorname{ch}\pi(\frac{t}{2} + r_j)}. \quad (5.2.14)$$

Letting ϕ be a function satisfying the conditions (5.2.1), multiply both sides of (5.2.14) by

$$\frac{2}{\pi^2}\, t \operatorname{sh} \pi t \int_0^\infty K_{it}(u)\phi(u)\frac{du}{u^2}$$

and integrate from 0 to ∞. We apply the Kontorovich-Lebedev transform ([P1, (35)] or [DL, (10.43.30-31)])

$$\frac{2}{\pi^2}\int_0^\infty K_{it}(x) t \operatorname{sh} \pi t \left(\int_0^\infty K_{it}(u)\phi(u)\frac{du}{u^2}\right) dt = \frac{\phi(x)}{x}$$

to the left-hand side of (5.2.14) and the transform [P1, (39)]

$$\int_0^\infty \frac{t \operatorname{sh} \pi t}{\operatorname{ch} \pi(\frac{t}{2}+r) \operatorname{ch} \pi(\frac{t}{2}-r)} \left(\int_0^\infty K_{it}(u)\phi(u)\frac{du}{u^2}\right) dt = \frac{2}{\operatorname{ch} \pi r}\check{\phi}(r)$$

(recalling the definition (5.2.2)) to the right-hand side. Then (5.2.14) becomes

$$\frac{i^{-1/2}}{4\pi\sqrt{\widetilde{m}|\widetilde{n}|}} \sum_{c>0} \frac{S(m,n,c,\chi)}{c}\phi\left(\frac{4\pi\sqrt{\widetilde{m}|\widetilde{n}|}}{c}\right) = \frac{2}{\pi}\sum_{r_j} \frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch} \pi r_j}\check{\phi}(r_j),$$

and Theorem 5.2.1 follows. □

5.3 A theta lift for Maass cusp forms

In this section we construct a version of the Shimura correspondence as in [Sa, §3] and [KS, §4] for Maass forms of weight 1/2 on $\Gamma_0(N)$ with the eta multiplier twisted by a Dirichlet character.

Throughout this section, $N \equiv 1 \bmod 24$ is a positive integer and ψ is an even Dirichlet character modulo N. When working with the Shimura correspondence and the Hecke operators, it is convenient to write the Fourier expansion of $G \in \mathcal{S}_{\frac{1}{2}}(N, \psi\chi, r)$ as

$$G(\tau) = \sum_{n\equiv 1(24)} a(n) W_{\frac{\operatorname{sgn}(n)}{4}, ir}\left(\frac{\pi|n|y}{6}\right) e\left(\frac{nx}{24}\right). \tag{5.3.1}$$

When $G = u_j$ as in (2.3.23) we have the relation

$$\rho_j\left(\frac{n+23}{24}\right) = a(n). \tag{5.3.2}$$

For each $t \equiv 1 \bmod 24$, the following theorem gives a map

$$S_t : \mathcal{S}_{\frac{1}{2}}(N, \psi\chi, r) \to \mathcal{S}_0(6N, \psi^2, 2r).$$

We will only apply this theorem in the case $N = 1$. The proof in the case $N = t = 1$ is simpler than the general proof given below; we must work in this generality since the proof in the case $t > 1$ requires the map S_1 on $\Gamma_0(t)$ forms.

Theorem 5.3.1. *Let* $N \equiv 1 \bmod 24$ *be squarefree. Suppose that* $G \in \mathcal{S}_{\frac{1}{2}}(N, \psi\chi, r)$ *with* $r \neq i/4$ *has Fourier expansion* (5.3.1). *Let* $t \equiv 1 \bmod 24$ *be a squarefree positive integer and define coefficients* $b_t(n)$ *by the relation*

$$\sum_{n=1}^{\infty} \frac{b_t(n)}{n^s} = L\left(s+1, \psi\left(\tfrac{t}{\bullet}\right)\right) \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) \frac{a(tn^2)}{n^{s-\frac{1}{2}}}, \tag{5.3.3}$$

where $L(s, \psi)$ *is the usual Dirichlet L-function. Then the function* $S_t(G)$ *defined by*

$$(S_t G)(\tau) := \sum_{n=1}^{\infty} b_t(n) W_{0,2ir}(4\pi n y) \cos(2\pi n x)$$

is an even Maass cusp form in $\mathcal{S}_0(6N, \psi^2, 2r)$.

An important property of the Shimura correspondence is Hecke equivariance. Since we only need this fact for $N = 1$, we state it only in that case.

Corollary 5.3.2. *Suppose that* $G \in \mathcal{S}_{\frac{1}{2}}(1, \chi, r)$ *with* $r \neq i/4$ *and that* $t \equiv 1 \bmod 24$ *is a squarefree positive integer. Then for any prime* $p \geq 5$ *we have*

$$T_p \, S_t(G) = \left(\frac{12}{p}\right) S_t(T_{p^2} G).$$

Proof. We prove this in the case $p \nmid t$; the other case is similar, and easier. Let G have coefficients $a(n)$ as in (5.3.1). The coefficients $b_t(n)$ of $S_t(G)$ are given by

$$b_t(n) = \sum_{jk=n} \left(\frac{t}{j}\right) \left(\frac{12}{k}\right) \frac{\sqrt{k}}{j} a(tk^2),$$

and the coefficients $A(n)$ of $T_{p^2} G$ are given by

$$A(n) = p\, a(p^2 n) + p^{-\frac{1}{2}} \left(\frac{12n}{p}\right) a(n) + p^{-1} a(n/p^2).$$

We must show that

$$p^{\frac{1}{2}} b_t(pn) + p^{-\frac{1}{2}} b_t(n/p) = \left(\frac{12}{p}\right) \sum_{jk=n} \left(\frac{t}{j}\right) \left(\frac{12}{k}\right) \frac{\sqrt{k}}{j} A(tk^2). \tag{5.3.4}$$

Writing $n = p^\alpha n'$ with $p \nmid n'$, the left-hand side of (5.3.4) equals

$$p^{-\frac{1}{2}} \sum_{\ell=0}^{\alpha+1} S_\ell + p^{\frac{1}{2}} \sum_{\ell=0}^{\alpha-1} S_\ell,$$

where

$$S_\ell = p^{-\alpha} \left(\frac{t}{p}\right)^{\alpha+1-\ell} \left(\frac{12}{p}\right)^\ell p^{\frac{3\ell}{2}} \sum_{jk=n'} \left(\frac{t}{j}\right) \left(\frac{12}{k}\right) \frac{\sqrt{k}}{j} a(tp^{2\ell}k^2).$$

A computation shows that the right-hand side of (5.3.4) equals

$$p^{-\frac{1}{2}} \sum_{\ell=0}^{\alpha} S_{\ell+1} + p^{-\frac{1}{2}} S_0 + p^{\frac{1}{2}} \sum_{\ell=1}^{\alpha} S_{\ell-1},$$

and the corollary follows. □

Using Theorem 5.3.1 we can rule out the existence of exceptional eigenvalues in $\mathcal{S}_{1/2}(1, \chi)$ and obtain a lower bound on the second smallest eigenvalue $\lambda_1 = \frac{1}{4} + r_1^2$ (we note that Bruggeman [Br1, Theorem 2.15] obtained $\lambda_1 > \frac{1}{4}$ using different methods). Theorem 5.3.1 shows that $2r_1$ is bounded below by the smallest spectral parameter for $\mathcal{S}_0(6, \mathbf{1})$. Huxley [H1, H2] studied the problem of exceptional eigenvalues in weight 0 for subgroups of $\mathrm{SL}_2(\mathbb{Z})$ whose fundamental domains are "hedgehog" shaped. On page 247 of [H1] we find a lower bound which, for $N = 6$, gives $2r_1 > 0.4$. Computations of Strömberg [LMF] suggest (since $2r_1$ corresponds to an even Maass cusp form) that $2r_1 \approx 3.84467$. Using work of Booker and Strömbergsson [BS] on Selberg's eigenvalue conjecture we can prove the following lower bound.

Corollary 5.3.3. *Let $\lambda_1 = \frac{1}{4} + r_1^2$ be as above. Then $r_1 > 1.9$.*

Proof. Let $\{\widetilde{r}_j\}$ denote the set of spectral parameters corresponding to even forms in $\mathcal{S}_0(6, \mathbf{1})$. We show that each $\widetilde{r}_j > 3.8$ using the method described in [BS, Section 4] in the case of level 6 and trivial character. We are thankful to the authors of that paper for providing the numerical details of this computation. Given a suitable test function h such that $h(t) \geq 0$ on $\mathbb{R}$ and $h(t) \geq 1$ on $[-3.8, 3.8]$, we compute via an explicit version of the Selberg trace formula [BS, Theorem 4] that

$$\sum_{\widetilde{r}_j} h(\widetilde{r}_j) < 1, \tag{5.3.5}$$

so we cannot have $\widetilde{r}_j \leq 3.8$ for any j. Let

$$f(t) = \left(\frac{\sin t/6}{t/6}\right)^2 \sum_{n=0}^{2} x_n \cos\left(\frac{nt}{3}\right),$$

where

$$x_0 = 1.167099688946872,$$
$$x_1 = 1.735437017086616,$$
$$x_2 = 0.660025094420283.$$

With $h = f^2$, we compute that the sum in (5.3.5) is approximately 0.976. $\square$

The proof of Theorem 5.3.1 occupies the remainder of this section. We first modify the theta functions introduced by Shintani [Sh] and Niwa [N]. Next, we construct the Shimura lift for $t = 1$ and derive the relation (5.3.3) in that case. The function $S_t(G)$ is obtained by applying the lift S_1 to the form $G(t\tau)$. We conclude the section by showing that $S_t(G)$ has the desired level.

5.3.1 The theta functions of Shintani and Niwa

In this subsection we adopt the notation of [N] for easier comparison with that paper. Let $Q = \frac{1}{12N}\begin{pmatrix} & & -2 \\ & 1 & \\ -2 & & \end{pmatrix}$, and for $x, y \in \mathbb{R}^3$ define

$$\langle x, y\rangle = x^T Q y = \frac{1}{12N}(x_2 y_2 - 2x_1 y_3 - 2x_3 y_1).$$

The signature of Q is $(2, 1)$. Let $L \subset L' \subset L^*$ denote the lattices

$$\begin{aligned} L &= N\mathbb{Z} \oplus 12N\mathbb{Z} \oplus 6N\mathbb{Z}, \\ L' &= \mathbb{Z} \oplus N\mathbb{Z} \oplus 6N\mathbb{Z}, \\ L^* &= \mathbb{Z} \oplus \mathbb{Z} \oplus 6\mathbb{Z}. \end{aligned}$$

Then L^* is the dual lattice of L, and for $x, y \in L$ we have $\langle x, y\rangle \in \mathbb{Z}$ and $\langle x, x\rangle \in 2\mathbb{Z}$. Let $f : \mathbb{R}^3 \to \mathbb{C}$ be a Schwarz function satisfying the conditions of [N, Corollary 0] for $\kappa = 1$. If ψ is a character mod N and $h = (h_1, Nh_2, 0) \in L'/L$, define

$$\psi_1(h) := \psi(h_1)\left(\frac{12}{h_2}\right).$$

For $z = u + iv$ we define

$$\theta(z, f, h) := v^{-\frac{1}{4}} \sum_{x\in L} [r_0(\sigma_z)f](x + h)$$

and

$$\theta(z, f) := \sum_{h\in L'/L} \overline{\psi}_1(h)\theta(z, f, h), \tag{5.3.6}$$

where $\sigma \mapsto r_0(\sigma)$ is the Weil representation (see [N, p. 149]) and

$$\sigma_z := \begin{pmatrix} v^{\frac{1}{2}} & uv^{-\frac{1}{2}} \\ 0 & v^{-\frac{1}{2}} \end{pmatrix}, \quad z = u + iv \in \mathbb{H}.$$

The following lemma gives the transformation law for $\theta(z, f)$.

Lemma 5.3.4. *Suppose that* $N \equiv 1 \pmod{24}$ *and that* $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. *With* $\theta(z, f)$ *defined as in* (5.3.6) *we have*

$$\theta(\sigma z, f) = \overline{\psi}(d)\left(\frac{N}{d}\right)\chi(\sigma)(cz + d)^{\frac{1}{2}}\theta(z, f), \tag{5.3.7}$$

where χ *is the eta multiplier* (2.1.2).

In the proof of Lemma 5.3.4 we will encounter the quadratic Gauss sum

$$G(a, b, c) = \sum_{x \bmod c} e\left(\frac{ax^2 + bx}{c}\right).$$

With $g = (a, c)$, we have

$$G(a, b, c) = \begin{cases} g\, G(a/g, b/g, c/g) & \text{if } g \mid b, \\ 0 & \text{otherwise.} \end{cases} \tag{5.3.8}$$

We require the following identity relating the Dedekind sums $s(d, c)$ to these

Gauss sums.

Lemma 5.3.5. *Suppose that* $(c,d)=1$ *and that* $k\in\mathbb{Z}$. *Let* $\overline{d}$ *satisfy* $d\overline{d}\equiv 1 \pmod{c}$. *Then*

$$\sqrt{12c}\,\left(\frac{12}{k}\right) e\left(\frac{\overline{d}(k^2-1)}{24c}\right) e^{\pi i s(d,c)} = \sum_{h \bmod 12} \left(\frac{12}{h}\right) e\left(\frac{-d(\frac{h^2-1}{24})}{c} - \frac{hk}{12c}\right) G(-6d, dh+k, c). \quad (5.3.9)$$

Proof. First suppose that $(k,6)=1$. If c is even and $d\overline{d}\equiv 1 \pmod{2c}$, or if c is odd, then by Lemma 4.3.1 and (4.3.1) we have

$$\sqrt{12c}\,\left(\frac{12}{k}\right) e\left(\frac{\overline{d}(k^2-1)}{24c}\right) e^{\pi i s(d,c)} = e\left(\frac{k}{12c}\right) \sum_{j \bmod 2c} e\left(\frac{-3dj^2-dj+j(c+k)}{2c}\right) + e\left(\frac{-k}{12c}\right) \sum_{j \bmod 2c} e\left(\frac{-3dj^2-dj+j(c-k)}{2c}\right). \quad (5.3.10)$$

If c is even and $d\overline{d}\not\equiv 1 \pmod{2c}$ then, applying (5.3.10) with $\overline{d}$ replaced by $\overline{d}+c$, we see that (5.3.10) is true in this case as well. Splitting each of the sums in (5.3.10) into two sums by writing $j=2x$ or $j=2x+1$ shows that (5.3.10) equals the right-hand side of (5.3.9).

It remains to show that the right-hand side of (5.3.9) is zero when $\delta := (k,6)>1$. If $(\delta,c)>1$ then (5.3.8) shows that $\left(\frac{12}{h}\right) G(-6d, dh+k, c)=0$. If $(\delta,c)=1$, write

$$A(h) = e\left(\frac{-d(\frac{h^2-1}{24})}{c} - \frac{hk}{12c}\right) G(-6d, dh+k, c).$$

If $\delta=2$ or 6 then a computation shows that, replacing x by $x-\overline{2}$ in $G(-6d,dh+k,c)$, we have $A(h)=A(h-6)$ for $(h,6)=1$. Similarly, if $\delta=3$ and c is odd then, replacing x by $x+\overline{6}h$, we find that $A(h)=-A(-h)$. Finally, if $\delta=3$ and c is even then, replacing x by $x-\overline{3}$, we find that $A(1)=A(5)$ and $A(7)=A(11)$. In each case we find that

$$\sum_{h \bmod 12} \left(\frac{12}{h}\right) A(h) = 0.$$

□

Proof of Lemma 5.3.4. Since $\sigma_{z+1} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \sigma_z$, Proposition 0 of [N] gives

$$\theta(z+1, f) = \sum_{h \in L'/L} e\Big(\frac{Nh_2^2}{24}\Big)\overline{\psi}_1(h)\theta(z, f, h) = e\left(\frac{1}{24}\right)\theta(z, f).$$

For $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ with $c > 0$ we have, by Corollary 0 of [N],

$$\theta(\sigma z, f) = (cz+d)^{\frac{1}{2}} \sum_{k \in L^*/L} \theta(z, f, k) \cdot \sqrt{-i} \sum_{h \in L'/L} \overline{\psi}_1(h) c(h, k)_\sigma,$$

where

$$c(h, k)_\sigma = \frac{1}{(Nc)^{\frac{3}{2}}\sqrt{12}}\, e\left(\frac{d\langle k, k\rangle}{2c}\right) \sum_{r \in L/cL} e\left(\frac{a\langle h+r, h+r\rangle}{2c} - \frac{\langle k, h+r\rangle}{c}\right). \tag{5.3.11}$$

For $h \in L'/L$, $k \in L^*/L$, and $r \in L/cL$, we write $h = (h_1, Nh_2, 0)$, $k = (k_1, k_2, 6k_3)$, and $r = (Nr_1, 12Nr_2, 6Nr_3)$. Then the sum on r in (5.3.11) equals

$$e\left(\frac{aNh_2^2 - 2h_2k_2}{24c}\right) G(6aN, aNh_2 - k_2, c)\, S(c), \tag{5.3.12}$$

where (recalling that $N \mid c$),

$$\begin{aligned} S(c) &= e\left(\frac{h_1k_3}{Nc}\right) \sum_{r_1, r_3 (c)} e\left(\frac{-aNr_1r_3 - ah_1r_3 + r_1k_3 + r_3k_1}{c}\right) \\ &= \begin{cases} Nc\, e\left(\dfrac{dk_1k_3'}{c}\right) & \text{if } k_3 = Nk_3' \text{ and } k_1 \equiv ah_1 \pmod{N}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{5.3.13}$$

So by (5.3.12), (5.3.13), and (5.3.8) we have

$$c(h, k)_\sigma = 0 \quad \text{ unless } \quad k = (k_1, Nk_2', 6Nk_3') \in L',$$

in which case $k_1 \equiv ah_1 \pmod{N}$. For such k we have

$$\begin{aligned} &\sum_{h \in L'/L} \overline{\psi}_1(h) c(h, k)_\sigma \\ &= \frac{\overline{\psi}(dk_1)}{\sqrt{12c'}} e\left(\frac{d(k_2')^2}{24c'}\right) \sum_{h_2 \bmod 12} \left(\frac{12}{h_2}\right) e\left(\frac{ah_2^2 - 2h_2k_2'}{24c'}\right) G(6a, ah_2 - k_2', c'). \end{aligned}$$

Applying Lemma 5.3.5 we have

$$\sqrt{-i}\sum_{h\in L'/L}\overline{\psi}_1(h)c(h,k)_\sigma=\overline{\psi}(d)\overline{\psi}_1(k)\sqrt{-i}\,e\left(\frac{a+d}{24c'}\right)e^{\pi is(-a,c')}.$$

By (68.4) and (68.5) of [R4] we have $s(-a,c')=-s(a,c')=-s(d,c')$. Therefore

$$\theta(\sigma z,f)=\overline{\psi}(d)\chi\left(\begin{pmatrix}a & Nb\\ c' & d\end{pmatrix}\right)(cz+d)^{\frac{1}{2}}\theta(z,f).$$

By (2.1.5) and the assumption $N\equiv 1 \pmod{24}$, we have $\chi\big(\left(\begin{smallmatrix}a & Nb\\ c' & d\end{smallmatrix}\right)\big)=\left(\frac{N}{d}\right)\chi(\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right))$, from which the lemma follows. □

For $w=\xi+i\eta\in\mathbb{H}$ and $0\le\theta<2\pi$, let

$$g=g(w,\theta)=\begin{pmatrix}1 & \xi\\ 0 & 1\end{pmatrix}\begin{pmatrix}\eta^{\frac{1}{2}} & 0\\ 0 & \eta^{-\frac{1}{2}}\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\in\mathrm{SL}_2(\mathbb{R}).$$

The action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{R}^3$ is given by $g.x=x'$, where

$$\begin{pmatrix}x_1' & x_2'/2\\ x_2'/2 & x_3'\end{pmatrix}=g\begin{pmatrix}x_1 & x_2/2\\ x_2/2 & x_3\end{pmatrix}g^T. \tag{5.3.14}$$

Since $\langle x,x\rangle=-\frac{1}{3N}\det\left(\begin{smallmatrix}x_1 & x_2/2\\ x_2/2 & x_3\end{smallmatrix}\right)$ we have $\langle gx,gx\rangle=\langle x,x\rangle$. The action of $\mathrm{SL}_2(\mathbb{R})$ on functions $f:\mathbb{R}^3\to\mathbb{C}$ is given by $gf(x):=f(g^{-1}x)$.

We specialize to $f=f_3$, with

$$f_3(x)=\exp\left(-\frac{\pi}{12N}\left(2x_1^2+x_2^2+2x_3^2\right)\right)$$

as in [N, Example 3], and we consider the function $\theta(z,gf_3)$. Let $k(\theta)=\left(\begin{smallmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{smallmatrix}\right)$. Since $f_3(k(\theta)x)=f_3(x)$, the function $\theta(z,gf_3)$ is independent of the variable θ. Therefore it makes sense to define

$$\vartheta(z,w):=\theta(z,g(w,0)f_3)=v^{\frac{1}{2}}\sum_{x\in L'}\overline{\psi}_1(x)e\left(\frac{u}{2}\langle x,x\rangle\right)f_3\left(\sqrt{v}\,\sigma_w^{-1}x\right), \tag{5.3.15}$$

where the second equality follows from

$$[r_0(\sigma_z)f](x)=v^{\frac{3}{4}}e\left(\tfrac{u}{2}\langle x,x\rangle\right)f(\sqrt{v}x).$$

To determine the transformation of $\vartheta(z,w)$ in the variable w, we use the relation

$$\sigma_{\gamma w} = \gamma\,\sigma_w\, k\big(\arg(cw+d)\big), \qquad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R}).$$

Suppose that $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(6N)$. If $x' = \gamma x$, then by (5.3.14) we have

$$x_1' = a^2\, x_1 + ab\, x_2 + b^2\, x_3, \tag{5.3.16}$$

$$x_2' = 2ac\, x_1 + (1+2bc)\, x_2 + 2bd\, x_3, \tag{5.3.17}$$

$$x_3' = c^2\, x_1 + cd\, x_2 + d^2\, x_3. \tag{5.3.18}$$

It follows that $\gamma L' = L'$ and that $\psi_1(x') = \psi^2(a)\psi_1(x)$. Thus

$$\vartheta(z,\gamma w) = \psi^2(d)\vartheta(z,w) \qquad \text{for all } \gamma \in \Gamma_0(6N). \tag{5.3.19}$$

A computation shows that replacing $w = \xi + i\eta$ by $w' = -\xi + i\eta$ in $f_3(\sqrt{v}\sigma_w^{-1}x)$ has the same effect as replacing x_2 by $-x_2$, so

$$\vartheta(z, -\xi + i\eta) = \vartheta(z, \xi + i\eta). \tag{5.3.20}$$

5.3.2 Proof of Theorem 5.3.1 in the case $t = 1$

Suppose that $F \in \mathcal{S}_{\frac{1}{2}}(N, \overline{\psi}\left(\frac{N}{\bullet}\right)\chi, r)$ with $r \neq i/4$. For $z = u + iv$ and $w = \xi + i\eta$ we define

$$\Psi_F(w) := \int_{\mathcal{D}} v^{\frac{1}{4}} \overline{\vartheta(z,w)} F(z) d\mu, \tag{5.3.21}$$

where $\mathcal{D} = \Gamma_0(N)\backslash\mathbb{H}$ and $d\mu = \frac{du\,dv}{v^2}$. Note that the integral is well-defined by Lemma 5.3.4. There is no issue with convergence since ϑ and F are both rapidly decreasing at the cusps.

Let $D^{(w)}$ denote the Casimir operator for $\mathrm{SL}_2(\mathbb{R})$ defined by

$$D^{(w)} = \eta^2 \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) - \eta \frac{\partial^2}{\partial \xi \partial \theta}.$$

To show that Ψ_F is an eigenform of $\Delta_0^{(w)}$, we use Lemmas 1.4 and 1.5 of [Sh], which give

$$v^{-\frac{1}{2}} r_0(\sigma_z) D^{(w)} f = \Delta_{\frac{1}{2}}^{(z)} \left[v^{-\frac{1}{2}} r_0(\sigma_z) f \right].$$

Since Ψ_F is constant with respect to θ, we have $\Delta_0^{(w)}\Psi_F = D^{(w)}\Psi_F$. By the lemma on p. 304 of [Sa], it follows that

$$\Delta_0^{(w)}\Psi_F + \left(\frac{1}{4} + (2r)^2\right)\Psi_F = 0.$$

This, together with the transformation law (5.3.19) shows that Ψ_F is a Maass form. The following lemma shows that Ψ_F is a cusp form. This is the only point where we use the assumption that N is squarefree. It would be possible to remove this assumption with added complications by arguing as in [C]. For the remainder of this section, to avoid confusion with the eta function, we write $w = \xi + iy$.

Lemma 5.3.6. *Suppose that $N \equiv 1 \pmod{24}$ is squarefree and that $F \in \mathcal{S}_{\frac{1}{2}}(N, \overline{\psi}\left(\frac{N}{\bullet}\right)\chi, r)$. Let $\eta_N(w) := y^{\frac{1}{4}}\eta(Nw)$. For each cusp $\mathfrak{a} = \gamma_{\mathfrak{a}}\infty$ there exists $c_{\mathfrak{a}} \in \mathbb{C}$ such that as $y \to \infty$ we have*

$$(\Psi_F\big|_0\gamma_{\mathfrak{a}})(iy) = \begin{cases} c_{\mathfrak{a}}\,\langle \eta_N, F\rangle\; y + O(1) & \text{if } \psi \text{ is principal,} \\ O(1) & \text{otherwise.}\end{cases} \tag{5.3.22}$$

Proof. For each $d \mid 6N$, let $f \mapsto f\big|_0 W_d$ be the Atkin-Lehner involution [AL, §2] given by any matrix $W_d \in \mathrm{SL}_2(\mathbb{R})$ of the form

$$W_d = \begin{pmatrix} \sqrt{d}\,\alpha & \beta/\sqrt{d} \\ 6N\gamma/\sqrt{d} & \sqrt{d}\,\delta \end{pmatrix}, \qquad \alpha, \beta, \gamma, \delta \in \mathbb{Z}.$$

Since $N \equiv 1 \pmod{24}$ is squarefree, every cusp of $\Gamma_0(6N)$ is of the form $W_d\infty$ for some d. Thus it suffices to establish (5.3.22) for $\gamma_{\mathfrak{a}} = W_d$.

By (5.3.16)–(5.3.18) we have $W_dL' = L'$, so

$$\vartheta_d(z,w) := \vartheta(z, W_d w) = v^{\frac{1}{2}} \sum_{x \in L'} \overline{\psi}_1(W_d x) e\left(\frac{u}{2}\langle x, x\rangle\right) f_3\left(\sqrt{v}\,\sigma_w^{-1} x\right).$$

To determine the asymptotic behavior of $\vartheta_d(z, iy)$ as $y \to \infty$, we follow the method of Cipra [C, §4.3]. We write

$$\vartheta_d(z, iy) = v^{\frac{1}{2}} \sum_{\substack{x \in L' \\ x_3 = 0}} \overline{\psi}_1(W_d x) e\Big(\frac{z}{24N}x_2^2\Big) \exp\Big(-\frac{\pi v}{6Ny^2}x_1^2\Big) + \varepsilon(z, y), \tag{5.3.23}$$

where

$$|\varepsilon(z,y)| \le v^{\frac12} \sum_{x_1\in\mathbb{Z}} \exp\Big(\frac{-\pi v}{6Ny^2}x_1^2\Big) \sum_{x_2\in\mathbb{Z}} \exp\Big(\frac{-\pi v}{12N}x_2^2\Big) \sum_{x_3\neq 0} \exp\Big(\frac{-\pi v y^2}{6N}x_3^2\Big)$$
$$\ll \Big(v+\frac{1}{v}\Big) y\, e^{-c_1 v y^2}$$

for some $c_1 > 0$ (see [C, Appendix B]). As in [C, Proposition B.3], the contribution of $\varepsilon(z,y)$ to $(\Psi_F\big|_0 W_d)(iy)$ is $o(1)$ as $y\to\infty$. For $x_3 = 0$ and $W_d x = (x_1', x_2', x_3')$ we have

$$x_1' \equiv d\alpha^2 x_1 \pmod N,$$
$$x_2' \equiv \Big(1+\frac{12N}{d}\gamma\beta\Big)x_2 \pmod{12}$$

(in particular, $\overline{\psi}_1(W_d x) = 0$ unless $d\in\{1,2,3,6\}$). Thus for some c_2 the main term of (5.3.23) equals

$$c_2\, v^{\frac12} \sum_{x_1\in\mathbb{Z}} \overline{\psi}(x_1) \exp\Big(-\frac{\pi v}{6Ny^2}x_1^2\Big) \sum_{x_2\in N\mathbb{Z}} \Big(\frac{12}{x_2}\Big)\, e\Big(\frac{z}{24N}x_2^2\Big)$$
$$= 2c_2\, \eta(Nz)\, v^{\frac12} \sum_{h \bmod N} \overline{\psi}(h)\theta_1\Big(\frac{iv}{12Ny^2}, h, N\Big),$$

where $\theta_1(\cdot, h, N)$ is as in [C, Theorem 1.10(i)]. By the first two assertions of that theorem we have

$$v^{\frac12} \sum_{h \bmod N} \overline{\psi}(h)\theta_1\Big(\frac{iv}{12Ny^2}, h, N\Big) = \sqrt{\frac{6}{N}}\, y \sum_{h \bmod N} \overline{\psi}(h) + O_N(\sqrt{v}).$$

The latter sum is zero unless ψ is principal. In that case we have

$$(\Psi_F\big|_0 W_d)(iy) = c_3\, y \int_{\mathcal{D}} v^{\frac14}\overline{\eta(Nz)}F(z)\, d\mu + O(1)$$

for some $c_3\in\mathbb{C}$; otherwise Ψ_F is $O(1)$ at the cusps. □

If the spectral parameter of F is not $i/4$, then F is orthogonal to η_N. Since a Maass form that is bounded at the cusps is a cusp form, we obtain the following proposition (recall (5.3.20)).

Proposition 5.3.7. *Suppose that* $F \in \mathcal{S}_{\frac12}(N, \overline{\psi}\left(\frac{N}{\bullet}\right)\chi, r)$*, where* $N \equiv 1 \pmod{24}$ *is squarefree and* $r \neq i/4$*. Then* Ψ_F *is an even Maass cusp form in*

$\mathcal{S}_0(6N, \overline{\psi}^2, 2r)$.

With G as in Theorem 5.3.1 let $F := G\big|_{\frac{1}{2}} w_N$, where

$$w_N = \begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix}$$

is the Fricke involution. To obtain the $t = 1$ case of Theorem 5.3.1 we will compute the Fourier expansion of

$$\Phi_F(w) := \Psi_F\left(-1/6Nw\right). \tag{5.3.24}$$

Once we show that the coefficients of Φ_F satisfy the relation (5.3.3), Proposition 5.3.7 completes the proof. We first show that F defined in this way satisfies the hypotheses of Proposition 5.3.7.

Lemma 5.3.8. *Suppose that* $N \equiv 1 \pmod{24}$. *If* $G \in \mathcal{S}_{\frac{1}{2}}(N, \psi\chi, r)$ *and* $F = G\big|_{\frac{1}{2}} w_N$, *then* $F \in \mathcal{S}_{\frac{1}{2}}\left(N, \overline{\psi}\left(\frac{N}{\bullet}\right)\chi, r\right)$.

Proof. Since the slash operator commutes with Δ_k, it suffices to show that F transforms with mutliplier $\overline{\psi}(\frac{N}{\bullet})\chi$. Writing $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and $\gamma' = w_N \gamma w_N^{-1} = \begin{pmatrix} d & -c/N \\ -bN & a \end{pmatrix}$ we have

$$F|_{\frac{1}{2}}\gamma = \overline{\psi}(d)\chi(\gamma')\, F.$$

Thus it suffices to show that $\chi(\gamma') = \left(\frac{N}{d}\right)\chi(\gamma)$. For this we argue as in [S, Proposition 1.4].

Write $w_N = SV$, where

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} \sqrt{N} & 0 \\ 0 & 1/\sqrt{N} \end{pmatrix}.$$

Let $\widetilde{\eta}(w) := y^{1/4}\eta(w)$ and let $g := \widetilde{\eta}\big|_{\frac{1}{2}} V$. Then by (2.1.5), g transforms on $\Gamma_0(N)$ with multiplier $\left(\frac{N}{\bullet}\right)\chi$. We compute

$$\begin{aligned} \chi(w_N\gamma w_N^{-1})\,\widetilde{\eta} = \widetilde{\eta}\big|_{\frac{1}{2}} w_N\gamma w_N^{-1} &= \sqrt{-i}\, g\big|_{\frac{1}{2}}\gamma w_N^{-1} \\ &= \sqrt{-i}\,\left(\frac{N}{d}\right)\chi(\gamma) g\big|_{\frac{1}{2}} w_N^{-1} = \left(\frac{N}{d}\right)\chi(\gamma)\,\widetilde{\eta}. \end{aligned}$$

This completes the proof. □

We will determine the Fourier coefficients of Φ_F by computing its Mellin transform $\Omega(s)$ in two ways. We have

$$\Omega(s) := \int_0^\infty \Phi_F(iy)y^s \frac{dy}{y} = (6N)^{-s} \int_0^\infty \Psi_F\left(i/y\right) y^s \frac{dy}{y}. \qquad (5.3.25)$$

For w purely imaginary, the sum in (5.3.15) simplifies to

$$\vartheta(z, iy) = \vartheta_1(z)\vartheta_2(z, y),$$

where $\vartheta_1(z) = 2\eta(Nz)$ and

$$\begin{aligned}\vartheta_2(z, y) &= v^{\frac{1}{2}} \sum_{m,n\in\mathbb{Z}} \overline{\psi}(m) e(-umn) \exp\left(-\pi v \left(\frac{m^2}{6Ny^2} + 6Ny^2n^2\right)\right) \\ &= \frac{1}{y\sqrt{6N}} \sum_{m,n\in\mathbb{Z}} \overline{\psi}(m) \exp\left(-\frac{\pi}{6Ny^2v}|mz+n|^2\right).\end{aligned}$$

The latter equality follows from Poisson summation on n.

Setting $A := \frac{\pi}{6Nv}|mz+n|^2$ for the moment, we find that

$$\Omega(s) = 2(6N)^{-s-\frac{1}{2}} \sum_{m,n} \psi(m) \int_{\mathcal{D}} v^{\frac{1}{4}}\overline{\eta(Nz)}F(z) \int_0^\infty y^s e^{-Ay^2}\, dy\, d\mu.$$

For $\mathrm{Re}(s) > 1$ the inner integral evaluates to $\frac{1}{2}A^{\frac{-s-1}{2}}\Gamma\left(\frac{s+1}{2}\right)$, so we have

$$\Omega(s) = \frac{\Gamma\left(\frac{s+1}{2}\right)}{(6N)^{\frac{s}{2}}\pi^{\frac{s+1}{2}}} \sum_{m,n} \psi(m) \int_{\mathcal{D}} v^{\frac{1}{4}}\overline{\eta(Nz)}F(z) \left(\frac{v}{|mz+n|^2}\right)^{\frac{s+1}{2}} d\mu.$$

Replacing z by $w_N z$, recalling that $G = F|_{\frac{1}{2}} w_N$ and using the fact that ψ is even, we obtain

$$\Omega(s) = \frac{\sqrt{i}N^{\frac{1}{4}}}{6^{\frac{s}{2}}\pi^{\frac{s+1}{2}}}\Gamma\left(\frac{s+1}{2}\right) \sum_{m,n} \psi(m) \int_{\mathcal{D}} v^{\frac{1}{4}}\overline{\eta(z)}G(z) \left(\frac{v}{|Nnz+m|^2}\right)^{\frac{s+1}{2}} d\mu.$$

For $(m, N) = 1$, write $m = gd$ and $Nn = gc$ with $g = (m, n) = (m, Nn)$ so that

$$\sum_{m,n} \psi(m) \left(\frac{v}{|Nnz+m|^2}\right)^{\frac{s+1}{2}} = L(s+1, \psi) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \psi(\gamma)\, \mathrm{Im}(\gamma z)^{\frac{s+1}{2}},$$

where $\psi(\gamma)=\psi(d)$ for $\gamma=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Thus

$$\Omega(s)=c(s)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)}\psi(\gamma)\int_{\mathcal{D}} v^{\frac{1}{4}}\overline{\eta(z)}G(z)\,\mathrm{Im}(\gamma z)^{\frac{s+1}{2}}\,d\mu,$$

where

$$c(s)=\frac{\sqrt{i}N^{\frac{1}{4}}}{6^{\frac{s}{2}}\pi^{\frac{s+1}{2}}}\Gamma\left(\frac{s+1}{2}\right)L(s+1,\psi).$$

Since $\psi(\gamma)v^{1/4}\overline{\eta(z)}G(z)=\mathrm{Im}(\gamma z)^{1/4}\overline{\eta(\gamma z)}G(\gamma z)$ for $\gamma\in\Gamma$, we have

$$\Omega(s)=c(s)\int_{\Gamma_\infty\backslash\mathbb{H}} v^{\frac{s+1}{2}+\frac{1}{4}}\overline{\eta(z)}G(z)\,d\mu.$$

Recall that G has the Fourier expansion

$$G(z)=\sum_{n\equiv 1(24)} a(n)W_{\frac{\mathrm{sgn}(n)}{4},ir}\left(\frac{\pi|n|v}{6}\right)e\left(\frac{nu}{24}\right).$$

Thus we have

$$\Omega(s)=c(s)\sum_{n=1}^{\infty}\left(\frac{12}{n}\right)a(n^2)\int_0^\infty v^{\frac{s}{2}-\frac{5}{4}}W_{\frac{1}{4},ir}\left(\frac{\pi n^2 v}{6}\right)e^{-\frac{\pi n^2}{12}v}\,dv.$$

By [DL, (13.23.4) and (16.2.5)], the integral evaluates to

$$\left(\frac{6}{\pi n^2}\right)^{\frac{s}{2}-\frac{1}{4}}\frac{\Gamma\left(\frac{s}{2}+\frac{1}{4}+ir\right)\Gamma\left(\frac{s}{2}+\frac{1}{4}-ir\right)}{\Gamma\left(\frac{s+1}{2}\right)},$$

so we conclude that

$$\Omega(s)=\sqrt{i}\left(\frac{\pi N}{6}\right)^{\frac{1}{4}}\pi^{-s-\frac{1}{2}}\Gamma\left(\frac{s}{2}+\frac{1}{4}+ir\right)\Gamma\left(\frac{s}{2}+\frac{1}{4}-ir\right)$$
$$\times L(s+1,\psi)\sum_{n=1}^{\infty}\left(\frac{12}{n}\right)\frac{a(n^2)}{n^{s-\frac{1}{2}}}.$$

On the other hand, since Ψ_F is an even Maass cusp form, it follows that Φ_F is also even. Since Φ_F has eigenvalue $\frac{1}{4}+(2r)^2$, it has a Fourier expansion of the form

$$\Phi_F(w)=2\sum_{n=1}^{\infty}b(n)W_{0,2ir}(4\pi ny)\cos(2\pi nx).$$

By the definition (5.3.25) of $\Omega(s)$ and [DL, (13.18.9) and (10.43.19)] we have

$$\begin{aligned}\Omega(s) &= 4\sum_{n=1}^{\infty}\sqrt{n}\,b(n)\int_0^{\infty}K_{2ir}(2\pi ny)y^{s-\frac12}\,dy\\ &= \pi^{-s-\frac12}\Gamma\left(\tfrac{s}{2}+\tfrac14+ir\right)\Gamma\left(\tfrac{s}{2}+\tfrac14-ir\right)\sum_{n=1}^{\infty}\frac{b(n)}{n^s}.\end{aligned}$$

So the coefficients $b(n)$ are given by the relation

$$\sum_{n=1}^{\infty}\frac{b(n)}{n^s}=\sqrt{i}\left(\frac{\pi N}{6}\right)^{\frac14}L(s+1,\psi)\sum_{n=1}^{\infty}\left(\frac{12}{n}\right)\frac{a(n^2)}{n^{s-\frac12}}. \tag{5.3.26}$$

This proves Theorem 5.3.1 in the case $t=1$ since $b(n)$ is a constant multiple of the function $b_1(n)$ in (5.3.3).

5.3.3 The case $t > 1$

For squarefree $t \equiv 1 \pmod{24}$ with $t>1$ we argue as in Section 3 of [N]. For a function f on $\mathbb{H}$ define $f_t(\tau) := f(t\tau)$. We apply Theorem 5.3.1 to

$$G_t(\tau)=\sum_{n\equiv1(24)}a(n/t)W_{\frac{\mathrm{sgn}(n)}{4},ir}\left(\frac{\pi|n|y}{6}\right)e\left(\frac{nx}{24}\right)\in\mathcal{S}_{\frac12}\left(Nt,\psi\left(\tfrac{t}{\bullet}\right)\chi,r\right).$$

The coefficients $c(n)$ of $S_1(G_t)\in\mathcal{S}_0(6Nt,\psi^2,2r)$ are given by

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{c(n)}{n^s} &= L\left(s+1,\psi\left(\tfrac{t}{\bullet}\right)\right)\sum_{n=1}^{\infty}\left(\frac{12}{n}\right)\frac{a(n^2/t)}{n^{s-\frac12}}\\ &= L\left(s+1,\psi\left(\tfrac{t}{\bullet}\right)\right)t^{-s+\frac12}\sum_{n=1}^{\infty}\left(\frac{12}{n}\right)\frac{a(tn^2)}{n^{s-\frac12}}.\end{aligned}$$

Thus $c(n)=0$ unless $t\mid n$, in which case $c(n)=\sqrt{t}\,b_t(n/t)$, where $b_t(n)$ are the coefficients of $S_t(G)$. We conclude that $S_1(G_t)=\sqrt{t}\,[S_t(G)]_t$. By a standard argument (see e.g. [N, Section 3]) we have

$$S_t(G)\in\mathcal{S}_0(6N,\psi^2,2r).$$

This completes the proof of Theorem 5.3.1.

5.4 Estimates for a K-Bessel transform

This section contains uniform estimates for the K-Bessel transform $\check{\phi}(r)$ which are required in Sections 5.5 and 5.7. Recall that

$$\check{\phi}(r) = \operatorname{ch} \pi r \int_0^\infty K_{2ir}(u)\phi(u)\, \frac{du}{u},$$

where ϕ is a suitable test function (see (5.2.1)). Given $a, x, T > 0$ with

$$T \leq \frac{x}{3} \quad \text{and} \quad T \asymp x^{1-\delta}, \qquad 0 < \delta < \frac{1}{2},$$

we choose $\phi = \phi_{a,x,T} : [0, \infty) \to [0, 1]$ to be a smooth function satisfying

(i) $\phi(t) = 1$ for $\frac{a}{2x} \leq t \leq \frac{a}{x}$,

(ii) $\phi(t) = 0$ for $t \leq \frac{a}{2x + 2T}$ and $t \geq \frac{a}{x - T}$,

(iii) $\phi'(t) \ll \left(\frac{a}{x - T} - \frac{a}{x}\right)^{-1} \ll \frac{x^2}{aT}$, and

(iv) ϕ and ϕ' are piecewise monotone on a fixed number of intervals (whose number is independent of a, x, and T).

In Theorem 5.2.1, the function $\check{\phi}(r)$ is evaluated at the spectral parameters r_j corresponding to the eigenfunctions u_j as in (2.3.23). In view of Corollary 5.3.3, we require estimates only for $r \geq 1$. We will prove the following theorem.

Theorem 5.4.1. *Suppose that a, x, T, and $\phi = \phi_{a,x,T}$ are as above. Then*

$$\check{\phi}(r) \ll \begin{cases} r^{-\frac{3}{2}} e^{-r/2} & \textit{if } 1 \leq r \leq \frac{a}{8x}, \\ r^{-1} & \textit{if } \max\left(\frac{a}{8x}, 1\right) \leq r \leq \frac{a}{x}, \\ \min\left(r^{-\frac{3}{2}}, r^{-\frac{5}{2}} \frac{x}{T}\right) & \textit{if } r \geq \max\left(\frac{a}{x}, 1\right). \end{cases} \tag{5.4.1}$$

To prove Theorem 5.4.1 we require estimates for $K_{iv}(vz)$ which are uniform for $z \in (0, \infty)$ and $v \in [1, \infty)$.

5.4.1 Uniform estimates for the K-Bessel function

We estimate $K_{iv}(vz)$ in the following ranges as $v \to \infty$:

(A) the oscillatory range $0 < z \le 1 - O(v^{-\frac{2}{3}})$,

(B) the transitional range $1 - O(v^{-\frac{2}{3}}) \le z \le 1 + O(v^{-\frac{2}{3}})$, and

(C) the decaying range $z \ge 1 + O(v^{-\frac{2}{3}})$.

Suppose that c is a positive constant. In the transitional range there is a significant "bump" in the K-Bessel function. By [BST, (14) and (21)] we have

$$e^{\frac{\pi v}{2}} K_{iv}(vz) \ll_c v^{-\frac{1}{3}} \qquad \text{for } z \ge 1 - cv^{-\frac{2}{3}}. \tag{5.4.2}$$

In the decaying range the K-Bessel function is positive and decreasing. By [BST, (14)] we have

$$e^{\frac{\pi v}{2}} K_{iv}(vz) \ll_c \frac{e^{-v\mu(z)}}{v^{\frac{1}{2}}(z^2-1)^{\frac{1}{4}}} \qquad \text{for } z \ge 1 + cv^{-\frac{2}{3}}, \tag{5.4.3}$$

where

$$\mu(z) := \sqrt{z^2-1} - \arccos\left(\frac{1}{z}\right).$$

The oscillatory range is much more delicate. Balogh [Ba] gives a uniform asymptotic expansion for $K_{iv}(vz)$ in terms of the Airy function Ai and its derivative Ai$'$. For $z \in (0,1)$ define

$$w(z) := \operatorname{arccosh}\left(\frac{1}{z}\right) - \sqrt{1-z^2}$$

and define ζ and ξ by

$$\frac{2}{3}\zeta^{\frac{3}{2}} = -i\, w(z), \qquad \xi = v^{\frac{2}{3}}\zeta.$$

Taking $m = 1$ in equation (2) of [Ba] we have

$$e^{\frac{\pi v}{2}} K_{iv}(vz) = \frac{\pi\sqrt{2}}{v^{\frac{1}{3}}}\left(\frac{-\zeta}{1-z^2}\right)^{\frac{1}{4}} \Bigg\{ \mathrm{Ai}(\xi)\left[1 + \frac{A_1(\zeta)}{v^2}\right] + \mathrm{Ai}'(\xi)\frac{B_0(\zeta)}{v^{\frac{4}{3}}} + \frac{e^{ivw(z)}}{1+|\xi|^{\frac{1}{4}}} O\left(v^{-3}\right) \Bigg\}, \tag{5.4.4}$$

uniformly for $v \in [1, \infty)$, where

$$A_1(\zeta) := \frac{455}{10368\, w(z)^2} - \frac{7(3z^2+2)}{1728(1-z^2)^{\frac{3}{2}} w(z)} - \frac{81z^4+300z^2+4}{1152(1-z^2)^3}, \quad (5.4.5)$$

$$B_0(\zeta) := \left(\frac{2}{3}\right)^{\frac{1}{3}} \frac{e^{2\pi i/3}}{w(z)^{\frac{1}{3}}} \left(\frac{3z^2+2}{24(1-z^2)^{\frac{3}{2}}} - \frac{5}{72 w(z)} \right). \quad (5.4.6)$$

A computation shows that $A_1(\zeta)$ and $B_0(\zeta)$ have finite limits as $z \to 0^+$ and as $z \to 1^-$, so both functions are $O(1)$ for $z \in (0,1)$.

Note that $\arg \xi = -\frac{\pi}{3}$. In order to work on the real line, we apply [DL, (9.6.2-3)] to obtain

$$\mathrm{Ai}(\xi) = \left(\tfrac{2}{3}\right)^{1/6} \frac{i}{2^{3/2}} (vw(z))^{\frac{1}{3}} H^{(1)}_{\frac{1}{3}}(vw(z)),$$
$$\mathrm{Ai}'(\xi) = -\left(\tfrac{3}{2}\right)^{1/6} \frac{i}{2^{3/2}} (vw(z))^{\frac{2}{3}} H^{(1)}_{\frac{2}{3}}(vw(z)),$$

where $H^{(1)}_{\frac{1}{3}}$ and $H^{(1)}_{\frac{2}{3}}$ are Hankel functions of the first kind. So we have

$$e^{\frac{\pi v}{2}} K_{iv}(vz) = \frac{\pi}{2} e^{2\pi i/3} \frac{w(z)^{\frac{1}{2}}}{(1-z^2)^{\frac{1}{4}}} H^{(1)}_{\frac{1}{3}}(vw(z)) \left[1 + \frac{A_1(\zeta)}{v^2} \right]$$
$$+ \left(\tfrac{3}{2}\right)^{1/3} \frac{\pi}{2} e^{-\pi i/3} \frac{w(z)^{\frac{5}{6}}}{(1-z^2)^{\frac{1}{4}}} H^{(1)}_{\frac{2}{3}}(vw(z)) \frac{B_0(\zeta)}{v} + O\left(\frac{v^{-\frac{7}{2}}}{(1-z^2)^{\frac{1}{4}}} \right). \quad (5.4.7)$$

Since $w(z) \to \infty$ as $z \to 0$ and $(1-z^2)^{-\frac{1}{4}} \to \infty$ as $z \to 1$, we derive more convenient expressions for (5.4.7) for z in the intervals $(0, 3/4]$ and $[3/16, 1 - cv^{-\frac{2}{3}})$.

Proposition 5.4.2. *Suppose that $z \in (0, 3/4]$ and that $v \geq 1$. Then*

$$e^{\frac{\pi v}{2}} K_{iv}(vz) = e^{\pi i/4} \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{e^{ivw(z)}}{v^{\frac{1}{2}} (1-z^2)^{\frac{1}{4}}} \left[1 - i \frac{3z^2+2}{24v(1-z^2)^{\frac{3}{2}}} \right] + O\big(v^{-\frac{5}{2}}\big). \quad (5.4.8)$$

Proof. Since $(1-z^2) \gg 1$ for $z \leq 3/4$, the error term in (5.4.7) is $\ll v^{-\frac{7}{2}}$. By

[DL, (10.17.5) and §10.17(iii)] we have

$$H^{(1)}_{\frac{1}{3}}(vw(z)) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{-5\pi i/12}\frac{e^{ivw(z)}}{(vw(z))^{\frac{1}{2}}}\left(1-\frac{5i}{72vw(z)}\right) + O\left((vw(z))^{-\frac{5}{2}}\right),$$
$$H^{(1)}_{\frac{2}{3}}(vw(z)) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{-7\pi i/12}\frac{e^{ivw(z)}}{(vw(z))^{\frac{1}{2}}} + O\left((vw(z))^{-\frac{3}{2}}\right).$$

In particular, this implies that

$$\frac{w(z)^{\frac{1}{2}}}{(1-z^2)^{\frac{1}{4}}}H^{(1)}_{\frac{1}{3}}(vw(z))\frac{A_1(\zeta)}{v^2} \ll v^{-\frac{5}{2}},$$

so by (5.4.7) we have

$$e^{\frac{\pi v}{2}}K_{iv}(vz) = e^{\pi i/4}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{e^{ivw(z)}}{v^{\frac{1}{2}}(1-z^2)^{\frac{1}{4}}}\Bigg[1-\frac{5i}{72vw(z)} + e^{-7\pi i/6}\left(\frac{3}{2}\right)^{\frac{1}{3}} w(z)^{\frac{1}{3}}\frac{B_0(\zeta)}{v}\Bigg] + O\big(v^{-\frac{5}{2}}\big).$$

Using (5.4.6), we obtain (5.4.8). □

We require some notation for the next proposition. Let $J_\nu(x)$ and $Y_\nu(x)$ denote the J and Y-Bessel functions, and define

$$M_\nu(x) = \sqrt{J_\nu^2(x) + Y_\nu^2(x)}.$$

Proposition 5.4.3. *Suppose that $c > 0$. Suppose that $v \geq 1$ and that $\frac{3}{16} \leq z \leq 1 - cv^{-\frac{2}{3}}$. Then*

$$e^{\frac{\pi v}{2}}K_{iv}(vz) = \frac{\pi}{2}e^{2\pi i/3}\frac{w(z)^{\frac{1}{2}}}{(1-z^2)^{\frac{1}{4}}}M_{\frac{1}{3}}(vw(z))e^{i\theta_{\frac{1}{3}}(vw(z))} + O_c(v^{-4/3}), \quad (5.4.9)$$

where $\theta_{\frac{1}{3}}(x)$ is a real-valued continuous function satisfying

$$\theta'_{\frac{1}{3}}(x) = \frac{2}{\pi x M^2_{\frac{1}{3}}(x)}. \quad (5.4.10)$$

Proof. Since $(1-z^2) \gg_c v^{-\frac{2}{3}}$ for $z \leq 1 - cv^{-\frac{2}{3}}$, the error term in (5.4.7) is

$\ll_c v^{-\frac{10}{3}}$. The modulus and phase of $H_\alpha^{(1)}(x)$ are given by [DL, §10.18]

$$H_\alpha^{(1)}(x) = M_\alpha(x)e^{i\theta_\alpha(x)},$$

where $M_\alpha^2(x)\theta_\alpha'(x) = 2/\pi x$. A straightforward computation shows that $w(z) \gg (1-z)^{\frac{3}{2}}$. It follows that $vw(z) \gg_c 1$ for $z \le 1 - cv^{-\frac{2}{3}}$, so by [DL, (10.18.17)] we obtain

$$M_\alpha(vw(z)) \ll_{\alpha,c} \frac{1}{(vw(z))^{\frac{1}{2}}}.$$

This, together with (5.4.7) and the fact that $A_1(\zeta)$ and $B_0(\zeta)$ are $O(1)$, gives (5.4.9). □

5.4.2 Estimates for $\check{\phi}(r)$

We treat each of the three ranges considered in Theorem 5.4.1 separately in the following propositions. We will make frequent use of an integral estimate which is an immediate corollary of the second mean value theorem for integrals.

Lemma 5.4.4. *Suppose that f and g are continuous functions on $[a,b]$ and that g is piecewise monotonic on M intervals. Then*

$$\left| \int_a^b f(x)g(x)\,dx \right| \le 2M \sup_{x\in[a,b]} |g(x)| \sup_{[\alpha,\beta]\subseteq[a,b]} \left| \int_\alpha^\beta f(x)\,dx \right|. \tag{5.4.11}$$

Proposition 5.4.5. *With the notation of Theorem 5.4.1, suppose that $1 \le r \le a/8x$. Then*

$$\check{\phi}(r) \ll r^{-\frac{3}{2}} e^{-r/2}. \tag{5.4.12}$$

Proof. Since $r \le a/8x$ and $T \le x/3$, we have $\frac{a}{4(x+T)r} \ge \frac{3}{2}$. Taking $c = 2^{-1/3}$ in (5.4.3) gives

$$\check{\phi}(r) = \operatorname{ch}\pi r \int_{\frac{a}{4(x+T)r}}^{\frac{a}{2(x-T)r}} K_{2ir}(2ry)\phi(2ry)\,\frac{dy}{y} \ll r^{-\frac{1}{2}} \int_{\frac{3}{2}}^\infty e^{-2r\mu(y)} \frac{dy}{y(y^2-1)^{\frac{1}{4}}}.$$

Since $\mu'(y) = \sqrt{y^2-1}/y$ we have

$$\check{\phi}(r) \ll r^{-\frac{3}{2}} \int_{\frac{3}{2}}^\infty \left(-e^{-2r\mu(y)}\right)' \frac{dy}{(y^2-1)^{\frac{3}{4}}} \ll r^{-\frac{3}{2}} e^{-2r\mu(3/2)},$$

which, together with $\mu(3/2) \approx .277$ proves the proposition. $\square$

Proposition 5.4.6. *With the notation of Theorem 5.4.1, suppose that*

$$\max(a/8x, 1) \leq r \leq a/x.$$

Then

$$\check{\phi}(r) \ll r^{-1}. \tag{5.4.13}$$

Before proving Proposition 5.4.6, we require a lemma describing the behavior of the function $M_{\frac{1}{3}}(x)$ in Proposition 5.4.3.

Lemma 5.4.7. *The function $\widetilde{M}(x) := xM_{\frac{1}{3}}^2(x)$ is increasing on $[0, \infty)$ with $\lim_{x\to\infty} \widetilde{M}(x) = \frac{2}{\pi}$.*

Proof. We will prove that $w(x) := \sqrt{\widetilde{M}(x)}$ is increasing. From [DL, (10.7.3-4), (10.18.17)] we have

$$\widetilde{M}(0) = 0, \qquad 0 \leq \widetilde{M}(x) \leq \frac{2}{\pi}, \quad \text{and} \quad \lim_{x\to\infty} \widetilde{M}(x) = \frac{2}{\pi}.$$

It is therefore enough to show that $w''(x) < 0$ for all $x > 0$. In view of the second order differential equation [DL, (10.18.14)] satisfied by w it will suffice to prove that

$$\widetilde{M}(x) > \frac{12x}{\pi\sqrt{36x^2 + 5}}. \tag{5.4.14}$$

The inequality (5.4.14) can be proved numerically, using the expansions of $\widetilde{M}(x)$ at 0 and ∞. By [DL, §10.18(iii)] we have

$$\widetilde{M}(x) \geq \frac{2}{\pi}\left(1 + \sum_{k=1}^{n} \frac{1\cdot 3\cdots(2k-1)}{2\cdot 4\cdots(2k)} \frac{(\frac{4}{9}-1)(\frac{4}{9}-9)\cdots(\frac{4}{9}-(2k-1)^2)}{(2x)^{2k}}\right) \tag{5.4.15}$$

for odd $n \geq 1$. Taking $n = 3$ in (5.4.15), we verify numerically that (5.4.14) holds for $x > 2.34$.

From [DL, (10.2.3)] we have

$$\widetilde{M}(x) = \frac{4x}{3}\left(J_{\frac{1}{3}}^2(x) - J_{\frac{1}{3}}(x)J_{-\frac{1}{3}}(x) + J_{-\frac{1}{3}}^2(x)\right).$$

So by [DL, (10.8.3)] it follows that the series for $\widetilde{M}(x)$ at 0 is alternating, and that $\widetilde{M}(x)$ is larger than the truncation of this series after the term with exponent 47/3. Moreover, this truncation is larger than $\frac{12x}{\pi\sqrt{36x^2+5}}$ for $x < 2.45$. The claim (5.4.14) follows. □

Proof of Proposition 5.4.6. Fix $c = \frac{1}{2}$ and write

$$\begin{aligned}\check{\phi}(r) &= \check{\phi}_1(r) + \check{\phi}_2(r) + \check{\phi}_3(r)\\ &= \left(\int_{\frac{a}{4(x+T)r}}^{1-cr^{-\frac{2}{3}}} + \int_{1-cr^{-\frac{2}{3}}}^{1+cr^{-\frac{2}{3}}} + \int_{1+cr^{-\frac{2}{3}}}^{\infty}\right) \operatorname{ch}\pi r K_{2ir}(2ry)\phi(2ry)\frac{dy}{y}.\end{aligned}$$

We will show that $\check{\phi}_i(r) \ll r^{-1}$ for $i = 1, 2, 3$. Note that $\frac{a}{4(x+T)r} \geq \frac{3}{16}$.

By Proposition 5.4.3 we have

$$\check{\phi}_1(r) \ll \left| \int_{\frac{a}{4(x+T)r}}^{1-cr^{-\frac{2}{3}}} e^{i\theta_{\frac{1}{3}}(2rw(y))} \frac{M_{\frac{1}{3}}(2rw(y))w(y)^{\frac{1}{2}}\phi(2ry)}{y(1-y^2)^{\frac{1}{4}}}\,dy \right| + r^{-\frac{4}{3}} \int_{\frac{3}{16}}^{1} \phi(2ry)\frac{dy}{y}. \quad (5.4.16)$$

The error term is $O(r^{-4/3})$. Using (5.4.10) and the fact that $w'(y) = -\sqrt{1-y^2}/y$, the first term in (5.4.16) equals

$$\frac{\pi}{2(2r)^{\frac{3}{2}}} \left| \int_{\frac{a}{4(x+T)r}}^{1-cr^{-\frac{2}{3}}} \left(e^{i\theta_{\frac{1}{3}}(2rw(y))}\right)' \frac{\left(2rw(y)M^2_{\frac{1}{3}}(2rw(y))\right)^{\frac{3}{2}}}{(1-y^2)^{\frac{3}{4}}} \phi(2ry)\,dy \right|.$$

Note that the function $w(y)$ is decreasing, so by Lemma 5.4.7 the function $2rw(y)M^2_{\frac{1}{3}}(2rw(y))$ is decreasing. We apply Lemma 5.4.4 three times; first with the decreasing function

$$g(y) = \left(2rw(y)M_{\frac{1}{3}}(2rw(y))\right)^{\frac{3}{2}} \ll 1,$$

next with the increasing function

$$g(y) = (1-y^2)^{-\frac{3}{4}} \ll r^{\frac{1}{2}},$$

and then with $g(y) = \phi(2ry)$. We conclude that $\check{\phi}_1(r) \ll r^{-1}$.

For $\check{\phi}_2(r)$ we apply (5.4.2) to obtain

$$\check{\phi}_2(r) \ll r^{-\frac{1}{3}} \int_{1-cr^{-\frac{2}{3}}}^{1+cr^{-\frac{2}{3}}} dy \ll r^{-1}.$$

For $\check{\phi}_3(r)$ we argue as in the proof of Proposition 5.4.5 to obtain

$$\check{\phi}_3(r) \ll r^{-\frac{3}{2}} \int_{1+cr^{-\frac{2}{3}}}^{\infty} \left(-e^{-2r\mu(y)}\right)' \frac{dy}{(y^2-1)^{\frac{3}{4}}}.$$

For $y \geq 1 + cr^{-\frac{2}{3}}$ we have $(y^2-1)^{\frac{3}{4}} \gg r^{-\frac{1}{2}}$, so $\check{\phi}_3(r) \ll r^{-1}$. □

Proposition 5.4.8. *Suppose that* $r \geq \max(a/x, 1)$. *Then*

$$\check{\phi}(r) \ll \min\left(r^{-\frac{3}{2}}, r^{-\frac{5}{2}}\frac{x}{T}\right). \tag{5.4.17}$$

Proof. Since $r \geq a/x$ and $T \leq x/3$, the interval on which $\phi(2ry) \neq 0$ is contained in $(0, 3/4]$. So by Proposition 5.4.2 we have

$$\check{\phi}(r) \ll r^{-\frac{1}{2}}|I_0| + r^{-\frac{3}{2}}|I_1| + r^{-\frac{5}{2}} \int_{\frac{a}{4(x+T)r}}^{\frac{a}{2(x-T)r}} \frac{dy}{y}, \tag{5.4.18}$$

where

$$I_\alpha = \int_{\frac{a}{4(x+T)r}}^{\frac{a}{2(x-T)r}} e^{2irw(y)} \frac{\phi(2ry)(3y^2+2)^\alpha}{y(1-y^2)^{\frac{1}{4}+\frac{3}{2}\alpha}}\, dy.$$

The third term in (5.4.18) equals

$$r^{-\frac{5}{2}} \log\frac{2(x+T)}{x-T} \ll r^{-\frac{5}{2}}.$$

For the other terms we use $w'(y) = -\sqrt{1-y^2}/y$ to write

$$I_\alpha \ll r^{-1} \left| \int_{\frac{a}{4(x+T)r}}^{\frac{a}{2(x-T)r}} \left(e^{2irw(y)}\right)' \frac{\phi(2ry)(3y^2+2)^\alpha}{(1-y^2)^{\frac{3}{4}+\frac{3}{2}\alpha}}\, dy \right|. \tag{5.4.19}$$

Since $y \leq 3/4$ we have

$$\frac{(3y^2+2)^\alpha}{(1-y^2)^{\frac{3}{4}+\frac{3}{2}\alpha}} \ll 1,$$

so by Lemma 5.4.4 we obtain $I_\alpha \ll r^{-1}$ and therefore $\check{\phi}(r) \ll r^{-\frac{3}{2}}$.

For large r, we obtain a better estimate by integrating by parts in (5.4.19) for $\alpha = 0$. Since $\phi(2ry) = 0$ at the limits of integration, we find that

$$I_0 \ll r^{-1} \left| \int_{\frac{a}{4(x+T)r}}^{\frac{a}{2(x-T)r}} e^{2irw(y)} \frac{y\phi(2ry)}{(1-y^2)^{\frac{7}{4}}}\, dy \right| + r^{-1} \left| \int_{\frac{a}{4(x+T)r}}^{\frac{a}{2(x-T)r}} e^{2irw(y)} \frac{r\phi'(2ry)}{(1-y^2)^{\frac{3}{4}}}\, dy \right|$$
$$=: J_1 + J_2.$$

As above, Lemma 5.4.4 gives

$$J_1 \ll r^{-2} \left| \int_{\frac{a}{4(x+T)r}}^{\frac{a}{2(x-T)r}} \left(e^{2irw(y)}\right)' \frac{y^2\phi(2ry)}{(1-y^2)^{\frac{9}{4}}}\, dy \right| \ll r^{-2}.$$

For J_2, we apply Lemma 5.4.4 with the estimates $\phi'(2ry) \ll x^2/aT$ and $y \ll a/rx$ to obtain

$$J_2 = r^{-1} \left| \int_{\frac{a}{4(x+T)r}}^{\frac{a}{2(x-T)r}} \left(e^{2irw(y)}\right)' \frac{y\phi'(2ry)}{(1-y^2)^{\frac{5}{4}}}\, dy \right| \ll r^{-2}\frac{x}{T}.$$

So $I_0 \ll r^{-2}x/T$. This, together with (5.4.18) and our estimate for I_1 above, gives (5.4.8). $\square$

5.5 Application to sums of Kloosterman sums

We are in a position to prove Theorem 1.2.3, which asserts that for $m > 0$, $n < 0$ we have

$$\sum_{c \le X} \frac{S(m,n,c,\chi)}{c} \ll \left(X^{\frac{1}{6}} + |mn|^{\frac{1}{4}}\right) |mn|^{\epsilon} \log X. \tag{5.5.1}$$

This will follow from an estimate for dyadic sums.

Proposition 5.5.1. *If $m > 0$, $n < 0$ and $x > 4\pi\sqrt{\widetilde{m}|\widetilde{n}|}$, then*

$$\sum_{x \le c \le 2x} \frac{S(m,n,c,\chi)}{c} \ll \left(x^{\frac{1}{6}} \log x + |mn|^{\frac{1}{4}}\right) |mn|^{\varepsilon}$$

To obtain Theorem 1.2.3 from the proposition, we estimate using (2.3.30) to see that the initial segment $c \le 4\pi\sqrt{\widetilde{m}|\widetilde{n}|}$ of (5.5.1) contributes $O(|mn|^{\frac{1}{4}+\epsilon})$.

We break the rest of the sum into dyadic pieces $x \le c \le 2x$ with $4\pi\sqrt{\widetilde{m}|\widetilde{n}|} < x \le X/2$. Estimating each of these with Proposition 5.5.1 and summing their contributions gives (5.5.1).

Proof of Proposition 5.5.1. Let

$$a := 4\pi\sqrt{\widetilde{m}|\widetilde{n}|}, \qquad T := x^{\frac{2}{3}},$$

and suppose that $x > a$. We apply Theorem 5.2.1 using a test function ϕ which satisfies conditions (i)–(iv) of Section 5.4. Corollary 1.1.6 and the mean value bound for the divisor function give

$$\begin{aligned} &\left|\sum_{c=1}^{\infty} \frac{S(m,n,c,\chi)}{c}\phi\left(\frac{a}{c}\right) - \sum_{x \le c \le 2x} \frac{S(m,n,c,\chi)}{c}\right| \\ &\qquad \le \sum_{\substack{x-T \le c \le x \\ 2x \le c \le 2x+2T}} \frac{|S(m,n,c,\chi)|}{c} \ll \frac{T \log x}{\sqrt{x}} |mn|^{\epsilon} \ll |mn|^{\epsilon} x^{\frac{1}{6}} \log x. \quad (5.5.2) \end{aligned}$$

Therefore, by Theorem 5.2.1, it will suffice to obtain the stated estimate for the quantity

$$\sum_{c=1}^{\infty} \frac{S(m,n,c,\chi)}{c}\phi\left(\frac{a}{c}\right) = 8\sqrt{i}\sqrt{\widetilde{m}|\widetilde{n}|} \sum_{1<r_j} \frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch} \pi r_j} \check{\phi}(r_j), \qquad (5.5.3)$$

where we have used Corollary 5.3.3 and the fact that the eigenvalue $r_0 = i/4$ makes no contribution since n is negative.

We break the sum (5.5.3) into dyadic intervals $A \le r_j \le 2A$. Using the Cauchy-Schwarz inequality together with Theorems 1.2.5 and 5.4.1 (recall that $a/x < 1 < r_j$) we obtain

$$\begin{aligned} &\sqrt{\widetilde{m}|\widetilde{n}|} \sum_{A \le r_j \le 2A} \left| \frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch} \pi r_j} \check{\phi}(r_j) \right| \\ &\ll \min\left(A^{-\frac{3}{2}}, A^{-\frac{5}{2}} x^{\frac{1}{3}}\right) \left(\widetilde{m} \sum_{A \le r_j \le 2A} \frac{|\rho_j(m)|^2}{\operatorname{ch} \pi r_j}\right)^{\frac{1}{2}} \left(|\widetilde{n}| \sum_{A \le r_j \le 2A} \frac{|\rho_j(n)|^2}{\operatorname{ch} \pi r_j}\right)^{\frac{1}{2}} \\ &\ll \min\left(A^{-\frac{3}{2}}, A^{-\frac{5}{2}} x^{\frac{1}{3}}\right) \left(A^{\frac{3}{2}} + m^{\frac{1}{2}+\varepsilon}\right)^{\frac{1}{2}} \left(A^{\frac{5}{2}} + |n|^{\frac{1}{2}+\varepsilon} A^{\frac{1}{2}}\right)^{\frac{1}{2}} \\ &\ll \min\left(A^{\frac{1}{2}}, A^{-\frac{1}{2}} x^{\frac{1}{3}}\right) \left(1 + m^{\frac{1}{4}+\epsilon} A^{-\frac{3}{4}} + |n|^{\frac{1}{4}+\epsilon} A^{-1} + |mn|^{\frac{1}{4}+\epsilon} A^{-\frac{7}{4}}\right). \end{aligned}$$

Summing the contribution from the dyadic intervals gives

$$\sqrt{\widetilde{m}|\widetilde{n}|}\sum_{1<r_j}\frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch}\pi r_j}\check{\phi}(r_j) \ll x^{\frac{1}{6}} + |mn|^{\frac{1}{4}+\epsilon},$$

and Proposition 5.5.1 follows. □

5.6 A second estimate for coefficients of Maass cusp forms

In the case $m = 1$ we can improve the estimate of Proposition 5.5.1 by using a second estimate for the sum of the Fourier coefficients of Maass cusp forms in $\mathcal{S}_{\frac{1}{2}}(1,\chi)$. The next theorem is an improvement on Theorem 1.2.5 only when n is much larger than x.

Theorem 5.6.1. *Suppose that $\{u_j\}$ is an orthonormal basis of $\mathcal{S}_{\frac{1}{2}}(1,\chi)$ with spectral parameters r_j and coefficients $\rho_j(n)$ as in* (2.3.23)*. Suppose that the u_j are eigenforms of the Hecke operators T_{p^2} for $p \nmid 6$. If $24n - 23$ is not divisible by 5^4 or 7^4 then*

$$\sum_{0<r_j\leq x}\frac{|\rho_j(n)|^2}{\operatorname{ch}\pi r_j} \ll |n|^{-\frac{4}{7}+\epsilon}\, x^{5-\frac{\operatorname{sgn} n}{2}}. \tag{5.6.1}$$

Remark. As the proof will show, the exponent 4 in the assumption on n in Theorem 5.6.1 can be replaced by any positive integer m. Increasing m has the effect of increasing the implied constant in (5.6.1).

We require an average version of an estimate of Duke [D1, Theorem 5]. Let ν_θ denote the weight 1/2 multiplier for $\Gamma_0(4)$ defined in (2.1.6).

Proposition 5.6.2. *Let D be an even fundamental discriminant, let N be a positive integer with $D \mid N$, and let*

$$(k,\nu) = \left(\frac{1}{2},\, \nu_\theta\left(\tfrac{|D|}{\bullet}\right)\right) \qquad \text{or} \qquad \left(\frac{3}{2},\, \overline{\nu}_\theta\left(\tfrac{|D|}{\bullet}\right)\right).$$

Suppose that $\{v_j\}$ is an orthonormal basis of $\mathcal{S}_k(N,\nu)$ with spectral parameters

r_j and coefficients $b_j(n)$. If $n > 0$ is squarefree then

$$\sum_{0\le r_j\le x} \frac{|b_j(n)|^2}{\operatorname{ch}\pi r_j} \ll_{N,D} n^{-\frac{4}{7}+\varepsilon} x^{5-k}. \tag{5.6.2}$$

Sketch of proof. Let $\phi, \widehat{\phi}$ be as in the proof of Theorem 5 of [D1]. Then $\widehat{\phi}(r) > 0$ and

$$\widehat{\phi}(r) \sim \frac{1}{2\pi^2}|r|^{k-5} \qquad \text{as } |r| \to \infty. \tag{5.6.3}$$

Set

$$V_1(n,n) := n \sum_{r_j} \frac{|b_j(n)|^2}{\operatorname{ch}\pi r_j} \widehat{\phi}(r_j).$$

By Theorem 2 of [D1] we have $|V_1(n,n)+V_2(n,n)| \ll |S_N| + |V_3(n,n)|$, where S_N is defined in the proof of Theorem 5 and V_2, V_3 are defined in Section 3 of that paper. Since $V_1(n,n)$ and $V_2(n,n)$ are visibly non-negative, we have

$$V_1(n,n) \ll |S_N| + |V_3(n,n)|. \tag{5.6.4}$$

The terms S_N and $V_3(n,n)$ are estimated by averaging over the level. For $P > (4\log 2n)^2$ let

$$\overline{Q} = \{pN : p \text{ prime}, \ P < p \le 2P, \ p \nmid 2n\}.$$

Summing (5.6.4) gives

$$\sum_{Q\in\overline{Q}} V_1^{(Q)}(n,n) \ll \sum_{Q\in\overline{Q}} |S_Q| + \sum_{Q\in\overline{Q}} |V_3^{(Q)}(n,n)|, \tag{5.6.5}$$

where $V_1^{(Q)}$, S_Q, and $V_3^{(Q)}$ are the analogues of V_1, S_N, and V_3 for $\Gamma_0(Q)$.

For each $Q \in \overline{Q}$, the functions $\{[\Gamma_0(N) : \Gamma_0(Q)]^{-1/2} u_j(\tau)\}$ form an orthonormal subset of $\mathcal{S}_k(Q,\nu)$. It follows that

$$V_1^{(Q)}(n,n) \ge \frac{n}{[\Gamma_0(N):\Gamma_0(Q)]} \sum_{r_j} \frac{|b_j(n)|^2}{\operatorname{ch}\pi r_j}\widehat{\phi}(r_j) = \frac{V_1(n,n)}{[\Gamma_0(N):\Gamma_0(Q)]}.$$

Since $[\Gamma_0(N) : \Gamma_0(Q)] \le p+1 \ll P$ we find that

$$V_1(n,n) \ll P\, V_1^{(Q)}(n,n) \quad \text{ for all } Q \in \overline{Q}.$$

Since $|\overline{Q}| \asymp P/\log P$ we conclude that

$$V_1(n,n) \ll \log P \sum_{Q\in\overline{Q}} V_1^{(Q)}(n,n). \tag{5.6.6}$$

In the proof of Theorem 5 of [D1], Duke gives the estimates

$$\sum_{Q\in\overline{Q}} |V_3^{(Q)}(n,n)| \ll_{N,D} n^{\frac{3}{7}+\epsilon} \quad \text{and} \quad \sum_{Q\in\overline{Q}} |S_Q| \ll_{N,D} \big((n/P)^{\frac{1}{2}} + (nP)^{\frac{3}{8}}\big) n^{\epsilon} \tag{5.6.7}$$

which follow from work of Iwaniec [I1]. By (5.6.5), (5.6.6), and (5.6.7) we conclude that

$$V_1(n,n) \ll_{N,D} \log P\big((n/P)^{\frac{1}{2}} + (nP)^{\frac{3}{8}} + n^{\frac{3}{7}}\big)n^{\epsilon}.$$

Choosing $P = n^{1/7}$, we find that $V_1(n,n) \ll_{N,D} n^{3/7+\epsilon}$. By (5.6.3) we have

$$n \sum_{0\le r_j\le x} \frac{|b_j(n)|^2}{\operatorname{ch}\pi r_j} \ll n \sum_{0\le r_j\le x} r_j^{5-k} \frac{|b_j(n)|^2}{\operatorname{ch}\pi r_j} \widehat{\phi}(r_j) \ll x^{5-k}\, V_1(n,n),$$

from which (5.6.2) follows. □

We turn to the proof of Theorem 5.6.1.

Proof of Theorem 5.6.1. Set $\alpha := [\Gamma_0(1) : \Gamma_0(24,24)]^{1/2}$. With $\{u_j\}$ as in the hypotheses, and recalling (2.3.20), define

$$v_j(\tau) := \tfrac{1}{\alpha} u_j(24\tau) = \sum_{n\equiv 1(24)} b_j(n) W_{\frac{1}{4}\operatorname{sgn} n, ir_j}(4\pi|n|y)e(nx),$$

$$v_j'(\tau) := \tfrac{1}{\alpha}\left(r_j^2 + \tfrac{1}{16}\right)^{-\frac{1}{2}} \overline{L_{\frac{1}{2}} u_j}(24\tau) = \sum_{n\equiv 23(24)} b_j'(n) W_{\frac{3}{4}\operatorname{sgn} n, ir_j}(4\pi|n|y)e(nx).$$

By (2.1.7) we have

$$v_j \in \mathcal{S}_{\frac{1}{2}}\left(576, \nu_\theta\left(\tfrac{12}{\bullet}\right)\right), \qquad v_j' \in \mathcal{S}_{\frac{3}{2}}\left(576, \overline{\nu_\theta}\left(\tfrac{12}{\bullet}\right)\right).$$

From (2.3.14) and (2.3.20) we see that $\{v_j\}$, $\{v_j'\}$ are orthonormal sets which can be extended to orthonormal bases for these spaces. Using (2.3.11) and

(2.3.13) and comparing Fourier expansions, we find for positive n that

$$b_j(n) = \frac{1}{\alpha}\,\rho_j\left(\frac{n+23}{24}\right), \qquad b_j'(n) = \frac{1}{\alpha}\,\left(r_j^2+\tfrac{1}{16}\right)^{-\frac{1}{2}}\,\overline{\rho_j\left(\frac{-n+23}{24}\right)}.$$

It follows from Proposition 5.6.2 that for squarefree n we have

$$\sum_{0<r_j\le x} \frac{\left|\rho_j\left(\frac{n+23}{24}\right)\right|^2}{\operatorname{ch}\pi r_j} \ll |n|^{-\frac{4}{7}+\epsilon}\,x^{5-\frac{\operatorname{sgn} n}{2}}. \tag{5.6.8}$$

To establish (5.6.8) for non-squarefree n we use the assumption that the $\{u_j\}$ are Hecke eigenforms. Recall the definition of the lift S_t from Theorem 5.3.1. For each j there exists some squarefree positive $t \equiv 1 \pmod{24}$ for which $S_t(u_j) \neq 0$. Denote by $\lambda_j(p)$ the eigenvalue of u_j under T_{p^2}. From Corollary 5.3.2 it follows that $\left(\frac{12}{p}\right)\lambda_j(p)$ is a Hecke eigenvalue of the weight zero Maass cusp form $S_t(u_j)$. For these eigenvalues we have the estimate

$$|\lambda_j(p)| \le p^{\frac{7}{64}} + p^{-\frac{7}{64}}$$

due to Kim and Sarnak [Ki, Appendix 2].

The Hecke action (2.3.26) gives

$$\begin{aligned} &p\,\rho_j\left(\frac{p^2n+23}{24}\right) \\ &= \lambda_j(p)\rho_j\left(\frac{n+23}{24}\right) - p^{-\frac{1}{2}}\left(\frac{12n}{p}\right)\rho_j\left(\frac{n+23}{24}\right) - p^{-1}\rho_j\left(\frac{n/p^2+23}{24}\right). \end{aligned} \tag{5.6.9}$$

Suppose that $p^2 \nmid n$. Then

$$\begin{aligned} \left|\rho_j\left(\frac{p^2n+23}{24}\right)\right| &\le \left(p^{-\frac{57}{64}} + p^{-\frac{71}{64}} + p^{-\frac{3}{2}}\right)\left|\rho_j\left(\frac{n+23}{24}\right)\right| \\ &\le \begin{cases} 1.25\,p^{-\frac{4}{7}} & \text{if } p = 5 \text{ or } 7, \\ p^{-\frac{4}{7}} & \text{if } p \ge 11. \end{cases} \end{aligned}$$

Thus (5.6.8) holds whenever n is not divisible by p^4 for any p.

To treat the remaining cases, assume that $p \ge 11$ and that for $p^2 \nmid n$ we have

$$\left|\rho_j\left(\frac{p^{2\ell}n+23}{24}\right)\right| \le p^{-\frac{4\ell}{7}}\left|\rho_j\left(\frac{n+23}{24}\right)\right|, \qquad \ell \le m-1.$$

Then for $m \geq 2$, the relation (5.6.9) gives

$$\left|\rho_j\left(\frac{p^{2m}n+23}{24}\right)\right| \leq \left(p^{-\frac{57}{64}}+p^{-\frac{71}{64}}\right)\left|\rho_j\left(\frac{p^{2m-2}n+23}{24}\right)\right| + p^{-2}\left|\rho_j\left(\frac{p^{2m-4}n+23}{24}\right)\right|$$
$$\leq \left[\left(p^{-\frac{57}{64}}+p^{-\frac{71}{64}}\right)p^{\frac{4}{7}}+p^{-2+\frac{8}{7}}\right] p^{-\frac{4m}{7}}\left|\rho_j\left(\frac{n+23}{24}\right)\right|.$$

For $p \geq 11$ the quantity in brackets is < 1. It follows that (5.6.8) holds for all n which are not divisible by 5^4 or 7^4. □

5.7 Sums of Kloosterman sums and Rademacher's formula

Throughout this section, n will denote a negative integer. In our application to the error term in Rademacher's formula we need a estimate for sums of the Kloosterman sums $S(1, n, c, \chi)$ which improves the bound of Theorem 1.2.3 with respect to n. We will also make the assumption throughout that

$$24n - 23 \text{ is not divisible by } 5^4 \text{ or } 7^4 \qquad (5.7.1)$$

so that we may apply Theorem 5.6.1 (see the remark after that theorem).

Theorem 5.7.1. *For* $0 < \delta < 1/2$ *and* $n < 0$ *satisfying* (5.7.1) *we have*

$$\sum_{c \leq X} \frac{S(1,n,c,\chi)}{c} \ll_\delta |n|^{\frac{13}{56}+\varepsilon} X^{\frac{3}{4}\delta} + \left(|n|^{\frac{41}{168}+\varepsilon} + X^{\frac{1}{2}-\delta}\right)\log X. \qquad (5.7.2)$$

Theorem 5.7.1 will follow from an estimate for dyadic sums.

Proposition 5.7.2. *Suppose that* $0 < \delta < 1/2$ *and that* $n < 0$ *satisfies* (5.7.1). *Then*

$$\sum_{x \leq c \leq 2x} \frac{S(1,n,c,\chi)}{c} \ll_\delta |n|^{\frac{41}{28}+\varepsilon} x^{-\frac{5}{2}} + |n|^{\frac{13}{56}+\varepsilon} x^{\frac{3}{4}\delta} + x^{\frac{1}{2}-\delta}\log x. \qquad (5.7.3)$$

Deduction of Theorem 5.7.1 from Proposition 5.7.2. We break the sum in (5.7.2) into an initial segment corresponding to $c \leq |n|^\alpha$ and dyadic intervals of the form (5.7.3) with $x \geq |n|^\alpha$. Estimating the initial segment using Lehmer's bound (1.2.5) and applying Proposition 5.7.2 to each of the dyadic intervals, we find that

$$\sum_{c\le X}\frac{S(1,n,c,\chi)}{c} \ll_{\delta,\alpha} |n|^{\frac{\alpha}{2}+\epsilon} + |n|^{\frac{13}{56}+\varepsilon}X^{\frac{3}{4}\delta} + \left(|n|^{\frac{41}{28}-\frac{5}{2}\alpha+\varepsilon} + X^{\frac{1}{2}-\delta}\right)\log X.$$

Theorem 5.7.1 follows upon setting $\alpha = 41/84$. □

Proof of Proposition 5.7.2. Let T satisfy

$$T \asymp x^{1-\delta} \quad \text{and} \quad T \le \frac{x}{3}, \tag{5.7.4}$$

where $0<\delta<1/2$ is a parameter to be chosen later, and set

$$a := 4\pi\sqrt{|\tilde{n}|}.$$

Arguing as in (5.5.2) (using Lehmer's bound (1.2.5) to remove the dependence on n) we have

$$\left|\sum_{c=1}^{\infty}\frac{S(1,n,c,\chi)}{c}\phi\left(\frac{a}{c}\right) - \sum_{x\le c\le 2x}\frac{S(1,n,c,\chi)}{c}\right| \ll \delta x^{\frac{1}{2}-\delta}\log x. \tag{5.7.5}$$

Let $\{u_j\}$ be an orthonormal basis for $\mathcal{S}_{\frac{1}{2}}(1,\chi)$ as in (2.3.23) which consists of eigenforms for the Hecke operators T_{p^2} with $p\nmid 6$. Theorem 5.2.1 gives

$$\sum_{c=1}^{\infty}\frac{S(1,n,c,\chi)}{c}\phi\left(\frac{a}{c}\right) = 8\sqrt{i}\sqrt{|\tilde{n}|}\sum_{r_j}\frac{\overline{\rho_j(1)}\rho_j(n)}{\operatorname{ch}\pi r_j}\check{\phi}(r_j). \tag{5.7.6}$$

We break the sum on r_j into ranges corresponding to the three ranges of the K-Bessel function:

(i) $r_j \le \frac{a}{8x}$,

(ii) $\frac{a}{8x} < r_j < \frac{a}{x}$,

(iii) $r_j \ge \frac{a}{x}$.

We may restrict our attention to $r_j > 1$ by Corollary 5.3.3 (the eigenvalue $r_0 = i/4$ does not contribute).

For the first range, Theorem 5.4.1 gives gives $\check{\phi}(r_j) \ll r_j^{-3/2}e^{-r_j/2}$. By

Theorem 5.6.1 we have

$$\frac{\rho_j(1)}{\sqrt{\operatorname{ch}\pi r_j}} \ll r_j^{\frac{9}{4}}, \qquad \frac{\rho_j(n)}{\sqrt{\operatorname{ch}\pi r_j}} \ll |n|^{-\frac{2}{7}+\epsilon} r_j^{\frac{11}{4}}.$$

Combining these estimates gives

$$\sqrt{|\widetilde{n}|} \sum_{1<r_j\le \frac{a}{8x}} \left| \frac{\overline{\rho_j(1)}\rho_j(n)}{\operatorname{ch}\pi r_j} \check{\phi}(r_j) \right| \ll |n|^{\frac{3}{14}+\epsilon} \sum_{1<r_j\le \frac{a}{8x}} r_j^{\frac{7}{2}} e^{-r_j/2} \ll |n|^{\frac{3}{14}+\epsilon}, \quad (5.7.7)$$

where we have used (2.3.22) to conclude that the latter sum over r_j is $O(1)$.

For the other ranges we need the mean value estimates of Theorem 1.2.5:

$$\sum_{1<r_j\le x} \frac{|\rho_j(1)|^2}{\operatorname{ch}\pi r_j} \ll x^{\frac{3}{2}}, \qquad (5.7.8)$$

$$|\widetilde{n}| \sum_{1<r_j\le x} \frac{|\rho_j(n)|^2}{\operatorname{ch}\pi r_j} \ll x^{\frac{5}{2}} + |n|^{\frac{1}{2}+\varepsilon} x^{\frac{1}{2}}, \qquad (5.7.9)$$

as well as the average version of Duke's estimate (Theorem 5.6.1):

$$|\widetilde{n}| \sum_{r_j\le x} \frac{|\rho_j(n)|^2}{\operatorname{ch}\pi r_j} \ll |n|^{\frac{3}{7}+\varepsilon} x^{\frac{11}{2}}. \qquad (5.7.10)$$

Using (5.7.8) and (5.7.9) with the Cauchy-Schwarz inequality, we find that

$$\sqrt{|\widetilde{n}|} \sum_{r_j\le x} \frac{|\overline{\rho_j(1)}\rho_j(n)|}{\operatorname{ch}\pi r_j} \ll x^{\frac{3}{4}} \left(x^{\frac{5}{2}} + |n|^{\frac{1}{2}+\varepsilon} x^{\frac{1}{2}} \right)^{\frac{1}{2}} \ll x^2 + |n|^{\frac{1}{4}+\varepsilon} x.$$

Using (5.7.8) and (5.7.10) we obtain

$$\sqrt{|\widetilde{n}|} \sum_{r_j\le x} \frac{|\overline{\rho_j(1)}\rho_j(n)|}{\operatorname{ch}\pi r_j} \ll x^{\frac{3}{4}} \left(|n|^{\frac{3}{7}+\varepsilon} x^{\frac{11}{2}} \right)^{\frac{1}{2}} \ll |n|^{\frac{3}{14}+\varepsilon} x^{\frac{7}{2}}. \qquad (5.7.11)$$

In the second range Theorem 5.4.1 gives $\check{\phi}(r_j) \ll r_j^{-1}$. It follows from (5.7.11) (assuming as we may that $a/x \ge 1$) that

$$\sqrt{|\widetilde{n}|} \sum_{\frac{a}{8x}<r_j<\frac{a}{x}} \left| \frac{\overline{\rho_j(1)}\rho_j(n)}{\operatorname{ch}\pi r_j} \check{\phi}(r_j) \right| \ll |n|^{\frac{3}{14}+\epsilon} \left(\frac{a}{x} \right)^{\frac{5}{2}} \ll |n|^{\frac{41}{28}+\epsilon} x^{-\frac{5}{2}}. \qquad (5.7.12)$$

In the third range Theorem 5.4.1 gives

$$\check{\phi}(r_j) \ll \min\left(r_j^{-\frac{3}{2}},\ r_j^{-\frac{5}{2}}\frac{x}{T}\right).$$

We use dyadic sums corresponding to intervals $A \le r_j \le 2A$ with $A \ge \max\left(\frac{a}{x}, 1\right)$. For such a sum we have

$$\begin{aligned}\sqrt{|\widetilde{n}|} \sum_{A \le r_j \le 2A} \left| \frac{\overline{\rho_j(1)}\rho_j(n)}{\operatorname{ch} \pi r_j} \check{\phi}(r_j) \right| \\ &\ll \min\left(A^{-\frac{3}{2}},\ A^{-\frac{5}{2}}\frac{x}{T}\right) \min\left(A^2 + |n|^{\frac{1}{4}+\varepsilon}A,\ |n|^{\frac{3}{14}+\varepsilon}A^{\frac{7}{2}}\right) \\ &\ll \min\left(A^{\frac{1}{2}}, A^{-\frac{1}{2}}\frac{x}{T}\right)\left(1 + |n|^{\frac{13}{56}+\varepsilon}A^{\frac{1}{4}}\right) \\ &\ll |n|^{\frac{13}{56}+\varepsilon} \min\left(A^{\frac{3}{4}}, A^{-\frac{1}{4}}\frac{x}{T}\right),\end{aligned}$$

where we have used the fact that for positive B, C, and D we have

$$\min(B, C + D) \le \min(B, C) + \min(C, D) \qquad \text{and} \qquad \min(B, C) \le \sqrt{BC}.$$

Combining the dyadic sums, we find that

$$\sqrt{|\widetilde{n}|} \sum_{r_j \ge \frac{a}{x}} \left| \frac{\overline{\rho_j(1)}\rho_j(n)}{\operatorname{ch} \pi r_j} \check{\phi}(r_j) \right| \ll |n|^{\frac{13}{56}+\varepsilon} \left(\frac{x}{T}\right)^{\frac{3}{4}} \ll_\delta |n|^{\frac{13}{56}+\varepsilon} x^{\frac{3}{4}\delta}. \tag{5.7.13}$$

Using (5.7.6) with (5.7.7), (5.7.12), and (5.7.13) we obtain

$$\sum_{c=1}^{\infty} \frac{S(1, n, c, \chi)}{c} \check{\phi}\left(\frac{a}{c}\right) \ll_\delta |n|^{\frac{41}{28}+\varepsilon} x^{-\frac{5}{2}} + |n|^{\frac{13}{56}+\varepsilon} x^{\frac{3}{4}\delta}.$$

Proposition 5.7.2 follows from this estimate together with (5.7.5). □

We now use Theorem 5.7.1 to prove Theorems 1.2.1 and 1.2.2. Recall that we wish to bound $R(n, N)$, where

$$p(n) = \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \sum_{c=1}^{N} \frac{A_c(n)}{c} I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6c}\right) + R(n, N),$$

and

$$A_c(n) = \sqrt{-i}\, S(1, 1-n, c, \chi).$$

To match the notation of the previous sections, we assume that $n < 0$ satisfies (5.7.1) and we provide a bound for

$$R(1-n,N) = \frac{2\pi\sqrt{-i}}{24^{\frac{3}{4}}}|\widetilde{n}|^{-\frac{3}{4}} \sum_{c>N} \frac{S(1,n,c,\chi)}{c} I_{\frac{3}{2}}\left(\frac{a}{c}\right), \tag{5.7.14}$$

where

$$a := \sqrt{\frac{2}{3}}\,\pi\sqrt{|\widetilde{n}|}. \tag{5.7.15}$$

We will apply partial summation to (5.7.14). For fixed $\nu, M > 0$ and $0 \le z \le M$, the asymptotic formula [DL, (10.30.1)] gives

$$I_\nu(z) \ll_{\nu,M} z^\nu. \tag{5.7.16}$$

We have a straightforward lemma.

Lemma 5.7.3. *Suppose that $a > 0$ and that $\alpha > 0$. Then for $\frac{t}{a} \ge \alpha$ we have*

$$\big(I_{\frac{3}{2}}(a/t)\big)' \ll_\alpha a^{\frac{3}{2}} t^{-\frac{5}{2}}.$$

Proof. By [DL, (10.29.1)] we have

$$I'_{\frac{3}{2}}(t) = \tfrac{1}{2}\big(I_{\frac{1}{2}}(t) + I_{\frac{5}{2}}(t)\big).$$

It follows that

$$\big(I_{\frac{3}{2}}(a/t)\big)' = -\frac{a}{2t^2}\big(I_{\frac{1}{2}}(a/t) + I_{\frac{5}{2}}(a/t)\big). \tag{5.7.17}$$

For $\frac{t}{a} \ge \alpha$, equations (5.7.16) and (5.7.17) give

$$\big(I_{\frac{3}{2}}(a/t)\big)' \ll_\alpha a^{\frac{3}{2}} t^{-\frac{5}{2}} + a^{\frac{7}{2}} t^{-\frac{9}{2}},$$

and the lemma follows. □

Proof of Theorems 1.2.1 and 1.2.2. Let a be as in (5.7.15). Then

$$R(1-n,N) \ll |\widetilde{n}|^{-\frac{3}{4}} \left| \sum_{c>N} \frac{S(1,n,c,\chi)}{c} I_{\frac{3}{2}}\left(\frac{a}{c}\right) \right|.$$

Let

$$S(n,X) := \sum_{c \le X} \frac{S(1,n,c,\chi)}{c},$$

so that Theorem 5.7.1 (for $0 < \delta < 1/2$) gives

$$S(n, X) \ll_\delta |n|^{\frac{13}{56}+\varepsilon} X^{\frac{3}{4}\delta} + \left(|n|^{\frac{41}{168}+\varepsilon} + X^{\frac{1}{2}-\delta}\right) \log X.$$

By partial summation using (5.7.16) we have

$$\sum_{c>N} \frac{S(1, n, c, \chi)}{c} I_{\frac{3}{2}}\left(\frac{a}{c}\right) = -S(n, N) I_{\frac{3}{2}}(a/N) - \int_N^\infty S(n, t)\big(I_{\frac{3}{2}}(a/t)\big)' \, dt.$$

Let $\alpha > 0$ be fixed and take

$$N = \alpha |n|^{\frac{1}{2}+\beta}$$

where $\beta \in [0, 1/2]$ is to be chosen. By (5.7.16) we have

$$I_{\frac{3}{2}}(a/N) \ll_\alpha |n|^{\frac{3}{4}} N^{-\frac{3}{2}}$$

and by Lemma 5.7.3 we have

$$\big(I_{\frac{3}{2}}(a/t)\big)' \ll_\alpha |n|^{\frac{3}{4}} t^{-\frac{5}{2}} \qquad \text{for } t \geq N.$$

Thus

$$S(n, N) I_{\frac{3}{2}}(a/N) \ll_{\delta,\alpha} |n|^{\frac{3}{4}} N^{-\frac{3}{2}} \left(|n|^{\frac{13}{56}} N^{\frac{3}{4}\delta} + |n|^{\frac{41}{168}} + N^{\frac{1}{2}-\delta}\right) |n|^{\varepsilon},$$

and the same bound holds for the integral term. Therefore

$$R(1-n, \alpha|n|^{\frac{1}{2}+\beta}) \ll_{\delta,\alpha} |n|^{(\frac{1}{2}+\beta)(\frac{3}{4}\delta-\frac{3}{2})+\frac{13}{56}+\varepsilon} + |n|^{-\frac{3}{2}(\frac{1}{2}+\beta)+\frac{41}{168}+\varepsilon} + |n|^{-(\frac{1}{2}+\beta)(1+\delta)+\varepsilon}. \quad (5.7.18)$$

When $\beta = 0$, the estimate (5.7.18) becomes

$$R(1-n, \alpha|n|^{\frac{1}{2}}) \ll_{\delta,\alpha} |n|^{\frac{3}{8}\delta-\frac{29}{56}+\varepsilon} + |n|^{-\frac{85}{168}+\varepsilon} + |n|^{-\frac{1}{2}-\frac{1}{2}\delta+\varepsilon}.$$

For any choice of $\delta \in [\frac{1}{84}, \frac{2}{63}]$ this gives

$$R(1-n, \alpha|n|^{\frac{1}{2}}) \ll_\alpha |n|^{-\frac{1}{2}-\frac{1}{168}+\varepsilon}.$$

Theorem 1.2.1 follows after replacing n by $1-n$.

To optimize, we choose $\beta = \frac{5}{252}$ and $\delta = \frac{4}{131}$, which makes all three exponents in (5.7.18) equal to $-\frac{1}{2} - \frac{1}{28}$. This gives Theorem 1.2.2. $\square$

6 The mock theta conjectures

In this chapter we prove the mock theta conjectures. The material in this chapter and in Section 1.3 of the Introduction appear in the paper [A3].

6.1 Definitions and transformations

In this section, we define the functions $M(\frac{a}{5}, q)$, $\theta_4(0, q)$, $\psi(q)$, $G(q)$, and $H(q)$ and describe the transformation behavior for these functions and the mock theta functions under the generators

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

of $\mathrm{SL}_2(\mathbb{Z})$. We employ the $\big|_k$ notation, defined for $k \in \frac{1}{2}\mathbb{Z}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ by

$$(f\big|_k \gamma)(z) := (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

For $k \in \mathbb{Z}$ we have $f\big|_k AB = f\big|_k A\big|_k B$, but for general k we have

$$f\big|_k AB = e^{ik(\arg j_A(Bi) + \arg j_B(i) - \arg j_{AB}(i))} f\big|_k A\big|_k B, \tag{6.1.1}$$

where $j_\gamma(z) = cz + d$ for $\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$ (see [I2, §2.6]). We always take $\arg z \in (-\pi, \pi]$. Much of the arithmetic here and throughout the paper takes place in the splitting field of the polynomial $x^4 - 5x^2 + 5$, which has roots

$$\alpha := \sqrt{\frac{5-\sqrt{5}}{2}} \quad \text{and} \quad \beta := \sqrt{\frac{5+\sqrt{5}}{2}}. \tag{6.1.2}$$

We begin by giving the modular transformations satisfied by the mock theta functions f_0, f_1, F_0, and F_1 which are given in Section 4.4 of [Zw]. The nonholomorphic completions are given in terms of the integral (see [Zw, Proposition 4.2])

$$R_{a,b}(z) := -i \int_{-\bar{z}}^{i\infty} \frac{g_{a,-b}(\tau)}{\sqrt{-i(\tau+z)}} d\tau,$$

where $g_{a,b}$ (see [Zw, §1.5]) is the unary theta function

$$g_{a,b}(z) := \sum_{\nu \in a+\mathbb{Z}} \nu e^{\pi i \nu^2 z + 2\pi i \nu b}.$$

We will simplify the components of $G_{5,1}(\tau)$ on page 75 of [Zw] by using the relation

$$g_{a,0}(z) - g_{a+\frac{1}{2},0}(z) = \frac{1}{2} e^{-2\pi i a} g_{2a,\frac{1}{2}}(z/4)$$

and Proposition 1.15 of [Zw]. As usual, $q := \exp(2\pi i z)$. We define

$$\widetilde{f_0}(z) := q^{-\frac{1}{60}} f_0(q) - \zeta_{10} \left(\zeta_{12}^{-1} R_{\frac{1}{30},\frac{1}{2}} + \zeta_{12} R_{\frac{11}{30},\frac{1}{2}} \right)(30z), \tag{6.1.3}$$

$$\widetilde{f_1}(z) := q^{\frac{11}{60}} f_1(q) - \zeta_5 \left(\zeta_{12}^{-1} R_{\frac{7}{30},\frac{1}{2}} + \zeta_{12} R_{\frac{17}{30},\frac{1}{2}} \right)(30z), \tag{6.1.4}$$

$$\widetilde{F_0}(z) := q^{-\frac{1}{120}} (F_0(q) - 1) + \frac{1}{2} \zeta_{10} \left(\zeta_{12}^{-1} R_{\frac{1}{30},\frac{1}{2}} + \zeta_{12} R_{\frac{11}{30},\frac{1}{2}} \right)(15z), \tag{6.1.5}$$

$$\widetilde{F_1}(z) := q^{\frac{71}{120}} F_1(q) + \frac{1}{2} \zeta_5 \left(\zeta_{12}^{-1} R_{\frac{7}{30},\frac{1}{2}} + \zeta_{12} R_{\frac{17}{30},\frac{1}{2}} \right)(15z). \tag{6.1.6}$$

The following is Proposition 4.10 of [Zw]. The vector (6.1.7) below equals the vector $F_{5,1}(\tau) - G_{5,1}(\tau)$ of that paper (some computation is required to see this for the fifth and sixth components).

Proposition 6.1.1. *The vector*

$$\boldsymbol{F}(z) := \left(\widetilde{f_0}(z),\ \widetilde{f_1}(z),\ \widetilde{F_0}\left(\tfrac{z}{2}\right),\ \widetilde{F_1}\left(\tfrac{z}{2}\right),\ \zeta_{240} \widetilde{F_0}\left(\tfrac{z+1}{2}\right),\ \zeta_{240}^{-71} \widetilde{F_1}\left(\tfrac{z+1}{2}\right) \right)^{\mathsf{T}} \tag{6.1.7}$$

satisfies the transformations

$$\boldsymbol{F}\big|_{\frac{1}{2}} T = M_T \boldsymbol{F} \quad \text{and} \quad \boldsymbol{F}\big|_{\frac{1}{2}} S = \zeta_8^{-1} \sqrt{\frac{2}{5}} M_S \boldsymbol{F},$$

where

$$M_T = \begin{pmatrix} \zeta_{60}^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta_{60}^{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta_{240}^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_{240}^{71} \\ 0 & 0 & \zeta_{240}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_{240}^{71} & 0 & 0 \end{pmatrix}$$

and

$$M_S = \begin{pmatrix} 0 & 0 & \alpha & \beta & 0 & 0 \\ 0 & 0 & \beta & -\alpha & 0 & 0 \\ \frac{\alpha}{2} & \frac{\beta}{2} & 0 & 0 & 0 & 0 \\ \frac{\beta}{2} & -\frac{\alpha}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\beta}{\sqrt{2}} & \frac{\alpha}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & \frac{\alpha}{\sqrt{2}} & -\frac{\beta}{\sqrt{2}} \end{pmatrix}.$$

Next we define the functions on the right-hand side of (1.3.1)–(1.3.4) and give their transformation properties. Following [BO2, GM1], we define, for $a \in \{1, 2, 3, 4\}$, the functions

$$M\left(\frac{a}{5}, z\right) := \sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(q^{\frac{a}{5}}; q)_n (q^{1-\frac{a}{5}}; q)_n}, \tag{6.1.8}$$

$$N\left(\frac{a}{5}, z\right) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(\zeta_5^a q; q)_n (\zeta_5^{-a} q; q)_n}. \tag{6.1.9}$$

Clearly we have $M(1-\frac{a}{5}, z) = M(\frac{a}{5}, z)$ and $N(1-\frac{a}{5}, z) = N(\frac{a}{5}, z)$. Bringmann and Ono [BO2] also define auxiliary functions $M(a, b, 5, z)$ and $N(a, b, 5, z)$ for $0 \leq a \leq 4$ and $1 \leq b \leq 4$. Together, the completed versions of these functions form a set that is closed (up to multiplication by roots of unity) under the action of $\mathrm{SL}_2(\mathbb{Z})$ (see [BO2, Theorem 3.4]). Garvan [G] made the definitions of these functions and their transformations more explicit, so in what follows we reference his paper.

The nonholomorphic completions for $M(\frac{a}{5}, z)$ and $N(\frac{a}{5}, z)$ are given in terms of integrals of weight 3/2 theta functions $\Theta_1(\frac{a}{5}, z)$ and $\Theta_1(0, -a, 5, z)$ (defined in Section 2 of [G]). A straightforward computation shows that

$$\Theta_1(0, -a, 5, z) = 15\sqrt{3}\,\zeta_{10}^a \left(\zeta_{12}^{-1}\, g_{\frac{6a-5}{30}, -\frac{1}{2}}(3z) + \zeta_{12}\, g_{\frac{6a+5}{30}, -\frac{1}{2}}(3z)\right).$$

Following (2.1), (2.2), (3.5), and (3.6) of [G], we define

$$\widetilde{M}\left(\frac{a}{5}, z\right) := 2q^{\frac{3a}{10}\left(1-\frac{a}{5}\right)-\frac{1}{24}} M\left(\frac{a}{5}, z\right) + \zeta_{10}^{a}\left(\zeta_{12}^{-1} R_{\frac{6a-5}{30}, \frac{1}{2}} + \zeta_{12} R_{\frac{6a+5}{30}, \frac{1}{2}}\right)(3z), \tag{6.1.10}$$

$$\widetilde{N}\left(\frac{a}{5}, z\right) := \csc\left(\frac{a\pi}{5}\right) q^{-\frac{1}{24}} N\left(\frac{a}{5}, z\right) + \frac{i}{\sqrt{3}} \int_{-\overline{z}}^{i\infty} \frac{\Theta_1(\frac{a}{5}, \tau)}{\sqrt{-i(\tau+z)}}\, d\tau. \tag{6.1.11}$$

The functions $\widetilde{M}(a, b, z) := \mathcal{G}_2(a, b, 5; z)$ and $\widetilde{N}(a, b, z) := \mathcal{G}_1(a, b, 5; z)$ are defined in (3.7) and (3.8) of that paper. By Theorems 3.1 and 3.2 of [G] we have

$$\widetilde{M}\left(\frac{a}{5}, z\right)\big|_{\frac{1}{2}} T^5 = \widetilde{M}\left(\frac{a}{5}, z\right) \times \begin{cases} \zeta_{120}^{-1} & \text{if } a = 1, \\ \zeta_{120}^{71} & \text{if } a = 2, \end{cases} \tag{6.1.12}$$

$$\widetilde{N}\left(\frac{a}{5}, z\right)\big|_{\frac{1}{2}} T = \zeta_{24}^{-1} \widetilde{N}\left(\frac{a}{5}, z\right), \tag{6.1.13}$$

and

$$\widetilde{M}\left(\frac{a}{5}, z\right)\big|_{\frac{1}{2}} S = \zeta_{8}^{-1} \widetilde{N}\left(\frac{a}{5}, z\right). \tag{6.1.14}$$

The theta functions $\theta_4(0, q)$ and $\psi(q)$ are defined by

$$\theta_4(0, q) := \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty} = \frac{\eta^2(z)}{\eta(2z)},$$
$$\psi(q) := \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} = q^{-\frac{1}{8}} \frac{\eta^2(2z)}{\eta(z)},$$

where $\eta(z) = q^{1/24}(q;q)_\infty$ is the Dedekind eta function. The transformation properties of these functions are easily obtained using the well-known transformation

$$\eta(-1/z) = \sqrt{-iz}\, \eta(z). \tag{6.1.15}$$

The Rogers-Ramanujan functions are defined by

$$G(q) := \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty},$$
$$H(q) := \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

It will be more convenient for us to use the functions

$$g(z) := q^{-\frac{1}{60}}G(q) \quad \text{and} \quad h(z) := q^{\frac{11}{60}}H(q). \tag{6.1.16}$$

They satisfy the transformations (see [GM1, p. 207])

$$g\big|_0 S = \alpha^{-1}g + \beta^{-1}h, \tag{6.1.17}$$

$$h\big|_0 S = \beta^{-1}g - \alpha^{-1}h. \tag{6.1.18}$$

Using the completed functions, the mock theta conjectures (1.3.1)–(1.3.4) are implied by the corresponding completed versions:

$$\widetilde{f}_0(z) = -\widetilde{M}\left(\frac{1}{5}, 10z\right) + \frac{\eta^2(5z)}{\eta(10z)}g(z), \tag{6.1.19}$$

$$\widetilde{f}_1(z) = -\widetilde{M}\left(\frac{2}{5}, 10z\right) + \frac{\eta^2(5z)}{\eta(10z)}h(z), \tag{6.1.20}$$

$$\widetilde{F}_0(z) = \frac{1}{2}\,\widetilde{M}\left(\frac{1}{5}, 5z\right) - \frac{\eta^2(10z)}{\eta(5z)}h(2z), \tag{6.1.21}$$

$$\widetilde{F}_1(z) = \frac{1}{2}\,\widetilde{M}\left(\frac{2}{5}, 5z\right) + \frac{\eta^2(10z)}{\eta(5z)}g(2z). \tag{6.1.22}$$

Motivated by (6.1.7) and (6.1.19)–(6.1.22), we define the vector

$$\boldsymbol{G}(z) := \begin{pmatrix} -\widetilde{M}\left(\frac{1}{5}, 10z\right) + \frac{\eta^2(5z)}{\eta(10z)}g(z) \\ -\widetilde{M}\left(\frac{2}{5}, 10z\right) + \frac{\eta^2(5z)}{\eta(10z)}h(z) \\ \frac{1}{2}\,\widetilde{M}\left(\frac{1}{5}, \frac{5z}{2}\right) - \frac{\eta^2(5z)}{\eta(\frac{5z}{2})}h(z) \\ \frac{1}{2}\,\widetilde{M}\left(\frac{2}{5}, \frac{5z}{2}\right) + \frac{\eta^2(5z)}{\eta(\frac{5z}{2})}g(z) \\ \frac{1}{2}\,\zeta_{240}\,\widetilde{M}\left(\frac{1}{5}, \frac{5z+5}{2}\right) - \zeta_{48}^{25}\frac{\eta^2(5z)}{\eta(\frac{5z+1}{2})}h(z) \\ \frac{1}{2}\,\zeta_{240}^{-71}\widetilde{M}\left(\frac{2}{5}, \frac{5z+5}{2}\right) + \zeta_{48}\frac{\eta^2(5z)}{\eta(\frac{5z+1}{2})}g(z) \end{pmatrix}, \tag{6.1.23}$$

where we have used $\eta(z+1) = \zeta_{24}\eta(z)$, $g(z+1) = \zeta_{60}^{-1}g(z)$, and $h(z+1) = \zeta_{60}^{11}h(z)$ to simplify the second terms of the fifth and sixth components.

To prove that $\boldsymbol{F} = \boldsymbol{G}$ we begin by showing that they transform in the same way.

Proposition 6.1.2. *The vector $\boldsymbol{G}(z)$ defined in* (6.1.23) *satisfies the transformations*

$$\boldsymbol{G}\big|_{\frac{1}{2}} T = M_T\, \boldsymbol{G} \quad \textit{and} \quad \boldsymbol{G}\big|_{\frac{1}{2}} S = \zeta_8^{-1} \sqrt{\frac{2}{5}}\, M_S\, \boldsymbol{G}, \tag{6.1.24}$$

where M_T and M_S are as in Proposition 6.1.1.

Before proving this proposition, we state two identities that will be indispensable in the proof. Equivalent identities can be found in [GM1, (3.8) and (3.9)], where they are proved using q-series methods. In Section 6.5 we provide a purely modular proof.

Lemma 6.1.3. *Let α and β be as in* (6.1.2)*. Then*

$$\begin{aligned}\widetilde{N}\left(\frac{1}{5}, z\right) + \alpha\, \widetilde{M}\left(\frac{1}{5}, 25z\right) + \beta\, \widetilde{M}\left(\frac{2}{5}, 25z\right) \\ = 2\frac{\eta^2(2z)}{\eta(z)} \left(\alpha^{-1} g(10z) + \beta^{-1} h(10z)\right) \\ - 2\,\frac{\eta^2(50z)}{\eta(25z)} \left(\beta\, g(10z) - \alpha\, h(10z)\right).\end{aligned} \tag{6.1.25}$$

Before proving Lemma 6.1.3 we deduce an immediate consequence. Note that the right-hand side of (6.1.25) is holomorphic; this implies that the non-holomorphic completion terms on the left-hand side sum to zero. By (6.1.9), the coefficients of $N(\frac{a}{5}, z)$ lie in $\mathbb{Q}(\zeta_5 + \zeta_5^{-1}) = \mathbb{Q}(\sqrt{5})$, and the automorphism $\sigma = (\sqrt{5} \mapsto -\sqrt{5})$ maps $N(\frac{1}{5}, z)$ to $N(\frac{2}{5}, z)$. By (6.1.11) and the fact that

$$\csc\left(\frac{\pi a}{5}\right) = \begin{cases} 2\alpha^{-1} & \text{if } a = 1, \\ 2\beta^{-1} & \text{if } a = 2, \end{cases}$$

it follows that the coefficients of both sides of (6.1.25) lie in $\mathbb{Q}(\alpha)$. The Galois group of $\mathbb{Q}(\alpha)$ is cyclic of order 4, generated by $\tau = (\alpha \mapsto \beta, \beta \mapsto -\alpha)$. Since $\sqrt{5} = \alpha\beta$, we have $\tau\,|_{Q(\sqrt{5})} = \sigma$. Applying τ to Lemma 6.1.3 gives the following identity.

Lemma 6.1.4. *Let α and β be as in* (6.1.2). *Then*

$$\begin{aligned}\widetilde{N}\left(\frac{2}{5},z\right)+\beta\,\widetilde{M}\left(\frac{1}{5},25z\right)-\alpha\,\widetilde{M}\left(\frac{2}{5},25z\right)&\\ =2\frac{\eta^2(2z)}{\eta(z)}\left(\beta^{-1}g(10z)-\alpha^{-1}h(10z)\right)&\\ +2\,\frac{\eta^2(50z)}{\eta(25z)}\left(\alpha\,g(10z)+\beta\,h(10z)\right).&\end{aligned}\tag{6.1.26}$$

Proof of Proposition 6.1.2. The transformation with respect to T follows immediately from (6.1.12).

Let $G_j(z)$ denote the j-th component of $\boldsymbol{G}(z)$. By (6.1.14), (6.1.15), and (6.1.17) we have

$$\begin{aligned}G_1(z)\big|_{\frac{1}{2}}S&=\zeta_8^{-1}\sqrt{\frac{2}{5}}\left(-\frac{1}{2}\,\widetilde{N}\left(\frac{1}{5},\frac{z}{10}\right)+\frac{\eta^2(\frac{z}{5})}{\eta(\frac{z}{10})}\left(\alpha^{-1}g(z)+\beta^{-1}h(z)\right)\right)\\&=\zeta_8^{-1}\sqrt{\frac{2}{5}}\left(\alpha\,G_3(z)+\beta\,G_4(z)\right),\end{aligned}$$

where we used Lemma 6.1.3 with z replaced by $\frac{z}{10}$ in the second line. For G_2, the situation is analogous, using Lemma 6.1.4. For G_3 and G_4 we note that the transformations for G_1 and G_2 imply that

$$\begin{aligned}G_3&=\zeta_8\sqrt{\frac{2}{5}}\,\left(\frac{\alpha}{2}\,G_1\big|_{\frac{1}{2}}S+\frac{\beta}{2}\,G_2\big|_{\frac{1}{2}}S\right),\\ G_4&=\zeta_8\sqrt{\frac{2}{5}}\,\left(\frac{\beta}{2}\,G_1\big|_{\frac{1}{2}}S-\frac{\alpha}{2}\,G_2\big|_{\frac{1}{2}}S\right).\end{aligned}$$

By (6.1.1) we have $f\big|_{\frac{1}{2}}S\big|_{\frac{1}{2}}S=f\big|_{\frac{1}{2}}(-I)=-if$, and we obtain the transformations for G_3 and G_4.

For G_5, we first observe that

$$\begin{pmatrix}1&5\\0&2\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}=T^3ST^2S\begin{pmatrix}1&1\\0&2\end{pmatrix}.$$

Using Theorems 3.1 and 3.2 of [G] we compute

$$\begin{aligned}\widetilde{M}\left(\frac{1}{5},z\right)\big|_{\frac{1}{2}}T^3ST^2S &= (-\zeta_{50}^{-3}\zeta_{24}^{-1})^3\widetilde{M}(1,3,z)\big|_{\frac{1}{2}}ST^2S\\ &= \zeta_{200}^{39}\zeta_8^{-1}\widetilde{N}(1,3,z)\big|_{\frac{1}{2}}T^2S\\ &= \zeta_{30}^{-7}\widetilde{N}(0,3,z)\big|_{\frac{1}{2}}S = \zeta_{120}^{77}\,\widetilde{M}(0,3,z).\end{aligned}$$

By (4.12) of [G] this equals $\zeta_{120}^{47}\,\widetilde{N}(\frac{2}{5},z)$, so we conclude that

$$\widetilde{M}\left(\frac{1}{5},\frac{5z+5}{2}\right)\big|_{\frac{1}{2}}S = \zeta_{120}^{47}\,\frac{1}{\sqrt{5}}\,\widetilde{N}\left(\frac{2}{5},\frac{z/5+1}{2}\right). \tag{6.1.27}$$

Similarly, we have $\eta(\frac{z+1}{2})\big|_{\frac{1}{2}}S = \zeta_8^{-1}\,\eta(\frac{z+1}{2})$ which, together with (6.1.15) and (6.1.18) gives

$$\zeta_{48}^{25}\frac{\eta^2(5z)}{\eta(\frac{5z+1}{2})}h(z)\big|_{\frac{1}{2}}S = \zeta_{48}^{19}\,\frac{1}{\sqrt{5}}\,\frac{\eta^2(\frac{z}{5})}{\eta(\frac{z}{10}+\frac{1}{2})}\left(\beta^{-1}g(z)-\alpha^{-1}h(z)\right). \tag{6.1.28}$$

Replacing z by $\frac{z}{10}+\frac{1}{2}$ in Lemma 6.1.4 and using (6.1.12) yields

$$\begin{aligned}&\widetilde{N}\left(\frac{2}{5},\frac{z}{10}+\frac{1}{2}\right)+\beta\zeta_{60}^{-1}\,\widetilde{M}\left(\frac{1}{5},\frac{5z+5}{2}\right)-\alpha\zeta_{60}^{11}\,\widetilde{M}\left(\frac{2}{5},\frac{5z+5}{2}\right)\\ &\quad= 2\frac{\eta^2(\frac{z}{5})}{\eta(\frac{z}{10}+\frac{1}{2})}\left(\beta^{-1}g(z)-\alpha^{-1}h(z)\right)-2\frac{\eta^2(5z)}{\eta(\frac{5z+1}{2})}\left(\alpha g(z)+\beta h(z)\right).\end{aligned} \tag{6.1.29}$$

Putting (6.1.27), (6.1.28), and (6.1.29) together we obtain

$$\begin{aligned}&\frac{1}{2}\zeta_{240}\,\widetilde{M}\left(\frac{1}{5},\frac{5z+5}{2}\right)-\zeta_{48}^{25}\frac{\eta^2(5z)}{\eta(\frac{5z+1}{2})}h(z)\big|_{\frac{1}{2}}S\\ &\qquad= \frac{1}{\sqrt{5}}\zeta_8^{-1}\Bigg(\zeta_{240}\,\frac{\beta}{2}\,\widetilde{M}\left(\frac{1}{5},\frac{5z+5}{2}\right)+\zeta_{240}^{-71}\,\frac{\alpha}{2}\,\widetilde{M}\left(\frac{2}{5},\frac{5z+5}{2}\right)\\ &\qquad\qquad -\zeta_{48}^{25}\frac{\eta^2(5z)}{\eta(\frac{5z+1}{2})}\left(\alpha\,g(z)+\beta\,h(z)\right)\Bigg)\\ &\qquad= \frac{1}{\sqrt{5}}\zeta_8^{-1}\left(\beta\,G_5(z)+\alpha\,G_6(z)\right).\end{aligned}$$

The transformation for $G_6(z)$ is similarly obtained by using Lemma 3. □

6.2 Vector-valued modular forms and the Weil representation

In this section we define vector-valued modular forms which transform according to the Weil representation, and we construct such a form from the components of $\boldsymbol{F} - \boldsymbol{G}$. A good reference for this material is [Br, §1.1].

Let $L = \mathbb{Z}$ be the lattice with associated bilinear form $(x, y) = -120xy$ and quadratic form $q(x) = -60x^2$. The dual lattice is $L' = \frac{1}{120}\mathbb{Z}$. Let $\{\mathfrak{e}_h : \frac{h}{120} \in \frac{1}{120}\mathbb{Z}/\mathbb{Z}\}$ denote the standard basis for $\mathbb{C}[L'/L]$. Let $\mathrm{Mp}_2(\mathbb{R})$ denote the metaplectic two-fold cover of $\mathrm{SL}_2(\mathbb{R})$; the elements of this group are pairs (M, ϕ), where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $\phi^2(z) = cz + d$. Let $\mathrm{Mp}_2(\mathbb{Z})$ denote the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under the covering map; this group is generated by

$$\widetilde{T} := (T, 1) \quad \text{and} \quad \widetilde{S} := (S, \sqrt{z}).$$

The Weil representation can be defined by its action on these generators, namely

$$\rho_L(T, 1)\mathfrak{e}_h := \zeta_{240}^{-h^2}\,\mathfrak{e}_h, \tag{6.2.1}$$

$$\rho_L(S, \sqrt{z})\mathfrak{e}_h := \frac{1}{\sqrt{-120i}} \sum_{h'(120)} \zeta_{120}^{hh'}\,\mathfrak{e}_{h'}. \tag{6.2.2}$$

A holomorphic function $\mathcal{F} : \mathbb{H} \to \mathbb{C}[L'/L]$ is a vector-valued modular form of weight $\frac{1}{2}$ and representation ρ_L if

$$\mathcal{F}(\gamma z) = \phi(z)\rho_L(\gamma, \phi)\mathcal{F}(z) \qquad \text{for all } (\gamma, \phi) \in \mathrm{Mp}_2(\mathbb{Z}) \tag{6.2.3}$$

and if $\mathcal{F}$ is holomorphic at ∞ (i.e. if the components of $\mathcal{F}$ are holomorphic at ∞ in the usual sense). The following lemma shows how to construct such forms from vectors that transform as in Propositions 6.1.1 and 6.1.2.

Lemma 6.2.1. *Suppose that $\boldsymbol{H} = (H_1, \ldots, H_6)$ satisfies*

$$\boldsymbol{H}\big|_{\frac{1}{2}} T = M_T\,\boldsymbol{H} \quad \textit{and} \quad \boldsymbol{H}\big|_{\frac{1}{2}} S = \zeta_8^{-1}\sqrt{\frac{2}{5}}\, M_S\,\boldsymbol{H},$$

where M_T and M_S are as in Proposition 6.1.1, and define

$$\mathcal{H}(z) := \sum_{\substack{0<h<60 \\ h\equiv\pm1(10) \\ (h,60)=1}} \left(a_h H_3(z) + b_h H_5(z)\right)(\mathfrak{e}_h - \mathfrak{e}_{-h}) - \sum_{\substack{0<h<60 \\ h\equiv\pm2(10) \\ (h,60)=2}} H_1(z)(\mathfrak{e}_h - \mathfrak{e}_{-h})$$
$$+ \sum_{\substack{0<h<60 \\ h\equiv\pm3(10) \\ (h,60)=1}} \left(a_h H_4(z) + b_h H_6(z)\right)(\mathfrak{e}_h - \mathfrak{e}_{-h}) - \sum_{\substack{0<h<60 \\ h\equiv\pm4(10) \\ (h,60)=2}} H_2(z)(\mathfrak{e}_h - \mathfrak{e}_{-h}),$$

where

$$a_h = \begin{cases} +1 & \text{if } 0 < h < 30, \\ -1 & \text{otherwise,} \end{cases}$$

and

$$b_h = \begin{cases} +1 & \text{if } h \equiv \pm1, \pm13 \pmod{60}, \\ -1 & \text{otherwise.} \end{cases}$$

Then $\mathcal{H}(z)$ satisfies (6.2.3)*.*

Proof. The proof is a straightforward but tedious verification involving (6.2.1) and (6.2.2) that is best carried out with the aid of a computer algebra system; the author used MATHEMATICA. □

6.3 Proof of the mock theta conjectures

Let $\boldsymbol{F}$ and $\boldsymbol{G}$ be as in Section 6.1. To prove (6.1.19)–(6.1.22) it suffices to prove that $\boldsymbol{F} = \boldsymbol{G}$. Let $\boldsymbol{H} := \boldsymbol{F} - \boldsymbol{G}$. It is easy to see that the nonholomorphic parts of $\boldsymbol{F}$ and $\boldsymbol{G}$ agree, as do the terms in the Fourier expansion involving negative powers of q. It follows that the function $\mathcal{H}$ defined in Lemma 6.2.1 is a vector-valued modular form of weight $\frac{1}{2}$ with representation ρ_L. By Theorem 5.1 of [EZ], the space of such forms is canonically isomorphic to the space $J_{1,60}$ of Jacobi forms of weight 1 and index 60. By a theorem of Skoruppa [Sk, Satz 6.1] (see also [EZ, Theorem 5.7]), we have $J_{1,m} = \{0\}$ for all m; therefore $\mathcal{H} = 0$. The mock theta conjectures (1.3.1)–(1.3.4) follow. □

6.4 The six remaining identities

Four of the six remaining identities, those involving the mock theta functions ψ_0, ψ_1, ϕ_0, and ϕ_1 (see [AG] for definitions), can be proved using the methods of Sections 6.1–6.3. For suitable completed nonholomorphic functions $\widetilde{\psi}_0$, $\widetilde{\psi}_1$, $\widetilde{\phi}_0$, and $\widetilde{\phi}_1$, these conjectures are (see [GM1, p. 206])

$$2\,\widetilde{\psi}_0(z) = \widetilde{M}\left(\frac{1}{5}, 10z\right) + 2\,\eta_{10,1}(z)\eta(10z)h(z), \tag{6.4.1}$$

$$2\,\widetilde{\psi}_1(z) = \widetilde{M}\left(\frac{2}{5}, 10z\right) + 2\,\eta_{10,3}(z)\eta(10z)g(z), \tag{6.4.2}$$

$$\widetilde{\phi}_0(z) = -\frac{1}{2}\,\widetilde{M}\left(\frac{1}{5}, 5z\right) + \frac{\eta(5z)\eta(2z)}{\eta(10z)}g^2(2z)h(z), \tag{6.4.3}$$

$$-\widetilde{\phi}_1(z) = -\frac{1}{2}\,\widetilde{M}\left(\frac{2}{5}, 5z\right) + \frac{\eta(5z)\eta(2z)}{\eta(10z)}h^2(2z)g(z), \tag{6.4.4}$$

where $\eta_{r,t}(z)$ is defined in [Rob2]. Here we have used that $g(z)h(z) = \frac{\eta(5z)}{\eta(z)}$ in the third and fourth formulas. Following Section 6.1, we construct two six-dimensional vectors $\boldsymbol{F}_1$ and $\boldsymbol{G}_1$ out of the functions on the left-hand and right-hand sides, respectively, of (6.4.1)–(6.4.4). The transformation properties of $\boldsymbol{F}_1$ are given in Proposition 4.13 of [Zw], and the corresponding properties of $\boldsymbol{G}_1$ follow from an argument similar to that given in the proof of Proposition 6.1.2. For the latter argument, we use the following identity (together with the identity obtained by applying the automorphism τ as in Lemma 6.1.4):

$$\begin{aligned}
&\widetilde{N}\left(\frac{1}{5}, 2z\right) + \alpha\,\widetilde{M}\left(\frac{1}{5}, 50z\right) + \beta\,\widetilde{M}\left(\frac{2}{5}, 50z\right) \\
&\quad = 2\,\frac{\eta(2z)\eta(5z)}{\eta(z)}\left(\alpha^{-1}g(5z) + \beta^{-1}h(5z)\right)^2\left(\alpha\,g(10z) - \beta\,h(10z)\right) \\
&\qquad - 2\eta(50z)\big(\alpha\,\eta_{10,1}(5z)h(5z) + \beta\,\eta_{10,3}(5z)g(5z)\big).
\end{aligned} \tag{6.4.5}$$

The proof of (6.4.5) is analogous to the proof of Lemma 6.1.3, and requires the transformation properties of $\eta_{10,1}$ and $\eta_{10,3}$, given in [Rob2]. The proof that $\boldsymbol{F}_1 = \boldsymbol{G}_1$ follows exactly as in Section 6.3.

The two remaining identities involve the mock theta functions χ_0 and χ_1. Using the relations (discovered by Ramanujan and proved by Watson [Wa,

(B_0) and (B_1)])

$$\chi_0(q) = 2F_0(q) - \phi_0(-q),$$
$$\chi_1(q) = 2F_1(q) + q^{-1}\phi_1(-q),$$

these mock theta conjectures (see [GM1, p. 206]) are implied by the identities

$$2\widetilde{F}_0(z) - \widetilde{\phi}_0(z) = \frac{3}{2}\,\widetilde{M}\left(\frac{1}{5}, 5z\right) - \eta(5z)\frac{g^2(z)}{h(z)}, \tag{6.4.6}$$

$$2\widetilde{F}_1(z) + \widetilde{\phi}_1(z) = \frac{3}{2}\,\widetilde{M}\left(\frac{2}{5}, 5z\right) + \eta(5z)\frac{h^2(z)}{g(z)}. \tag{6.4.7}$$

By (6.1.21), (6.1.22), (6.4.3), and (6.4.4), equations (6.4.6) and (6.4.7) follow from the identities (see [Rob1, (1.25) and (1.26)] for a proof using modular forms)

$$g^2(z)h(2z) - h^2(z)g(2z) = 2h(z)h^2(2z)\frac{\eta^2(10z)}{\eta^2(5z)},$$
$$g^2(z)h(2z) + h^2(z)g(2z) = 2g(z)g^2(2z)\frac{\eta^2(10z)}{\eta^2(5z)}.$$

This completes the proof of the remaining mock theta conjectures.

6.5 Proof of Lemma 6.1.3

Let $L(z)$ and $R(z)$ denote the left-hand and right-hand sides of (6.1.25), respectively. Let Γ denote the congruence subgroup

$$\Gamma = \Gamma_0(50) \cap \Gamma_1(5) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \bmod 50 \text{ and } a, d \equiv 1 \bmod 5 \right\}.$$

We claim that

$$\eta(z)L(z), \eta(z)R(z) \in M_1(\Gamma), \tag{6.5.1}$$

where $M_k(G)$ (resp. $M_k^!(G)$) denotes the space of holomorphic (resp. weakly holomorphic) modular forms of weight k on $G \subseteq \mathrm{SL}_2(\mathbb{Z})$. We have

$$\frac{1}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma] = 15,$$

so once (6.5.1) is established it suffices to check that the first 16 coefficients of $\eta(z)L(z)$ and $\eta(z)R(z)$ agree. A computation shows that the Fourier expansion of each function begins

$$\frac{2}{\sqrt{5}}\big(\beta - \alpha^2\beta\, q^2 - \alpha\beta^2\, q^3 + \beta^3\, q^5 - \alpha^2\beta\, q^7 \\ + 2\alpha^2\beta\, q^{10} - \alpha^2\beta\, q^{12} - \alpha\beta^2\, q^{13} + 2\alpha\beta^2\, q^{15} + \dots\big).$$

To prove (6.5.1), we first note that Theorem 5.1 of [G] shows that $\eta(25z)L(z) \in M_1^!(\Gamma_0(25) \cap \Gamma_1(5))$; since $\eta(z)/\eta(25z) \in M_0^!(\Gamma_0(25))$ it follows that $\eta(z)L(z) \in M_1^!(\Gamma)$. Suppose that $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, and let

$$\gamma_n := \begin{pmatrix} a & nb \\ c/n & d \end{pmatrix}.$$

Then $\left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right)\gamma = \gamma_n \left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right)$. By a result of Biagioli [B, Proposition 2.5] we have

$$g(10z)\big|_0\gamma = v_\eta^{14}(\gamma_{10})\, g(10z) \quad \text{and} \quad h(10z)\big|_0\gamma = v_\eta^{14}(\gamma_{10})\, h(10z), \tag{6.5.2}$$

where v_η is the multiplier system for $\eta(z)$ (see [B, (2.5)]). For d odd we have

$$v_\eta^2(\gamma) = (-1)^{\frac{d-1}{2}}\zeta_{12}^{-ac(d^2-1)+d(b-c)}. \tag{6.5.3}$$

We have $\eta^2(2z)\big|_1\gamma = v_\eta^2(\gamma_2)\eta^2(2z)$ and $\eta^2(50z)\big|_1\gamma = v_\eta^2(\gamma_{50})\eta^2(50z)$. A computation involving (6.5.3) shows that

$$v_\eta^2(\gamma_2)v_\eta^{14}(\gamma_{10}) = v_\eta^2(\gamma_{50})v_\eta^{14}(\gamma_{10}) = 1.$$

It follows that $\eta(z)R(z) \in M_1^!(\Gamma)$.

It remains to show that $\eta(z)L(z)$ and $\eta(z)R(z)$ are holomorphic at the cusps. Using MAGMA we compute a set of Γ-inequivalent cusp representatives:

$$\Big\{\infty,\, 0,\, \frac{1}{8},\, \frac{2}{15},\, \frac{1}{7},\, \frac{3}{20},\, \frac{1}{6},\, \frac{1}{5},\, \frac{13}{50},\, \frac{4}{15},\, \frac{11}{40},\, \frac{7}{25},\, \frac{3}{10}, \\ \frac{7}{20},\, \frac{9}{25},\, \frac{11}{30},\, \frac{2}{5},\, \frac{8}{15},\, \frac{11}{20},\, \frac{3}{5},\, \frac{7}{10},\, \frac{29}{40},\, \frac{11}{15},\, \frac{4}{5}\Big\}. \tag{6.5.4}$$

Given a cusp $\mathfrak{a} \in \mathbb{P}^1(\mathbb{Q})$ and a meromorphic modular form f of weight k with Fourier expansion $f(z) = \sum_{n\in\mathbb{Q}} a(n)q^n$, the invariant order of f at $\mathfrak{a}$ is defined as

$$\operatorname{ord}(f,\infty) := \min\{n : a(n) \neq 0\},$$
$$\operatorname{ord}(f,\mathfrak{a}) := \operatorname{ord}(f\big|_k \gamma_{\mathfrak{a}},\infty),$$

where $\gamma_{\mathfrak{a}} \in \mathrm{SL}_2(\mathbb{Z})$ sends ∞ to $\mathfrak{a}$. For $N \in \mathbb{N}$, we have the relation (see e.g. [B, (1.7)])

$$\operatorname{ord}(f(Nz), \tfrac{r}{s}) = \frac{(N,s)^2}{N} \operatorname{ord}(f, \tfrac{Nr}{s}). \tag{6.5.5}$$

We extend this definition to functions f in the set

$$S := \{\widetilde{M}(\tfrac{a}{5},z), \widetilde{N}(\tfrac{a}{5},z) : a = 1,2\} \cup \{\widetilde{M}(a,b,z), \widetilde{N}(a,b,z) : 0 \leq a \leq 4, 1 \leq b \leq 4\}$$

by defining the orders of these functions at ∞ to be the orders of their holomorphic parts at ∞ (see Section 6.2 and (2.1)–(2.4) of [G]); that is,

$$\operatorname{ord}\left(\widetilde{M}\left(\frac{a}{5},z\right),\infty\right) = \operatorname{ord}\left(\widetilde{M}(a,b,z),\infty\right) := \frac{3a}{10}\left(1-\frac{a}{5}\right) - \frac{1}{24}, \tag{6.5.6}$$

$$\operatorname{ord}\left(\widetilde{N}\left(\frac{a}{5},z\right),\infty\right) := -\frac{1}{24}, \tag{6.5.7}$$

$$\operatorname{ord}\left(\widetilde{N}(a,b,z),\infty\right) := \frac{b}{5}k(b,5) - \frac{3b^2}{50} - \frac{1}{24}, \tag{6.5.8}$$

where $k(b,5) = 1$ if $b \in \{1,2\}$ and 2 if $b \in \{3,4\}$. Lastly, for $f \in S$ we define

$$\operatorname{ord}(f,\mathfrak{a}) := \operatorname{ord}(f\big|_{\frac{1}{2}}\gamma_{\mathfrak{a}},\infty). \tag{6.5.9}$$

This is well-defined since S is closed (up to multiplication by roots of unity) under the action of $\mathrm{SL}_2(\mathbb{Z})$. By this same fact we have

$$\min_{\text{cusps } \mathfrak{a}} \operatorname{ord}\left(\widetilde{N}(\tfrac{a}{5},z),\mathfrak{a}\right) \geq \min_{f\in S} \operatorname{ord}(f,\infty) = -\frac{1}{24}, \tag{6.5.10}$$

from which it follows that

$$\operatorname{ord}\left(\eta(z)\widetilde{N}(\tfrac{a}{5},z),\mathfrak{a}\right) \geq 0 \qquad \text{for all } \mathfrak{a}.$$

To determine the order of $\eta(z)\widetilde{M}(\frac{a}{5},25z)$ at the cusps of Γ, we write

$$\eta(z)\widetilde{M}(\tfrac{a}{5},25z)=\frac{\eta(z)}{\eta(25z)}m(25z),\qquad \text{where } m(z)=\eta(z)\widetilde{M}(\tfrac{a}{5},z).$$

The cusps of $\Gamma_0(25)$ are ∞ and $\frac{r}{5}$, $0\le r\le 4$. By (6.5.5) the function $\eta(z)/\eta(25z)$ is holomorphic at every cusp except for those which are $\Gamma_0(25)$-equivalent to ∞ (the latter are $\frac{13}{50}$, $\frac{7}{25}$, and $\frac{9}{25}$ in (6.5.4)); there we have $\mathrm{ord}(\eta(z)/\eta(25z),\infty)=-1$. Since $\widetilde{M}(\frac{a}{5},z)$ also satisfies (6.5.10), it suffices to check $\frac{13}{50}$, $\frac{7}{25}$, and $\frac{9}{25}$. By (6.5.5), [G, Theorems 3.1 and 3.2], the fact that $\left(\begin{smallmatrix}13&6\\2&1\end{smallmatrix}\right)=T^6S^{-1}T^{-2}S$, and (6.1.1), we have

$$\begin{aligned}\mathrm{ord}\left(m(25z),\frac{13}{50}\right)&=25\,\mathrm{ord}\left(m(z),\frac{13}{2}\right)\\&=25\left(\frac{1}{24}+\mathrm{ord}\left(\widetilde{M}(3a \bmod 5,a,z),\infty\right)\right)=\begin{cases}9 & \text{if } a=1,\\ 6 & \text{if } a=2.\end{cases}\end{aligned}$$

A similar computation shows that $\mathrm{ord}(m(25z),\frac{7}{25}),\mathrm{ord}(m(25z),\frac{9}{25})\ge 4$. Since $L(z)$ is holomorphic on $\mathbb{H}$, we have, for each cusp $\mathfrak{a}$, the inequality

$$\begin{aligned}&\mathrm{ord}(\eta(z)L(z),\mathfrak{a})\ge\\&\quad\min\left\{\mathrm{ord}\left(\eta(z)f(z),\mathfrak{a}\right):f(z)=\widetilde{N}(\tfrac{1}{5},z),\widetilde{M}(\tfrac{1}{5},25z),\widetilde{M}(\tfrac{2}{5},25z)\right\}\ge 0.\end{aligned}$$

We turn to $\eta(z)R(z)$. For this we require Theorem 3.3 of [B]:

$$\mathrm{ord}(g,\tfrac{r}{s})=\begin{cases}\frac{11}{60} & \text{if } 5\mid s \text{ and } r\equiv\pm2 \pmod 5,\\ -\frac{1}{60} & \text{otherwise,}\end{cases}\tag{6.5.11}$$

$$\mathrm{ord}(h,\tfrac{r}{s})=\begin{cases}\frac{11}{60} & \text{if } 5\mid s \text{ and } r\equiv\pm1 \pmod 5,\\ -\frac{1}{60} & \text{otherwise.}\end{cases}\tag{6.5.12}$$

Here we have corrected a typo in (3.2) of [B] (see (2.9) and Lemma 3.2 of that paper). By (6.5.5), (6.5.11), and (6.5.12) we have

$$\mathrm{ord}\left(\eta(z)R(z),\tfrac{r}{s}\right)\ge-\frac{(10,s)^2}{600}+\min\left\{\frac{1}{24}+\frac{(50,s)^2-(25,s)^2}{600},\frac{(2,s)^2}{24}\right\}.$$

Since the latter expression is nonnegative for all $s\mid 50$, it follows that $\eta(z)R(z)$ is holomorphic at the cusps. This completes the proof. □

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