ACTION OF BRAID GROUPS ON DETERMINANTAL IDEALS, COMPACT SPACES AND A STRATIFICATION OF TEICHMÜLLER SPACE

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Abstract. We show that for $n \geq 3$ there is an action of the braid group B_n on the determinantal ideals of a certain $n \times n$ symmetric matrix with algebraically independent entries off the diagonal and 2s on the diagonal. We show how this action gives rise to an action of B_n on certain compact subspaces of some Euclidean spaces of dimension $\binom{n}{2}$. These subspaces are real semi-algebraic varieties and include spheres of dimension $\binom{n}{2} - 1$ on which the kernel of the action of B_n is the centre of B_n. We investigate the action of B_n on these subspaces. We also show how a finite number of disjoint copies of the Teichmüller space for the n-punctured disc is naturally a subset of this $\mathbb{R}^{\binom{n}{2}}$ and how this cover (in the broad sense) of Teichmüller space is a union of non-trivial B_n-invariant subspaces. The action of B_n on this cover of Teichmüller space is via polynomial automorphisms. For the case $n = 3$ we show how to define modular forms on the 3-dimensional Teichmüller space relative to the action of B_3.

§1. Introduction.

The braid group B_n is the group with (standard) generators $\sigma_1, \dots, \sigma_{n-1}$ and relations [Bi, p. 18]

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}, &&\text{for } i = 1, \dots, n-2, \\ \sigma_i\sigma_j &= \sigma_j\sigma_i, &&\text{for } |i-j| > 1. \end{aligned}$$

One of the main results of this paper is the following:

Theorem 1. *For $n \geq 3$ let U_n be the following symmetric matrix:*

$$U_n = \begin{pmatrix} 2 & a_{21} & \dots & a_{i1} & \dots & a_{n-1\,1} & a_{n1} \\ a_{21} & 2 & \dots & a_{i2} & \dots & a_{n-1\,2} & a_{n2} \\ \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots \\ a_{i1} & a_{i2} & \dots & 2 & \dots & a_{n-1\,i} & a_{ni} \\ \vdots & \vdots & & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1\,i} & \dots & 2 & a_{n\,n-1} \\ a_{n1} & a_{n2} & \dots & a_{n\,i} & \dots & a_{n,n-1} & 2 \end{pmatrix}$$

where the $a_{ij}, 1 \leq j < i \leq n$, are algebraically independent indeterminates generating a polynomial ring R'_n over a commutative ring R with identity. The ring R'_n is acted upon by B_n with kernel the cyclic centre $Z(B_n)$ of B_n.

For $r \geq 0$ let $\mathcal{I}_{nr}$ be the determinantal ideal (of R'_n) generated by all of the $(r+1)\times(r+1)$-minors of U_n. Then the braid group acts on each $\mathcal{I}_{nr}$.

The braid group also acts on the quotients $R'_n/\mathcal{I}_{nr}$.
(i) Suppose that 2 is invertible in R. Then for $n > 3$ the kernel of the action of $B_n/Z(B_n)$ on $R'_n/\mathcal{I}_{n\,1}$ contains the non-trivial normal subgroup generated by $(\sigma_1)^4$. The action of $B_n/Z(B_n)$ on $R'_n/\mathcal{I}_{n1}$ determines an epimorphism $B_n \to W(D_n)$ where $W(D_n)$ is the Coxeter group of type D_n.
(ii) Suppose that 2 is invertible in R. Then for $n > 3$ the kernel of the action of $B_n/Z(B_n)$ on $R'_n/\mathcal{I}_{n\,2}$ contains the non-trivial normal subgroup generated by $(\sigma_1\sigma_2)^6$.
(iii) For any commutative ring R the action of $B_n/Z(B_n)$ on $R'_n/\mathcal{I}_{n\,n-1}$ has trivial kernel.
(iv) For $R = \mathbb{Z}$ or R a field having characteristic 2, the action of $B_n/Z(B_n)$ on $R'_n/\mathcal{I}_{n\,r}$ has trivial kernel for all $1 \leq r < n$.

We will also show how this result enables us to find various compact subsets of $\mathbb{R}^{\binom{n}{2}}$ which are acted upon by B_n. We then show that certain subsets of this $\mathbb{R}^{\binom{n}{2}}$ can be identified with a finite number of disjoint copies of the Teichmüller space for the punctured disc and combining this with results of [H2] we will obtain a non-trivial stratification of this cover (in the broad sense) of Teichmüller space by B_n-invariant subsets. We now explain where this action of B_n on $\mathbb{R}^{\binom{n}{2}}$ comes from.

Let D_n be the disc with n punctures $\pi_1, \ldots, \pi_n$. Then B_n acts as (isotopy classes of) diffeomorphisms of D_n [Bi, Ch.1]. Further, for $2 \leq m \leq n$, B_n acts transitively on the set of isotopy classes of positively oriented simple closed curves on D_n which surround m of the punctures. The generator σ_i acts as a half-twist [Bi] on D_n interchanging π_i and π_{i+1} and has a representative diffeomorphism which is supported in a tubular neighbourhood of an arc a_i joining π_i to π_{i+1} (see Figure 1).

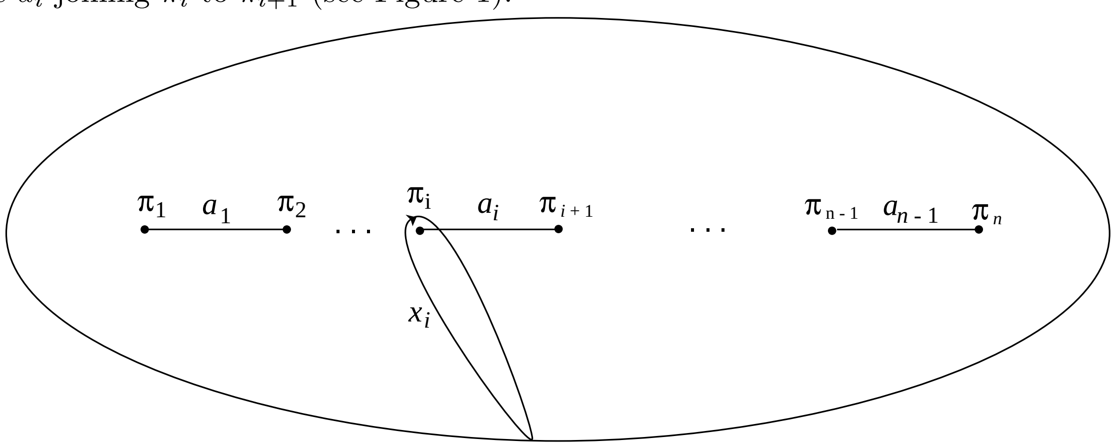

Figure 1

In Figure 1 we have shown the arcs a_i. This fact allows one to construct [Bi] a faithful representation of B_n as automorphisms of a free group $F_n =< x_1, \ldots, x_n >$, which we identify with the fundamental group $\pi_1(D_n)$. Here the x_i are (homotopy classes of) simple closed curves surrounding one of the punctures and based at a fixed point of the boundary of D_n, as in Figure 1. A characterisation of the image of B_n in $Aut(F_n)$ was given by Artin as follows: $\phi \in Aut(F_n)$ is the image of a braid if and only if

(i) for all $1 \le i \le n$, $\phi(x_i)$ is a conjugate of some x_j; and
(ii) $\phi(x_1 x_2 \dots x_n) = x_1 x_2 \dots x_n$.
The action of B_n on the generators x_i is as follows: let $1 \le j < n$; then

$$\sigma_j(x_i) = x_i \quad \text{if} \quad i \neq j, j+1, \quad \sigma_j(x_j) = x_j x_{j+1} x_j^{-1}, \quad \sigma_j(x_{j+1}) = x_j.$$

Let R be a commutative ring with identity and let

$$R_n = R[a_{12}, a_{13}, \dots, a_{1n}, a_{21}, a_{23}, \dots, a_{2n}, \dots, a_{n1}, \dots, a_{n\,n-1}]$$

be a polynomial ring in commuting indeterminates a_{ij}, $1 \le i \neq j \le n$. It will be convenient to put $a_{ii} = 0$ for all $i \le n$. In a previous paper [H1] we have shown that by representing the free group $\pi_1(D_n)$ using transvections (see below) and looking at certain traces we obtain an action of B_n on the ring R_n i.e. we have a homomorphism

$$\psi_n : B_n \to Aut(R_n);$$

the kernel of ψ_n is the centre of B_n [H1]. This is understood as follows.

For fixed $n \ge 2$ we let $\Pi_n = T_1 T_2 \dots T_n$ where the T_i are certain $n \times n$ matrices (transvections):

$$T_i = \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots \\ a_{i1} & a_{i2} & \dots & 1 & \dots & a_{in-1} & a_{in} \\ \vdots & \vdots & \dots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Here the non-zero off-diagonal entries of T_i occur in the ith row. One way of defining a *transvection* [A] is as a matrix $M = I_n + A$ where I_n is the $n \times n$ identity matrix, $det(M) = 1, rank(A) = 1$ and $A^2 = 0$. In particular, conjugates of transvections are transvections. That $< T_1, \dots, T_n >$ is a free group of rank n is shown in [H2]. This allows us to identify x_i and T_i for $i = 1, \dots, n$ and so to identify F_n and $< T_1, \dots, T_n >$.

Now, since we are identifying $x_i \in F_n$ with $T_i \in < T_1, \dots, T_n >$, we see, by Artin's characterisation (i) and (ii) above, that the matrix Π_n is invariant under the action of B_n and so that $\Pi_n + \lambda I_n$ is also invariant under the action of B_n for any choice of $\lambda \in R$. Thus the action fixes the various determinantal ideals $\mathcal{I}_{n,r,\lambda}$ determined by $\Pi_n + \lambda I_n$. Here $\mathcal{I}_{n,r,\lambda}$ is the ideal generated by the determinants of all $(r+1) \times (r+1)$ submatrices of $\Pi_n + \lambda I_n$. These are the points at which $\Pi_n + \lambda I_n$ has rank r (except in the characteristic 2 case). For general properties of such rings see [BV, DEP1, DEP2]. We will put $\mathcal{I}_{n,r} = \mathcal{I}_{n,r,1}$.

Now, since Π_n is invariant under the action of B_n, it follows that the characteristic polynomial

$$\chi_n(x) = \sum_{i=0}^{n} c_{ni} x^i$$

of the matrix Π_n has coefficients $c_{ni} = c_{ni}(a_{12}, \dots, a_{n\,n-1}) \in R_n$ which are invariant under the action of B_n (this was first noted in [H2, Theorem 2.8]). We there also noted that the c_{ni} are non-homogeneous polynomials of degree n for $1 \leq i < n$. If $n < 6$, then they generate the ring of invariants for the action of B_n on the subring Y_n of R_n defined in the next paragraph [H3]. Note that we have $c_{nn} = 1$ and $c_{n0} = \pm 1$.

For $i, j, k, \ldots, r, s \in \{1, 2, \ldots, n\}$ let $c_{ijk\ldots rs}$ denote the *cycle* $a_{ij}a_{jk}\ldots a_{rs}a_{si} \in R_n$. Then the cycles generate a subalgebra of R_n denoted Y_n. A cycle $c_{ijk\ldots rs}$ will be called *simple* if $i, j, k, ..., r, s$ are all distinct. The ring Y_n is generated by the (finite number of) simple cycles and is B_n-invariant.

The representation of B_n in $Aut(R_n)$ can be thought of in the following way. Note that the action of B_n on D_n fixes the boundary of D_n. This fact gives Artin's condition (ii) above. Let $\mathcal{C}_n$ denote the set of all oriented simple closed curves on D_n. Now, from the above, B_n acts by automorphisms on F_n in such a way that for $\alpha \in B_n$ the matrix $\alpha(T_i)$ is a conjugate of some T_j, $1 \leq j \leq n$ i.e. $\alpha(T_i)$ is also a transvection. Further, if $c \in \mathcal{C}_n$, then c represents a conjugacy class in F_n and so its *trace* is well-defined (the trace of the corresponding product of transvections in $F_n =< T_1, \ldots, T_n >$). In fact one easily sees that $trace(c) \in Y_n$ [H1]. Then a map $\phi = \phi_n : \mathcal{C}_n \to R_n$ is defined by

$$\phi_n(c) = trace(c) - n.$$

Thus ϕ_n can be thought of as being defined on certain conjugacy classes of elements of F_n (namely those representing simple closed curves). The map ϕ can be extended to act on all of F_n, by the requirement that for $s \in F_n$ we have $\phi(s) = trace(s) - n$. It is easy to see that if $w = T_{i_1}^{e_1} \ldots T_{i_r}^{e_r} \in< T_1, \ldots, T_n >$ is cyclically reduced as written with $e_i \neq 0$, $i_k \neq i_{k+1}, i_r \neq i_1$, and $r > 1$, then $\phi(w)$ is a polynomial in Y_n of degree r (see §2).

Now for $m \geq n$ and $s \in F_n$ we may also consider s as an element of F_m under the natural inclusion of F_n in F_m. In this case we note that $\phi(s)$ has the same value whether we consider s as an element of F_n or F_m.

A fundamental property of the transvections T_i is that for all $1 \leq i, j \leq n$ we have $trace(T_iT_j) = a_{ij}a_{ji} + n$ and in general if $A, B \in F_n$, then

$$trace(AT_iA^{-1}BT_jB^{-1}) = b_{ij}b_{ji} + n, \tag{1.1}$$

where $b_{ij} \in R_n$ (see [H1]). It is also easy to see that there is a natural choice so that

$$b_{ij} = \pm a_{ij} + \text{terms of higher degree.}$$

For example $trace((T_1T_2T_1^{-1})T_3) = (a_{23} - a_{21}a_{13})(a_{32} + a_{31}a_{12}) + n$. This is explained in detail in §2.

Now for $\alpha \in B_n$ the image $\alpha(T_i)$ is a conjugate AT_jA^{-1} for some $A \in F_n$ and $1 \leq j \leq n$ (by Artin's condition (i) above). Here the action of α on the a_{ij} is defined by

$$\phi(\alpha(T_i)\alpha(T_j)) = \alpha(a_{ij})\alpha(a_{ji}),$$

(see §2 for more details) so that it has the following naturality property (with respect to the action of B_n on F_n): for all $w \in< T_1, \ldots, T_n >, \alpha \in B_n$, we have

$$\phi(\alpha(w)) = \alpha(\phi(w)).$$

For example the action of the generator $\sigma_i, 1 \leq i < n$, is given by

$$\begin{aligned} &\sigma_i(a_{i\,i+1}) = a_{i+1\,i}, \quad \sigma_i(a_{i+1\,i}) = a_{i\,i+1}, \\ &\sigma_i(a_{h\,i}) = a_{h\,i+1} + a_{hi}a_{i\,i+1}, \quad \sigma_i(a_{h\,i+1}) = a_{h\,i}, \\ &\sigma_i(a_{i\,h}) = a_{i+1\,h} - a_{i+1\,i}a_{ih}, \quad \sigma_i(a_{i+1\,h}) = a_{i\,h}, \end{aligned} \tag{1.2}$$

where $1 \leq h \leq n$ and $h \neq i, i+1$.

It follows from [H1, Theorem 2.5 and Theorem 6.2] that the kernel of the action of B_n on R_n is the centre of B_n and that if B_n and R_n are thought of as sub-objects of B_{n+1} and R_{n+1} (respectively), then the action of B_n on R_{n+1} is faithful.

We note as in [H2] that there is a natural ring involution $*$ on R_n which commutes with the action of B_n, so that for $\alpha \in B_n$ we have

$$\alpha(w)^* = \alpha(w^*) \tag{1.3}$$

for all $w \in R_n$. This involution is determined by its action on the generators a_{ij} which is as follows:

$$a_{ij}^* = -a_{ji}.$$

(Thus to check (1.3) one need only consider the situation where $\alpha = \sigma_i$ and $w = a_{rs}$.) This involution has the following property:

$$trace(A^{-1}) = trace(A)^*,$$

for all $A \in F_n$. Thus for $c \in \mathcal{C}_n$ we have $\phi(c^{-1}) = \phi(c)^*$, where c^{-1} is the curve c with its orientation reversed. We also have $b_{ji} = -b_{ij}^*$, for b_{ij}, b_{ji} as in (1.1).

Now factoring out by the action of the above involution leads us to consider the situation where $a_{ij} = -a_{ji}$ for all $1 \leq i \neq j \leq n$. Let R'_n denote the corresponding quotient of R_n and T'_i the corresponding transvections etc. The ring R'_n is isomorphic to the subring of R_n generated by the a_{ij} with $i > j$ and so is a polynomial ring. Then B_n also acts on R'_n. The fact that $< T'_1, \dots, T'_n >$ is still a free group and that the kernel of the action of B_n on R'_n is still the centre $Z(B_n)$ were noted in [H1, H2]. We will let $\Pi'_n = T'_1T'_2 \dots T'_n$ and

$$\chi'_n(x) = \sum_{i=0}^{n} c'_{ni} x^i$$

the characteristic polynomial of the matrix Π'_n. Again the coefficients c'_{ni} are invariant under the action of B_n on R'_n.

We will primarily be interested in the situation where $R = \mathbb{R}$ (due in part to the connection with Teichmüller space); however the presence of the 2s on the diagonal of U_n will force us to distinguish between rings R where 2 is invertible or where 2 is not invertible. In the case $R = \mathbb{R}$ each ideal $\mathcal{I}_{n,r,\lambda}$ determines a real algebraic variety $V_{n,r,\lambda} \subset \mathbb{R}^{\binom{n}{2}}$. These varieties are invariant under the action of B_n. For general properties of real algebraic varieties see [BCR]. Another reason for looking at the case $R = \mathbb{R}$ is that in this case there is a compact piece of $V_{n,r} = V_{n,r,1}$ which is also B_n-invariant. We will determine the nature of this compact set. Recall that a *semi-algebraic set* in $\mathbb{R}^n$ is (roughly speaking) a set of points determined by a set of algebraic equalities and inequalities (including all finite unions of finite intersections of such). For example we prove

Theorem 2. *Suppose that $R = \mathbb{R}$. For all $k \leq n$ there is a compact real algebraic set $V^{(2)}_{n,k,1} \subset V_{n,k,1}$ which is invariant under the above action of B_n.*

For $n = 3$ the semi-algebraic subset $V^{(2)}_{3,2,1}$ is homeomorphic to a 2-sphere, and is smooth except at 4 singular points.

For $n \geq 4$ the semi-algebraic subset $V^{(2)}_{n,2,1}$ is homeomorphic to the quotient

$$S^1 \times S^1 \times \cdots \times S^1/\alpha,$$

where α is the inverse map

$$\alpha(z_1, \dots, z_{n-1}) = (z_1^{-1}, \dots, z_{n-1}^{-1}),$$

on $(S^1)^{n-1}$. Here $V^{(2)}_{n,2,1}$ has 2^{n-1} singular points and is otherwise smooth. For $n = 4$ we find a presentation for the image of B_4 in $Aut(R'_4/\mathcal{I}_{4,2,1})$ and show that this group has a faithful 3×3 linear representation over $\mathbb{C}$.

For all $n \geq 3$ the group B_n acts on a one-parameter family of smooth topological spheres of dimension $\binom{n}{2} - 1$, each such sphere being B_n-invariant. These spheres come from level sets of $det(U_n)$. Each point of these spheres corresponds naturally to a positive definite symmetric matrix. For a dense set of the parameter values the kernel of this action of B_n on the corresponding sphere is the cyclic centre $Z(B_n)$.

When $n = 4$ there are two convex 5-balls in $\mathbb{R}^6$ the boundary of whose intersection is a B_4-invariant 4-sphere.

More details of these varieties and the B_n-actions will be given in the rest of this paper.

In the following we will need to recall [Bi] that there is an epimorphism $\pi : B_n \to S_n$, where S_n is the symmetric group on n objects, which is induced by sending σ_i to the transposition $(i, i+1) \in S_n$ for each $i = 1, \dots, n-1$. The kernel of π is P_n the group of *pure* or *coloured braids* on n strings.

In relation to the action of B_n on the determinantal rings indicated in Theorem 1 we should also mention the paper [H4] where we show that B_4 acts on an ordinal Hodge algebra or algebra with straightening law. One hope is that using either approach we may be able to say something about the representation theory of B_n or P_n using (standard) methods as in [BV, DEP1, DEP2, JPW]. The present paper is more topological in nature, however we investigate the representation theory of the action given above more fully in [H5].

Suppose that R contains the rational numbers. Now in [H3] we proved that for any $\alpha \in P_n$ there is a derivation $D(\alpha)$ of the power series algebra $\bar{R}_n$ of R_n such that

$$\alpha(x) = exp(D(\alpha))(x)$$

for all $x \in \bar{R}_n$. For example we showed that

$$\begin{aligned} D(\sigma_1^2) = \frac{arcsinh\left(\sqrt{a_{12}a_{21} + (a_{12}a_{21})^2/4}\right)}{\sqrt{a_{12}a_{21} + (a_{12}a_{21})^2/4}} &((a_{32}a_{21} + a_{31}a_{12}a_{21}/2)\frac{\partial}{\partial a_{31}} \\ &+ (-a_{12}a_{23} + a_{12}a_{21}a_{13}/2)\frac{\partial}{\partial a_{13}} + (a_{31}a_{12} - a_{32}a_{21}a_{12}/2)\frac{\partial}{\partial a_{32}} \\ &+ (-a_{21}a_{13} - a_{21}a_{12}a_{23}/2)\frac{\partial}{\partial a_{23}}). \end{aligned}$$

These derivations $D(\alpha)$ have the property that $D(\alpha)(c_{ni}) = 0$ for all $i \leq n$ (this being equivalent to the invariance of the c_{ni}). We obtain a group

$$\mathcal{P}_n =< exp(tD(\alpha)) | \alpha \in P_n, t \in \mathbb{R} > \quad \subset \quad Aut(\bar{R}_n)$$

using composition of functions, or equivalently, using the Campbel-Baker-Hausdorff formula [J]. We can extend $\mathcal{P}_n$ to

$$\mathcal{B}_n =< \mathcal{P}_n, \sigma_1, \ldots, \sigma_{n-1} > \quad \subset \quad Aut(\bar{R}_n)$$

to obtain a continuous group with $B_n/Z(B_n)$ as a subgroup. We will prove:

Theorem 3. *Suppose that R contains the rational numbers. Then the group $\mathcal{B}_n$ acts on each of the spaces V_{nk} and $V_{nk}^{(2)}$ (if $R = \mathbb{R}$) and also on the level sets of the functions c_{ni}.*

We assume that $R = \mathbb{R}$ for the following discussion, so that topologically we have $R'_n \cong \mathbb{R}^{\binom{n}{2}}$. The action of B_n on R'_n comes, as described above, from considerations of trace algebras i.e. character varieties. One motivation for looking at actions of groups on character varieties was to give coordinates for Teichmüller space. This approach originated with Fricke and Klein [FK] and there have been many subsequent attempts at ways of giving these real analytic trace coordinates (see for example [K1, K2, O, Sa] and references therein). Thus it is not surprising to see that there is a connection with our invariant varieties and Teichmüller space, which we now describe.

Let $G < PSL_2(\mathbb{R})$ be a Fuchsian group acting discontinuously on the upper half plane $\mathbb{H}$ and such that $\mathbb{H}/G$ is conformally equivalent to a Riemann surface of genus g with n ordered points and m conformal discs removed; we say that G has *type* (g, n, m). We choose a *marking* on G by specifying a set of generators $a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n, e_1, \ldots, e_m$ for G. For example one can do so such that the only relator is

$$e_m \ldots e_1 c_n \ldots c_1 b_g^{-1} a_g^{-1} b_g a_g \ldots b_1^{-1} a_1^{-1} b_1 a_1 = id.$$

There is a non-degeneracy condition that is required for the following to work, namely that $6g + 3m + 2n - 6 > 0$. Any two marked Fuchsian groups are *conformally equivalent* if they are conjugate in $PSL_2(\mathbb{R})$. The set of such equivalence classes of Fuchsian groups of type (g, n, m) is called the *Teichmüller space* of type (g, n, m) and is denoted by $\mathcal{T}_{g,n,m}$. It is well known that $\mathcal{T}_{g,n,m}$ is a real analytic manifold of dimension $6g + 3m + 2n - 6$ [Ab]. Taking traces of a finite number of products of the images of these generators results in a (finite) set of real analytic coordinates for $\mathcal{T}_{g,n,m}$ [FK, K1, K2, O]. In our case we are interested in the situation where $g = 0, m = 1$. Usually one associates an element $\nu_i, i = 1, \ldots, n$, of the set $\{2, 3, \ldots, \infty\}$ to each deleted point, however we will only be interested in the situation where $\nu_i = \infty$ for all $i = 1, \ldots, n$. We also note that each c_i is a parabolic element (its squared trace is 4) and that any non-identity element not conjugate to some power of some c_i is hyperbolic (has squared trace greater than 4). Now from the discussion of transvections above it appears natural to solve

$$trace(c_i c_j) = 2 - a_{ij}^2 \tag{1.4}$$

for all $1 \leq j < i \leq n$. We will show that the B_n actions on both sides of the equation $\sqrt{2 - trace(c_i c_j)} = \pm a_{ij}$ are compatible at least for some choices of the sign of $\pm a_{ij}$. We thus obtain a representation of Teichmüller space as a subspace of our coordinate space $\mathbb{R}^{\binom{n}{2}}$. Actually we get a finite disjoint cover of Teichmüller space corresponding to some of the choices of sign that we get when solving (1.4), so that each connected component of this cover is isomorphic to Teichmüller space via the covering projection.

For $n = 3, 4$ it will follow from dimensional considerations that $\bar{\mathcal{T}}_{0,n,1}$ has a non-trivial stratification by B_n-invariant subsets. For $n \geq 5$ we see that $\bar{\mathcal{T}}_{0,n,1}$ has dimension $2n-3$; however there are at most $\lfloor \frac{n}{2} \rfloor$ independent invariants: $c'_{51}, \ldots, c'_{5\lfloor \frac{n}{2} \rfloor}$ (see Lemma 2.5). Since $\bar{\mathcal{T}}_{0,n,1} \subset \mathbb{R}^{\binom{n}{2}}$ it does not follow from such dimensional considerations that we have a non-trivial stratification (by level sets of the c_{ni}) in this case. However we will show that there actually is such a non-trivial stratification by showing that $c'_{n\,n-1}$ is not constant on each component of $\bar{\mathcal{T}}_{0,n,1}$. We do this by exhibiting a 1-parameter family of matrices generating a 1-parameter family of Fuchsian groups, each giving a point of Teichmüller space and showing that $c'_{n\,n-1}$ is not constant on any of the lifts of this family to $\bar{\mathcal{T}}_{0,n,1}$.

Theorem 4. *There is a real analytic subset $\bar{\mathcal{T}}_{0,n,1}$ of $\mathbb{R}^{\binom{n}{2}}$ which is a union of 2^{n-1} disjoint copies of the Teichmüller space $\mathcal{T}_{0,n,1}$ of the punctured disc D_n. The set $\bar{\mathcal{T}}_{0,n,1}$ is a union of B_n-invariant pieces (strata) corresponding to the level sets of the invariants c'_{ni}. The strata corresponding to the invariant $c'_{n1} = \pm c'_{n\,n-1}$ are of codimension 1 in $\bar{\mathcal{T}}_{0,n,1}$.*

For $n = 3$ we determine a fundamental domain for the action of $B_3/Z(B_3)$ on certain of the 2-dimensional level sets in $\mathbb{R}^3$ of the invariant function c'_{31}. For $t < 4$ and $|a_{ij}| > 2$ there are 4 connected components of the level set $c'_{31} = t$ and we may identity them using the identifications $(x, y, z) \equiv (-x, -y, z) \equiv (-x, y, -z)$ (see §11). With this understood we have the following result which says that the vanishing of the gradient of the invariant function c'_{31} cuts out a region which is a fundamental domain for the action of a finite index subgroup of $B_3/Z(B_3) \cong PSL(2, \mathbb{Z})$ on the level surfaces of c'_{31}, thus indicating a close relationship between the group and the level surfaces:

Theorem 5. *Consider the action of $B_3/Z(B_3)$ on $\mathbb{R}^3 = \mathbb{R}[a_{21}, a_{31}, a_{32}]$. For $t < 0$ there is a fundamental domain for the action of a subgroup $H_3 < B_3/Z(B_3)$ of index 3 on the level sets*

$$c'_{31} = a_{21}^2 + a_{31}^2 + a_{32}^2 - a_{21}a_{31}a_{32} = t.$$

In fact for such values of t the functions

$$\frac{\partial c'_{31}}{\partial a_{21}} = 2a_{21} - a_{31}a_{32}, \quad \frac{\partial c'_{31}}{\partial a_{31}} = 2a_{31} - a_{21}a_{32}, \quad \frac{\partial c'_{31}}{\partial a_{32}} = 2a_{32} - a_{31}a_{21},$$

cut out a region of the level set $c'_{31} = t$ which is a fundamental domain for this action.

The subgroup H_3 is freely generated by 3 involutions. Each of the curves determined by the above equations is fixed by one of these three involutions.

See Figure 8 for a representation of this fundamental domain. In proving the above result we give a natural generalisation of results relating to the Markoff equation and the Markoff tree [CF, Mo].

As indicated above there are various kinds of "natural" coordinates that can be defined on Teichmüller space, including such coordinates coming from traces of various products of generators. Relative to these coordinates the action of the mapping class groups (in our case the braid groups) is at best via rational maps (in equation (11.4) we have written down such rational maps for the action of the generators of B_4 for some such choice of coordinates). However we now have the following indication that the a_{ij} also give a very natural set of coordinates for $\mathcal{T}_{0,n,1}$:

Theorem 6. *There is an embedding of a disjoint union of 2^{n-1} copies of the Teichmüller space in a Euclidean space $\mathbb{R}^{\binom{n}{2}}$ (as in Theorem 4) and the action of B_n on this cover is via polynomial automorphisms.*

In §2 we give more details on the action of B_n on R_n and R'_n. In §3 we consider the rank 1 case. In §4 we investigate the case where $n = 3$. In §5 we give various results on symmetric matrices that will be used later. In §6 we investigate the case where $n = 4$. In §7 we look at the rank 2 case. §8 is devoted to the rank $n - 1$ case. In §9 we study faithfulness questions for the action of B_n on various quotients of R'_n. In §10 we prove Theorem 3. The part of this paper concerning Teichmüller space is to be found in §11 and is largely independent of §§$5 - 10$.

§2 Action of B_n on R_n continued

In this section we describe in greater detail the action of B_n on R_n and on R'_n so as to be able to give explicit formulae for the action of certain braids. In general [A] a transvection in $SL(Q^n)$ (for a commutative ring Q with identity) can be defined as a pair $T = (\phi, d)$ where $d \in Q^n$ and ϕ is an element of the dual space of Q^n satisfying $\phi(d) = 0$. The action is given by

$$T(x) = x + \phi(x)d \quad \text{for all} \quad x \in Q^n.$$

Then we have [H1, Lemma 2.1]

Lemma 2.1. *Let $T = (\phi, d)$ and $U = (\psi, e)$ be two transvections. Then for all $\lambda \in \mathbb{Z}$ we have*

$$U^\lambda T U^{-\lambda} = (\phi - \lambda\phi(e)\psi, U^\lambda(d)). \quad \square$$

Let $T = \{T_1 = (\phi_1, d_1), \dots, T_n = (\phi_n, d_n)\}$ be a fixed set of transvections in $SL((R_n)^n)$ where $\phi_i(d_j) = a_{ij}$ for all $1 \le i \ne j \le n$ as in the above. For any set of transvections

$$T' = \{T'_1 = (\phi'_1, e'_1), \dots, T'_n = (\phi'_n, e'_n)\}$$

we let $M(T')$ denote the $n \times n$ matrix $(\phi'_i(e'_j))$ and we call $M(T')$ the *M-matrix* of the set of transvections T'.

Any monomial in R_n that can be written in the form $a_{j_1j_2}a_{j_2j_3}\dots a_{j_{r-2}j_{r-1}}a_{j_{r-1}j_r}$ (with $j_i \ne j_{i+1}$ for $1 \le i < r$) will be called a j_1j_r-word. Note that by (1.2) if $\alpha \in B_n$ and $1 \le i \ne j \le n$, then $\alpha(a_{ij})$ is a sum of rs-words, where $\alpha(T_i)$ is a conjugate of T_r and $\alpha(T_j)$ is a conjugate of T_s. Let $\alpha \in B_n$ where $\alpha(T_i) = w_iT_jw_i^{-1}$ in freely reduced form for $i = 1, ..., n$ and where $w_i = w_i(T_1, \dots, T_n)$. Then for $i = 1, \dots, n$ we have $w_iT_jw_i^{-1} = (\psi_i, f_i)$ for some ψ_i, f_i determined by Lemma 2.1, which result in fact shows that

$$\psi_i = q_1\phi_1 + \dots + q_n\phi_n \quad \text{and} \quad f_i = p_1d_1 + \dots + p_nd_n, \tag{2.1}$$

where $p_1,\dots,p_n,q_1,\dots,q_n \in R_n$. Since the a_{ij} are algebraically independent the ϕ_i and d_j are linearly independent and so the above representation is unique. We define the action of B_n on R_n by

$$\alpha(a_{ij}) = \psi_i(f_j).$$

One can check that this agrees with the previous definition. Thus the M-matrix is acted upon naturally by B_n:

$$\alpha(M(T)) = M(\{\alpha(T_1),\dots,\alpha(T_n)\}).$$

From Lemma 2.3 of [H1] we have:

Lemma 2.2. *Let $\alpha \in B_n$ where $\alpha(T_i) = C_1T_kC_1^{-1}, \alpha(T_j) = C_2T_pC_2^{-1}$, with $C_1, C_2 \in < T_1,\dots,T_n >$ and let $C = C_1^{-1}C_2 = T_{j_1}^{q_1}\dots T_{j_r}^{q_r}$ be freely reduced with $j_r \neq p,\ j_1 \neq k, q_s \neq 0$ for $s = 1,\dots,r$ and $j_s \neq j_{s+1}$, for $s = 1,\dots,r-1$. Then*

$$\alpha(a_{ij}) = \sum_{h=1}^{n} A_h a_{hp}$$

where A_h is equal to the sum of all the products of the form

$$q_{r_1}q_{r_2}\dots q_{r_m}a_{kj_{r_1}}a_{j_{r_1}j_{r_2}}\dots a_{j_{r_{m-1}}j_{r_m}}$$

where $1 \le r_1 < r_2 < \dots < r_m \le r$ and $j_{r_m} = h$. If $p \neq j_r$, then the summand of $\alpha(a_{ij})$ of highest degree is unique and is equal to

$$\pm q_1q_2\dots q_ra_{kj_1}a_{j_1j_2}\dots a_{j_{r-1}j_r}a_{j_rp}. \quad \square$$

For example, if $\alpha(T_1) = T_3T_2^{-1}T_1T_2T_3^{-1}$ and $\alpha(T_2) = T_2^{-1}T_3T_2$, then we would have $C = T_2T_3^{-1}T_2^{-1}$ and

$$\alpha(a_{12}) = a_{13} + a_{13}a_{32}a_{23} + a_{12}a_{23}a_{32}a_{23}.$$

We showed in [H1, Lemma 2.10] that for $\alpha \in B_n$ the freely reduced form of $\alpha(T_i) \in < T_1,\dots,T_n >$ has no subword of the form $T_j^{\pm 2}$. In fact if $c \in \mathcal{C}_n$ is a simple closed curve, then any cyclically reduced word in $< T_1,\dots,T_n >$ which represents c has no subword of the form $T_j^{\pm 2}$. This allows one to sharpen the conclusion of Lemma 2.2:

Lemma 2.3. *If $\alpha \in B_n$ and $1 \le i \neq j \le n$, then the coefficient of the unique monomial of highest degree in $\alpha(a_{ij})$ is ± 1.* $\square$

One can be more specific about the coefficients in (1.2), namely

Lemma 2.4. *Let $C \in < T_1,\dots,T_n >$. Then $CT_iC^{-1} = (\psi, f)$, where $\psi = \sum_i \lambda_i\phi_i, f = \sum_i \mu_i d_i$, where $\lambda_i, \mu_i \in R_n$ satisfy $\lambda_i^* = \mu_i$ for all i.*

Proof. This uses Lemma 2.1 and is by induction on the length of C in the standard generators $T_1,\dots,T_n$. $\square$

Thus if $i \neq j$ and $A = C_1T_iC_1^{-1} = (\psi_1, f_1), B = C_2T_jC_2^{-1} = (\psi_2, f_2) \in < T_1,\dots,T_n >$, then f_1, f_2 are linearly independent and relative to this basis we have

$$A = \begin{pmatrix} 1 & \psi_1(f_2) \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ \psi_2(f_1) & 1 \end{pmatrix},$$

where by Lemma 2.4 we have

$$\psi_1 = \sum_i \lambda_i \phi_i, \quad f_1 = \sum_i \lambda_i^* d_i, \quad \psi_2 = \sum_i \mu_i \phi_i, \quad f_2 = \sum_i \mu_i^* d_i.$$

Thus $trace(AB) = 2 + \psi_1(f_2)\psi_2(f_1) = 2 + \phi(AB)$ and we have

$$\psi_1(f_2) = \sum_{i,j=1}^{n} \lambda_i \mu_j^* \phi_i(d_j), \quad \psi_2(f_1) = \sum_{i,j=1}^{n} \mu_j \lambda_i^* \phi_j(d_i).$$

Now let us refer back to (1.1). Since $\phi_i(d_j) = a_{ij}$ we see that if we let $b_{ij} = \psi_1(f_2), b_{ji} = \psi_2(f_1)$, that we have $trace(AB) = 2 + b_{ij}b_{ji}$ and

$$b_{ij}^* = \psi_1(f_2)^* = -\psi_2(d_1) = -b_{ji},$$

thus proving (1.1) and the relation $b_{ij}^* = -b_{ji}$.

Lemma 2.5. *For all $i \leq n$ we have $c'_{ni} = \pm c'_{n\,n-i}$.*

Proof. By [Hu2, Corollary 2.7] we see that the characteristic polynomial $\chi'_n(x)$ satisfies

$$\chi'_n(x) = (-x)^n \chi'_n(1/x)^*,$$

so that, up to a sign, the list $c'_{n1}, c'_{n2}, \ldots, c'_{n\,n-2}, c'_{n\,n-1}$ is the same forwards and backwards when we are in the ring R'_n where $a_{ij} = -a_{ji}$. □

§3 The rank 1 case

We first note that the matrix U_n can never have rank 0.

Lemma 3.1. *The i,j entry of Π_n is the sum of all monomials of the form*

$$a_{i_1 i_2} a_{i_2 i_3} a_{i_3 i_4} \ldots a_{i_{r-1} i_r},$$

where $i_1 = i, i_r = j$ and $i = i_1 < i_2 < i_3 < \cdots < i_{r-1} \leq n$.

Proof. This is by induction on $n \geq 2$. □

For an $n \times n$ matrix A and $1 \leq i_1, \ldots, i_r, j_1, \ldots, j_r \leq n$ we let $A([i_1, \ldots, i_r], [j_1, \ldots, j_r])$ denote the determinant of the $r \times r$ submatrix of A where we use only the rows $i_1, \ldots, i_r$ and columns $j_1, \ldots, j_r$ from the matrix A. If $[i_1, \ldots, i_r] = [j_1, \ldots, j_r]$ then we use the notation $A[i_1, \ldots, i_r]$. We also let $A_{[i_1,\ldots,i_r],[j_1,\ldots,j_r]}$ denote the determinant of the $(n-r) \times (n-r)$ submatrix of A where we use only the rows numbered $\{1, \ldots, n\} \setminus \{i_1, \ldots, i_r\}$ and columns numbered $\{1, \ldots, n\} \setminus \{j_1, \ldots, j_r\}$. If $[i_1, \ldots, i_r] = [j_1, \ldots, j_r]$, then we use the notation $A_{[i_1,\ldots,i_r]}$.

A key observation that simplifies many calculations is:

Proposition 3.2. *For all $n \geq 2$ we have $det(\Pi'_n + I_n) = det(U_n)$ where U_n is as defined in Theorem 1. In fact all of the determinantal ideals determined by $\Pi'_n + I_n$ and U_n are the same.*

Proof. As we noted in the proof of [H2, Theorem 2.8], for any $\lambda \in R$ the matrix $\Pi_n - \lambda I_n$ can be row-reduced to the following matrix (using only elementary matrices of determinant 1), which thus has the same determinantal ideals as $\Pi_n - \lambda I_n$:

$$V_n = \begin{pmatrix} 1-\lambda & \lambda a_{12} & \dots & \lambda a_{1i} & \dots & \lambda a_{1,n-1} & \lambda a_{1n} \\ a_{21} & 1-\lambda & \dots & \lambda a_{2i} & \dots & \lambda a_{2,n-1} & \lambda a_{2n} \\ \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots \\ a_{i1} & a_{i2} & \dots & 1-\lambda & \dots & \lambda a_{i,n-1} & \lambda a_{in} \\ \vdots & \vdots & & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,i} & \dots & 1-\lambda & \lambda a_{n-1,n} \\ a_{n1} & a_{n2} & \dots & a_{ni} & \dots & a_{n,n-1} & 1-\lambda \end{pmatrix} \qquad (3.1.)$$

Putting $\lambda = -1$ and $a_{ij} = -a_{ji}$ for $i < j$ we obtain the matrix U_n. □

Proposition 3.3. *Suppose that 2 is invertible in R. The matrix $\Pi_n + I_n$ has rank 1 if and only if we have*

$$a_{in}a_{ni} = -4, \quad \textit{and} \quad 2a_{ij} = a_{in}a_{nj},\ 2a_{ji} = -a_{jn}a_{ni}$$

for all $1 \leq i < j < n$. Further, the point where $a_{ij} = 2$ for $i > j$ and $a_{ij} = -2$ for $i < j$ satisfies the above conditions.

Proof. The last statement follows easily from the first. The basic fact here is that a matrix has rank 1 if and only if all of its 2×2 minors are zero. By Proposition 3.2 it suffices to consider the matrix V_n defined by (3.1) with $\lambda = -1$. In this case one easily sees that for $i < j < n$ we have

$$V_n([i,j]) = 4 + a_{ij}a_{ji}, \quad \text{and} \quad V_n([i,n],[j,n]) = 2a_{ij} - a_{in}a_{nj}.$$

Now for example if $i < j < k < m \leq n$, then we have $V_n([i,j],[k,m]) = a_{ik}a_{jm} - a_{im}a_{jk}$, and modulo the ideal generated by the above relations this is equal to

$$\frac{1}{4}(a_{in}a_{nk}a_{jn}a_{nm} - a_{in}a_{nm}a_{jn}a_{nk}) = 0.$$

All other cases are similar. The result follows. □

In the antisymmetric case $a_{ij} = -a_{ji}$ we see that there are only 2^{n-1} real solutions to these equations. Each such solution gives a coordinate vector $(a_{21}, a_{32}, \dots, a_{n\,n-1})$ where each entry is ± 2. Let $\mathcal{S}_n$ denote the 2^{n-1} points determined in Proposition 3.3 at which $\Pi'_n + I_n$ has rank 1. Note that the point where $a_{ij} = 2, a_{ji} = -2$ for $1 \leq i > j \leq n$ is in $\mathcal{S}_n$.

Lemma 3.4. *Suppose that 2 is invertible in R. Then there are exactly 2^{n-1} points where $\Pi'_n + I_n$ has rank 1 and the braid group B_n acts transitively on them.*

Proof. We have already noted that there are 2^{n-1} such points and that each point in $\mathcal{S}_n$ is completely determined by the $n-1$ coordinates corresponding to the variables $a_{ni}, 1 \leq i < n$.

Thus we will denote each point of $\mathcal{S}_n$ by a list of $n-1$ integers $a_{n1}, a_{n2}, \ldots, a_{n\,n-1}$ each of which is in $\{\pm 2\}$. Thus the B_n-action on these $n-1$ coordinates can easily be calculated (for the generators $\sigma_i, i < n$) using (1.2) as follows: for $1 \leq i < n-1$ we have

$$\sigma_i(a_{ni}) = -a_{n\,i+1}; \quad \sigma_i(a_{n\,i+1}) = a_{ni}; \quad \sigma_i(a_{nj}) = a_{nj} \quad j \neq i, i+1. \tag{3.2}$$

The action of σ_{n-1} is:

$$\sigma_{n-1}(a_{ni}) = \frac{a_{n\,n-1}a_{ni}}{2} \quad \text{for} \quad i \leq n-2 \quad \text{and} \quad \sigma_{n-1}(a_{n\,n-1}) = a_{n-1\,n} = -a_{n\,n-1}. \tag{3.3}$$

We now show that each point of $\mathcal{S}_n$ is in the orbit of $\pi_2 = (2, 2, \ldots, 2)$. Suppose that $p = (p_1, \ldots, p_{n-1}) \in \mathcal{S}_n$. If $p \neq \pi_2$, then there is $1 \leq i \leq n$ such that $p_1 = \cdots = p_{i-1} = 2, p_i = -2$. Define $\mu(p) = i$. If $\mu(p) > 1$, then using the above action and the relations satisfied by the a_{ij} given in Proposition 3.2 one checks that $\mu(\sigma_{i-1}(p)) > \mu(p)$. Continuing we see that p is in the orbit of π_2.

If $\mu(p) = 1$, then either $p = -\pi_2$ or there is a unique $j > 1$ such that $p_1 = p_2 = \cdots = p_{j-1} = -2, p_j = 2$. Let $\nu(p) = j$. If $p \neq -\pi_2$, then again one checks that $\nu(\sigma_{j-1}(p)) > \nu(p)$ and so in each case we see that p is in the orbit of $-\pi_2$. But we now have $\sigma_1(-\pi_2) = (2, -2, -2, \ldots, -2)$, which puts us back into the situation where $\mu(\sigma_1(-\pi_2)) = 2 > 1$. □

Let $W(D_n) < W(B_n)$ denote the Weyl or Coxeter groups of types B_n, D_n [GB, §5.3]. Thus $W(B_n)$ is the group of all "signed permutations" of $\{\pm 1, \ldots, \pm n\}$ and $W(D_n)$ is the subgroup where the product of the signs equals the sign of the permutation. As we pointed out in §1, the symmetric group S_n, which is the Coxeter group of type A_n, is a quotient of B_n. The above result is connected to the following result which shows that $W(D_n)$ is also a quotient of B_n:

Proposition 3.5. *For all $n \geq 2$ there is an epimorphism $B_n \to W(D_n)$. Let κ_i denote the image of σ_i. Then the following is a presentation of $W(D_n)$ with these generators:*

$$< \kappa_1, \ldots, \kappa_{n-1} | \kappa_i\kappa_{i+1}\kappa_i = \kappa_{i+1}\kappa_i\kappa_{i+1}, \quad \kappa_i^2 = \kappa_{i+1}\kappa_i^2\kappa_{i+1}, \text{ for } i < n,$$
$$\kappa_i^4 = 1, \quad \kappa_i\kappa_j = \kappa_j\kappa_i \text{ for } |i-j| > 1 > .$$

The group with the above presentation is isomorphic to the image of the action of B_n on the points of $\mathcal{S}_n$ if 2 is invertible in R.

Proof. For $1 \leq i \leq n$ we let $\tau_i = (i\,i+1) \in S_n \subset W(B_n)$ denote the transposition and let $\epsilon_i \in W(B_n)$ denote multiplication of i by -1. Then one has the standard relations between these generators of $W(D_n)$ and one checks that the elements $\kappa_i = \epsilon_i\tau_i \in W(D_n)$ satisfy the braid relations and so a homomorphism $B_n \to W(D_n)$ is determined by $\sigma_i \mapsto \kappa_i$.

To see that we have an epimorphism we note that $\kappa_i^2 = \epsilon_i\epsilon_{i+1}$ and that the subgroup $< \kappa_1^2, \ldots, \kappa_{n-1}^2 >$ of $W(D_n)$ is equal to $W(D_n) \cap < \epsilon_1, \ldots, \epsilon_n >$ and has order 2^{n-1}. Further there is an epimorphism $< \kappa_1, \ldots, \kappa_{n-1} > \to S_n$ which kills $< \kappa_1^2, \ldots, \kappa_{n-1}^2 >$. Thus the order of $< \kappa_1, \ldots, \kappa_{n-1} >$ is at least $n!2^{n-1} = |W(D_n)|$. However as $< \kappa_1, \ldots, \kappa_{n-1} >$ is a subgroup of $W(D_n)$ it must be equal to it. Thus the homomorphism $B_n \to W(D_n)$ is onto.

Now we consider the given presentation. The relations show that $< \kappa_1^2, \ldots, \kappa_{n-1}^2 >$ is a normal abelian subgroup of the group with the given presentation and that it has order 2^{n-1}. One further sees that the quotient by this normal subgroup is isomorphic to S_n. Since the relations given are easily shown to be satisfied by the images of σ_i under the homomorphism $B_n \to W(D_n)$ we see that the given presentation is a presentation for $W(D_n)$.

Now using the action of the generators σ_i on the elements of $\mathcal{S}_n$ (see (3.2) and (3.3)) one can show that these generators satisfy the same relations as do the κ_i. One can check for example that the action of σ_i^2 on $\mathcal{S}_n$ is to multiply the ith and $i+1$th coordinates of elements of $\mathcal{S}_n$ by -1, showing that the images of $\sigma_1^2, \dots, \sigma_{n-1}^2$ generate a normal abelian subgroup of order 2^{n-1}. The rest follows easily. $\square$

Remark 3.6. The set of singular points $\mathcal{S}_3$ was considered by Goldman [Go, §6.1 case (d)] and corresponded to characters of representations into the centre of $SL(2, \mathbb{R})$.

§4 The case $n = 3$

Before embarking upon a general analysis of the rank 2 case we look at the situation where $n = 3$ and $R = \mathbb{R}$. Here we have a single generator for the ring of invariants, namely

$$c'_{31} = a_{21}^2 + a_{31}^2 + a_{32}^2 - a_{21}a_{32}a_{31} + 3.$$

Now $det(U_3) = -2(c_{31} + 1)$ and solving $c'_{31} + 1 = 0$ for (say) a_{21} we get two solutions:

$$a_{21}^{\pm} = \frac{a_{31}a_{32} \pm \sqrt{(a_{31}^2 - 4)(a_{32}^2 - 4)}}{2}.$$

We will thus think of each $a_{21}^{\pm}$ as being a function of a_{31}, a_{32}. Now as we are looking for real solutions we see that we need $(a_{31}^2 - 4)(a_{32}^2 - 4) \geq 0$. This defines a domain $E \subset \mathbb{R}^2$ for each $a_{21}^{\pm}$. The interior of E consists of 5 components, one of which has closure the region S determined by $-2 \leq a_{31}, a_{32} \leq 2$. Thus the solution set of $c'_{31} + 1 = 0$ over S consists of two smooth topological discs D_-, D_+ in $\mathbb{R}^3$ which meet along the image of the subset of E where $a_{21}^+ = a_{21}^-$ i.e. at all points where $(a_{31}^2 - 4)(a_{32}^2 - 4) = 0$. But this set is just the boundary of S. We will use coordinates for $\mathbb{R}^3$ in the order (a_{21}, a_{31}, a_{32}). Then the boundary of S is a union of 4 straight line segments where $a_{31}, a_{32} = \pm 2$. One checks that if $a_{31} = 2$, then $a_{21} = a_{32}$ and so we get a straight line segment from $(2, 2, 2)$ to $(-2, 2, -2)$ in this case. The other cases are similar and we get line segments connecting the points $(2, 2, 2), (-2, 2, -2), (2, -2, -2), (-2, -2, 2)$ of $\mathcal{S}_3$. These line segments give the 1-skeleton of a regular tetrahedron in $\mathbb{R}^3$. It follows from the above formula for $a_{21}^{\pm}$ that the discs D_-, D_+ meet along a piecewise-linear curve. Let $D = D_- \cup D_+$.

Now one can check that the line segment between $(2, 2, 2)$ and $(2, -2, -2)$ is in D, and that similarly the line segment between $(-2, 2, -2)$ and $(-2, -2, 2)$ is in D. Further the smooth nature of the equations defining $D_\pm$ shows that all the points on these two lines are smooth (except possibly at the end points). Clearly every point of the interior of D_- and the interior of D_+ is a smooth point. Thus the only possibility of a non-smooth point is a point of the piecewise linear curve $D_- \cap D_+$. However we can also do the above analysis by solving for a_{31} or a_{32} (instead of a_{21}). Doing so, and repeating the above argument, shows that in fact the only possible singular points are at the four points $(2, 2, 2), (-2, 2, -2), (2, -2, -2), (-2, -2, 2)$.

One can also check that D meets the 3-ball $[-2, 2]^3$ only at the six line segments joining the points $(2, 2, 2), (-2, 2, -2), (2, -2, -2), (-2, -2, 2)$.

Now B_3 acts on the set V'_{31} of real solutions of $c'_{31}+1=0$. The projection of V'_{31} onto the a_{31}, a_{32}-axis gives a region consisting of S together with four regions $Q_1, \ldots, Q_4$ determined by $|a_{31}|, |a_{32}| \geq 2$. Each of these four regions is a plane meeting S at a single point. The part of V'_{31} above each Q_i is thus an open disc which meets D at one of the four points $(2,2,2), (-2,2,-2), (2,-2,-2), (-2,-2,2)$. Again one easily sees that it is smooth except at these points. One gets a very nice picture of V'_{31} upon drawing it using something like maple. One clearly sees that D has similarities with the standard tetrahedron embedded in $[-2,2]^3 \subset \mathbb{R}^3$. See Figure 2 for a graphical representation of D. A similar drawing may be found in [Go, Fig. 2].

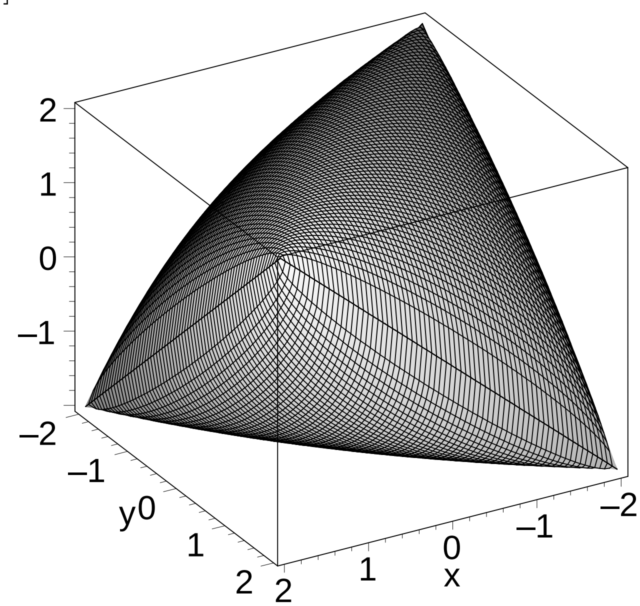

Figure 2.

Now D bounds a 3-ball $B \subset \mathbb{R}^3$ and the action of B_3 clearly fixes B since B_3 fixes the origin and D. Further for all $0 < t < 4$ we consider the set of solutions to

$$a_{21}a_{31}a_{32} - a_{21}^2 - a_{31}^2 - a_{32}^2 + 4 - t = 0 \tag{4.1}$$

in $\mathbb{R}^3$. This equation occurs frequently in the literature; see for example [Go, K1]. As in the above we can solve for a_{21} to get:

$$a_{21}^{\pm}(t) = \frac{a_{31}a_{32} \pm \sqrt{(a_{31}^2-4)(a_{32}^2-4)-4t}}{2}.$$

Thus for $0 < t < 4$ we see that a part of the solution set is a smooth 2-sphere $D^{(t)} \subset B$. We have $D = D^{(0)}$. The fact that this 2-sphere is smooth for $t \in (0,4)$ is seen by looking at the Jacobian of the equation (4.1) and noting that the only singular points are at the four points $(2,2,2), (-2,2,-2), (2,-2,-2), (-2,-2,2)$, together with $(0,0,0)$.

From [H1] we know that the representation $B_3 \to Aut(\mathbb{Q}'_3)$ has kernel equal to its centre $Z(B_3) =< (\sigma_1\sigma_2)^3 >\cong \mathbb{Z}$. Since B is a 3-ball there are points $(b_{21}, b_{31}, b_{32}) \in B$ for which the real numbers b_{21}, b_{31}, b_{32} are algebraically independent. Thus the action of $B_3/Z(B_3)$ at these points is faithful. Now the set of points $(b_{21}, b_{31}, b_{32}) \in B$ with algebraically independent coordinates is dense in the ball B. Thus there are a dense set of values of $t \in (0,4)$ such that the action of $B_3/Z(B_3)$ is faithful on the spheres $D^{(t)}$.

We now consider the question of convexity. This we will prove by showing that the Hessians $H_4^{\pm}$ of the function

$$a_{21}^{\pm} = a_{31}a_{32}/2 \pm \sqrt{(a_{31}^2-4)(a_{32}^2-4)}/2$$

are positive and negative definite (except on a set of measure 0). We will do H_4^-, showing that it is positive definite, the other case being similar. This Hessian is

$$H_4^- = \begin{pmatrix} 2\,\frac{{a_{32}}^2-4}{({a_{31}}^2-4)\sqrt{({a_{32}}^2-4)({a_{31}}^2-4)}} & -1/2\,\frac{a_{31}a_{32}-\sqrt{({a_{32}}^2-4)({a_{31}}^2-4)}}{\sqrt{({a_{32}}^2-4)({a_{31}}^2-4)}} \\ -1/2\,\frac{a_{31}a_{32}-\sqrt{({a_{32}}^2-4)({a_{31}}^2-4)}}{\sqrt{({a_{32}}^2-4)({a_{31}}^2-4)}} & 2\,\frac{{a_{31}}^2-4}{({a_{32}}^2-4)\sqrt{({a_{32}}^2-4)({a_{31}}^2-4)}} \end{pmatrix}.$$

Now since $-2 \le a_{31}, a_{32} \le 2$ we see that the diagonal entries are positive on the interior of the domain. Thus we need only show that $det(H_4^-)$ is positive. Now we have

$$det(H_4^-) = -1/2\,\frac{{a_{31}}^2{a_{32}}^2 - a_{31}a_{32}\sqrt{({a_{32}}^2-4)({a_{31}}^2-4)} - 2\,{a_{31}}^2 - 2\,{a_{32}}^2}{({a_{32}}^2-4)({a_{31}}^2-4)}.$$

Solving $det(H_4^-) = 0$ gives the solutions $a_{31} = \pm a_{32}$. Call this set Z and notice that $[-2,2]^2 \setminus Z$ has components on the interior of each of which $det(H_4)$ is positive.

We summarise these results as follows:

Theorem 4.1. *The quotient $B_3/Z(B_3) \cong PSL_2(\mathbb{Z})$ acts faithfully on a convex topological ball $B \subset [-2,2]^3 \subset \mathbb{R}^3$ with boundary D containing the four points*

$$(2,2,2), (-2,2,-2), (2,-2,-2), (-2,-2,2).$$

The 2-sphere D is smooth except at these points. The six line segments connecting these points are also in D and form the 1-skeleton of a regular tetrahedron in $\mathbb{R}^3$. Further, D meets the 3-ball $[-2,2]^3$ only at these six line segments.

Moreover, for all $0 < t < 4$ there is a component $D^{(t)}$ of the solution set of

$$a_{21}a_{31}a_{32} - a_{21}^2 - a_{31}^2 - a_{32}^2 + 4 - t = 0$$

that is a $B_3/Z(B_3)$-invariant smooth 2-sphere inside B. There is a dense set of values of such t for which this action of $B_3/Z(B_3)$ on $D^{(t)}$ is faithful. $\square$

§5 Results on symmetric matrices

Proposition 5.1. *Let $U = (u_{ij})$ be any symmetric $n \times n$ matrix. Then for all $1 \le j < i \le n$ the determinant $det(U)$ is quadratic in u_{ij} and its discriminant relative to u_{ij} is equal to $4U_{[i]}U_{[j]}$.*

In particular, if $U = \Pi'_n + I_n$, then for all $1 \le j < i \le n$ the determinant $det(\Pi'_n + I_n)$ is quadratic in a_{ij} and its discriminant relative to a_{ij} is equal to $4U_{[i]}U_{[j]}$.

Proof. The second statement will follow from the first, which we now prove. Since there is an action of S_n on the entries of U which does not change the determinants we need only prove the result for $i = 2, j = 1$. Since u_{21} only occurs in the $(1,2)$ and $(2,1)$ positions in U, $det(U)$ is clearly quadratic in u_{21}. Now from [M; p. 370] we see that

$$U_{[1]}U_{[2]} - (U_{[1],[2]})^2 = det(U)U_{[1,2]}. \tag{5.1}$$

Now

$$U_{[1],[2]} = u_{21}U_{[1,2],[1,2]} - u_{31}U_{[1,2],[1,3]} + u_{41}U_{[1,2],[1,4]} - \cdots = u_{21}U_{[1,2],[1,2]} + X.$$

Since $U_{[1]}, U_{[2]}$ and $U_{[1,2],[1,2]}$ are constant relative to u_{21} it follows that the discriminant of

$$\begin{aligned} -U_{[1]}U_{[2]} + (U_{[1],[2]})^2 &= -U_{[1]}U_{[2]} + (u_{21}U_{[1,2]} + X)^2 \\ &= u_{21}^2(U_{[1,2]})^2 + 2XU_{[1,2]}u_{21} - U_{[1]}U_{[2]} + X^2 \end{aligned}$$

relative to the variable u_{21} is

$$4X^2(U_{[1,2]})^2 - 4(U_{[1,2]})^2(X^2 - U_{[1]}U_{[2]}) = 4(U_{[1,2]})^2U_{[1]}U_{[2]}.$$

Now the appearance of the extra factor $U_{[1,2]}$ on the right hand side of (5.1) shows that the discriminant of $det(U)$ relative to u_{21} is $4U_{[1]}U_{[2]}$. $\square$

Proposition 5.2. *Let $1 < r \le n, r \in 2\mathbb{Z}$ and let R be a ring of characteristic 2. Then there is a symmetric matrix of rank r over R'_n with 0s on the diagonal, all of whose off-diagonal entries are non-zero.*

Proof. Let $m = n - r$ and let $Q_r = (a_{ij})$ be a symmetric $r \times r$ matrix with 0s on the diagonal. Let $B = (b_{ij})$ be a generic $m \times r$ matrix where $b_{ij} = a_{i+r\,j}$. Note that Q_r has rank r since r is even. Also define $n \times n$ matrices

$$M_n = \begin{pmatrix} Q_r & 0 \\ 0 & 0 \end{pmatrix}, \quad E_n = \begin{pmatrix} I_r & 0 \\ B & I_m \end{pmatrix}.$$

Clearly M_n has rank r. We claim that $E_nM_nE_n^t$ satisfies the requirements of Proposition 5.2. Now we note that $E_nM_nE_n^t$ is symmetric; further we have

$$\begin{aligned} E_nM_nE_n^t &= \begin{pmatrix} I_r & 0 \\ B & I_m \end{pmatrix}\begin{pmatrix} Q_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I_r & B^t \\ 0 & I_m \end{pmatrix} \\ &= \begin{pmatrix} Q_r & 0 \\ BQ_r & 0 \end{pmatrix}\begin{pmatrix} I_r & B^t \\ 0 & I_m \end{pmatrix} = \begin{pmatrix} Q_r & Q_rB^t \\ BQ_r & BQ_rB^t \end{pmatrix}. \end{aligned}$$

The ps entry of BQ_r is $\sum_{i=1}^r b_{pi}a_{is} \neq 0$ and the pt entry of BQ_rB^t is $\sum_{s=1}^r \sum_{i=1}^r b_{pi}a_{is}b_{ts}$. It follows that all the off-diagonal entries of $E_nM_nE_n^t$ are non-zero and that all the diagonal entries are zero. $\square$

§6 The $n = 4$ case

The case $n = 4$ has some features that do not seem to arise in the $n > 4$ cases, so we look at these in this section. The reason for the exceptional behaviour is the existence of the well-known epimorphism $B_4 \to B_3$, $(\sigma_1 \mapsto \sigma_1, \sigma_2 \mapsto \sigma_2, \sigma_3 \mapsto \sigma_1)$. The most notable difference is the existence of a pair of subspaces that are permuted by B_4:

Theorem 6.1. *For $n = 4$ the braid group B_4 acts on a disjoint union $S^{2+} \cup S^{2-}$ of two 2-spheres. The union of these spheres contain the singular points $\mathcal{S}_4$. For each $\epsilon = \pm$ there is a smooth embedding $\iota_\epsilon : \mathbb{R}^3 \to \mathbb{R}^6$ such that the images P_ϵ of these maps are permuted by B_4 and contain the two 2-spheres i.e. $S^{2\pm} \subset P_\pm$. We have $P_- \cap P_+ = \{0\}$.*

Proof. We define the embedded planes relative to the coordinates $(a_{21}, a_{31}, a_{32}, a_{41}, a_{42}, a_{43})$ as follows:

$$P_+ = \{(a, b, c, c, d, a) \in \mathbb{R}^6 | b + d = ac\};$$
$$P_- = \{(a, b, c, -c, -d, -a) \in \mathbb{R}^6 | b + d = ac\}.$$

It is easy to check that $\sigma_i(P_\epsilon) = P_{-\epsilon}$ for all $1 \le i < 4$ and $\epsilon = \pm$; for example if $(a, b, c, c, d, a) \in P_+$, then

$$\sigma_1(a, b, c, c, d, a) = (-a, c - ab, b, d - ac, c, a) = (a', b', c', -c', -d', -a'),$$

where $b' + d' = (c - ab) + (-c) = -ab = a'c'$ and so $\sigma_1(a, b, c, c, d, a) \in P_-$.

One now also checks that each point in $\mathcal{S}_4$ is contained in one of the $P_\pm$. Next note that if we require that the a, b, c, d in the above also satisfy

$$abc - a^2 - b^2 - c^2 + 4 = 0,$$

then we obtain the 2-spheres $S^{2+} \subset P_+, S^{2-} \subset P_-$. That these are 2-spheres follows from the arguments of §3. These are contained in the same level set of the invariants c'_{41}, c'_{42} as the points of $\mathcal{S}_4$. One easily checks that $P_- \cap P_+ = \{0\}$ and so that S^{2+}, S^{2-} are disjoint. Next one checks that the union of these spheres is also fixed by the action of B_4. To do this one only needs to check the action of the generators $\sigma_i, i = 1, 2, 3$ and this is easy. $\square$

§7 The rank 2 case

In this section we assume that $R = \mathbb{R}$ and we investigate the nature of the subset $V_{n2} \subset \mathbb{R}^{\binom{n}{2}}$ where the rank of the matrix $\Pi'_n + I_n$ is 2. This set is invariant under the action of B_n by Proposition 3.2.

Let $\lambda = (\lambda_{21}, \lambda_{31}, \lambda_{32}, \ldots, \lambda_{nn-1}) \in \mathbb{R}^{\binom{n}{2}}$. Then λ is called a *tetrahedral point* if for all $1 \le i < j < k \le n$ we have

$$\lambda_{kj}\lambda_{ki}\lambda_{ji} - \lambda_{kj}^2 - \lambda_{ki}^2 - \lambda_{ji}^2 + 4 = 0.$$

Lemma 7.1. *Let $\lambda = (\lambda_{21}, \lambda_{31}, \ldots, \lambda_{nn-1}) \in \mathbb{R}^{\binom{n}{2}}$ be a point at which rank$(\Pi'_n + I_n)$ is at most 2. Then λ is a tetrahedral point.*

Proof. Here we need only consider the minors $U_n([i, j, k])$; these give exactly the tetrahedral condition. $\square$

Lemma 7.2. *The only singular points on V_{42} are the points in the set $\mathcal{S}_4$.*

Proof. For any square matrix M the singular points of the determinantal ideal of all 3×3 minors is given by the determinantal ideal of all 2×2 minors [ACGH, p. 69]. This is exactly the set $\mathcal{S}_n$ (see Lemma 3.4). $\square$.

Remark 7.3. We have seen above that the set $\mathcal{S}_n$ is fixed by the action of B_n, however these are not the only finite orbits: For $n = 3$ the orbit of $(\sqrt{2}, 0, -1)$ has 36 elements and the action of B_3 gives a group of order $2654208 = 2^{15}3^4$. The orbit of $(-\sqrt{2+\sqrt{2}}, -\sqrt{2+\sqrt{2}}, 2)$ has 96 elements and the action of B_3 gives a group of order $1536 = 2^9 3$. The B_4-orbit of the tetrahedral point $(\sqrt{3}, -\sqrt{3}, -1, 0, 1, 1) \in V_{42}$ consists of 288 points; the action of B_4 on these points gives a group of order $165888 = 2^{11}3^4$.

Now for $1 \le i < j < k \le n$ there is a projection $\pi_{ijk} : \mathbb{R}^{\binom{n}{2}} \to \mathbb{R}^3$ which forgets those coordinates a_{rs} with $r, s \notin \{i, j, k\}$. The fact that points $v \in V_{n2}$ satisfy the tetrahedral condition shows that the image of each such π_{ijk} consists of points $(x, y, z) \in \mathbb{R}^3$ such that $xyz - x^2 - y^2 - z^2 + 4 = 0$. The nature of this solution set was investigated in §3 from which it is natural to define:

$$V_{n2}^{(2)} = V_{n2} \cap [-2, 2]^{\binom{n}{2}}.$$

Note that $V_{n2}^{(2)}$ is a compact set and is invariant under the action of B_n.

Now we solve the tetrahedral equations. For any $1 \le j < i < n$ the coordinates a_{ij}, a_{ni}, a_{nj} satisfy the tetrahedral relation and so we can solve for a_{ij} as a function of a_{ni}, a_{nj}:

$$a_{ij} = a_{ij}^{\epsilon_{ij}}(a_{n1}, a_{n2}, \ldots, a_{n\,n-1}) = a_{nj}a_{ni}/2 + \epsilon_{ij}\sqrt{(a_{nj}^2 - 4)(a_{ni}^2 - 4)}/2, \tag{7.1}$$

where $\epsilon_{ij} = \epsilon_{ji} = \pm 1$. For convenience we also put $a_{ni}^{\pm 1}(a_{n1}, a_{n2}, \ldots, a_{n\,n-1}) = a_{ni}$.

Lemma 7.4. *For any $z = (a_{n1}, a_{n2}, \ldots, a_{n\,n-1}) \in [-2, 2]^{n-1}$ and $\epsilon = (\epsilon_{21}, \epsilon_{31}, \ldots, \epsilon_{n\,n-1}) \in \{\pm 1\}^{\binom{n}{2}}$ the point $f_\epsilon(z) = (a_{21}^{\epsilon_{21}}(z), a_{31}^{\epsilon_{31}}(z), \ldots, a_{n\,n-1}^{\epsilon_{n\,n-1}}(z)) \in \mathbb{R}^{\binom{n}{2}}$ with the $a_{ij}^{\epsilon_{ij}}, 1 \le j < i < n$, given by (7.1) is in V_{n2} if and only if the following conditions are satisfied:*

$$\epsilon_{ij}\epsilon_{jk}\epsilon_{ik} = -1 \quad \textit{for all distinct} \quad 1 \le i, j, k < n. \tag{7.2}$$

Proof. To prove the Lemma we need to show that any $(a_{21}^{\epsilon_{21}}, a_{31}^{\epsilon_{31}}, \ldots, a_{n\,n-1}^{\epsilon_{n\,n-1}})$ gives a point in $R^{\binom{n}{2}}$ (with the $a_{ij}^{\epsilon_{ij}}, 1 \le j < i < n$ given by (7.1)) satisfying all of the equations $U_n([i_1, i_2, i_3], [j_1, j_2, j_3]) = 0$. Now the symmetric group acts naturally on U_n by permutation of subscripts and the subgroup S_{n-1} fixes any point where the ϵ_{ij} are given by (7.2). Thus this S_{n-1} acts on the $U_n([i_1, i_2, i_3], [j_1, j_2, j_3])$ and we need only check $U_n([i_1, i_2, i_3], [j_1, j_2, j_3]) = 0$ for $U_n([1, 2, 3], [j_1, j_2, j_3])$ and $U_n([1, 2, n], [j_1, j_2, j_3])$. In fact even here there are only a small number of cases to check depending on the cardinality of the set $\{1, 2, 3\} \cap \{j_1, j_2, j_3\}$ or $\{1, 2, n\} \cap \{j_1, j_2, j_3\}$ (respectively). For example we have (using $\epsilon_{ij}^2 = 1$)

$$\begin{aligned}
U_n([1, 2, 3]) = {} & (a_{n1}^2 - 4)(a_{n2}^2 - 4)(a_{n3}^2 - 4)(\epsilon_{12}\epsilon_{13}\epsilon_{23} + 1)/8 \\
& + a_{n2}a_{n3}\sqrt{(a_{n2}^2 - 4)(a_{n3}^2 - 4)}(a_{n1}^2 - 4)(\epsilon_{23} + \epsilon_{12}\epsilon_{13})/8 \\
& + a_{n1}a_{n3}\sqrt{(a_{n1}^2 - 4)(a_{n3}^2 - 4)}(a_{n2}^2 - 4)(\epsilon_{13} + \epsilon_{23}\epsilon_{12})/8 \\
& + a_{n1}a_{n2}\sqrt{(a_{n1}^2 - 4)(a_{n2}^2 - 4)}(a_{n3}^2 - 4)(\epsilon_{12} + \epsilon_{13}\epsilon_{23})/8.
\end{aligned}$$

This is zero if and only if $\epsilon_{12}\epsilon_{13}\epsilon_{23} = -1$ and so we get $\epsilon_{ij}\epsilon_{ik}\epsilon_{jk} = -1$ for all distinct $1 \leq i, j, k < n$. Now one similarly checks that with these conditions satisfied we also have $U_n([1, 2, 3], [1, 2, 4] = 0, U_n([1, 2, 3], [3, 4, 5] = 0, U_n([1, 2, 3], [4, 5, 6] = 0, U_n([1, 2, n], [1, 2, 3] = 0$ etc. For example $U_n([1, 2, 3], [1, 2, 4]$ can be written as:

$$\begin{aligned}
&\frac{1}{8}\sqrt{(a_{n1}^2-4)(a_{n3}^2-4)}a_{n1}a_{n4}(a_{n2}^2-4)(\epsilon_{13}+\epsilon_{12}\epsilon_{23})\\
&+\frac{1}{8}\sqrt{(a_{n2}^2-4)(a_{n3}^2-4)}a_{n2}a_{n4}(a_{n1}^2-4)(\epsilon_{23}+\epsilon_{12}\epsilon_{13})\\
&+\frac{1}{8}\sqrt{(a_{n1}^2-4)(a_{n4}^2-4)}a_{n1}a_{n3}(a_{n2}^2-4)(\epsilon_{14}+\epsilon_{12}\epsilon_{24})\\
&+\frac{1}{8}\sqrt{(a_{n2}^2-4)(a_{n4}^2-4)}a_{n2}a_{n3}(a_{n1}^2-4)(\epsilon_{24}+\epsilon_{12}\epsilon_{14})\\
&-\frac{1}{8}\sqrt{(a_{n1}^2-4)(a_{n2}^2-4)(a_{n3}^2-4)(a_{n4}^2-4)}a_{n1}a_{n2}(2\epsilon_{12}\epsilon_{34}-\epsilon_{14}\epsilon_{23}-\epsilon_{13}\epsilon_{24})\\
&+\frac{1}{8}\sqrt{(a_{n3}^2-4)(a_{n4}^2-4)}[16\epsilon_{24}(\epsilon_{23}+\epsilon_{12}\epsilon_{13})+16\epsilon_{23}(\epsilon_{24}+\epsilon_{12}\epsilon_{14})\\
&+a_{51}^2a_{52}^2(\epsilon_{12}\epsilon_{13}\epsilon_{24}+\epsilon_{12}\epsilon_{23}\epsilon_{14}-2\epsilon_{34})-4a_{n1}^2(\epsilon_{13}\epsilon_{14}+\epsilon_{12}\epsilon_{14}\epsilon_{23}+\epsilon_{12}\epsilon_{13}\epsilon_{24}-\epsilon_{34})\\
&-4a_{n2}^2(\epsilon_{23}\epsilon_{24}+\epsilon_{12}\epsilon_{23}\epsilon_{14}+\epsilon_{12}\epsilon_{13}\epsilon_{24}-\epsilon_{34})].
\end{aligned}$$

The other cases are similar. This proves the result. $\square$

Conjecture 7.5. Now if $1 < r \leq n$, then for $1 \leq j < i \leq n-r+1$ we can solve the quadratic equation $U_n[i, j, n+2-r, n+3-r, \ldots, n] = 0$ for a_{ij} to get a solution depending on a parameter $\epsilon_{ij} = \pm 1$. We conjecture that doing so for all such i, j will give a matrix of rank r if and only if we have $\epsilon_{ij}\epsilon_{jk}\epsilon_{ik} = (-1)^{r+1}$. What we have proved above is that this is true for $r = 2$. As indicated in the above proof each case requires only a finite amount of checking and we have also checked that the conjecture holds for $r = 3, 4$.

Returning to the case $r = 2$ one can check that the conditions (7.2) are satisfied if we have

$$\epsilon_{ij} = -\epsilon_{1i}\epsilon_{1j}$$

for all $1 < i < j < n$. Thus there are 2^{n-2} choices corresponding to the values of $\epsilon_{21}, \epsilon_{31}, \ldots, \epsilon_{n-1,1}$ and we will now let $\epsilon = (\epsilon_{21}, \epsilon_{31}, \ldots, \epsilon_{n-1,1})$. Thus by (7.1) for each ϵ satisfying (7.2) we have a function

$$f_\epsilon = f_\epsilon(a_{n1}, \ldots, a_{nn-1}) : [-2, 2]^{n-1} \to \mathbb{R}^{\binom{n}{2}}.$$

The image of $[-2, 2]^{n-1}$ is an $(n-1)-$*ball* in $\mathbb{R}^{\binom{n}{2}}$. We will call it an $\epsilon-$*cube*.

These ϵ-cubes meet along only their faces and we now describe these identifications and a certain fundamental groupoid. Let $p_\epsilon = f_\epsilon(0, 0, \ldots, 0)$. This is the *centre* of the ϵ-cube. By a *face* of the ϵ-cube we will mean the subset determined by $a_{ni} = \pm 2$ for some $1 \leq i < n$.

For $1 \leq i < n$ and $\mu \in \{\pm 1\}$ we let $F_{i,\epsilon,\mu}$ be the face of the ϵ-cube determined by $a_{ni} = \mu 2$. The following two conditions are checked using (7.1):

(7.i) If ϵ' differs from ϵ only in the ϵ_{1i} place ($i > 1$), then $F_{i,\epsilon,\mu}$ is canonically identified with $F_{i,\epsilon',\mu}$: $f_\epsilon(z) = f_{\epsilon'}(z)$ for all z with $a_{ni} = \mu 2$.

(7.ii) If $\epsilon' = -\epsilon$, then $F_{1,\epsilon,\mu}$ is similarly identified with $F_{1,\epsilon',\mu}$.

These are the only ways that the faces are identified. Thus the faces are identified in pairs. For example, when $n = 4$ the $\epsilon = (+,+)$ cube is as shown in Figure 3:

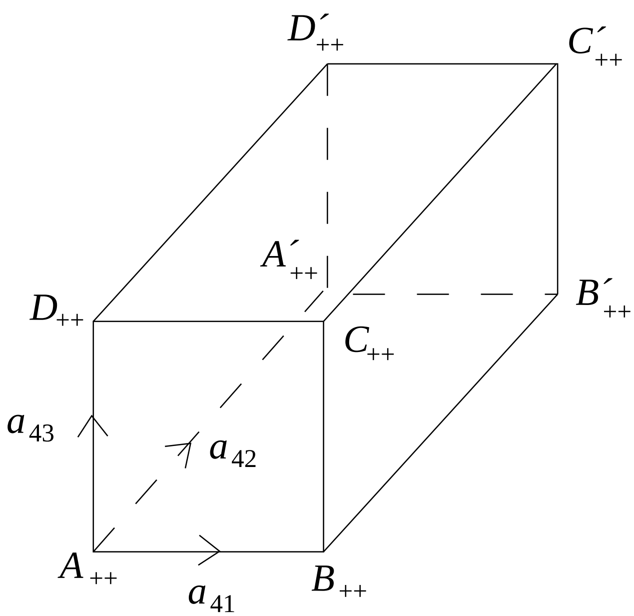

Figure 3

Here we have indicated the direction in which each of a_{41}, a_{42}, a_{43} increases and $-2 \leq a_{41}, a_{42}, a_{43} \leq 2$. Thus, for example, $F_{1,(+,+),+1}$ is the face $B_{++}B'_{++}C'_{++}C_{++}$. There are three other such cubes whose vertices we label similarly and they have faces which are identified as follows:

$$\begin{aligned}
ABB'A'_{++} &\equiv ABB'A'_{+-}; \quad DCC'D'_{++} \equiv DCC'D'_{+-} \\
ADD'A'_{++} &\equiv ADD'A'_{--}; \quad BB'C'C_{++} \equiv BB'C'C_{--} \\
ABCD_{++} &\equiv ABCD_{-+}; \quad A'B'C'D'_{++} \equiv A'B'C'D'_{-+}; \\
AA'D'D_{+-} &\equiv AA'D'D_{-+}; \quad BB'C'C_{+-} \equiv BB'C'C_{-+}; \\
ABCD_{+-} &\equiv ABCD_{--}; \quad A'B'C'D'_{+-} \equiv A'B'C'D'_{--}; \\
ABB'A'_{-+} &\equiv ABB'A'_{--}; \quad DCC'D'_{-+} \equiv DCC'D'_{--}.
\end{aligned}$$

Now making the $ABB'A'_{++} \equiv ABB'A'_{+-}, DCC'D'_{++} \equiv DCC'D'_{+-}$ identifications in the above list gives a solid torus, as does making the $ABB'A'_{-+} \equiv ABB'A'_{--}, DCC'D'_{-+} \equiv DCC'D'_{--}$ identifications. Now the rest of the identifications give the way of identifying the boundaries of these two solid tori. See Figure 4:

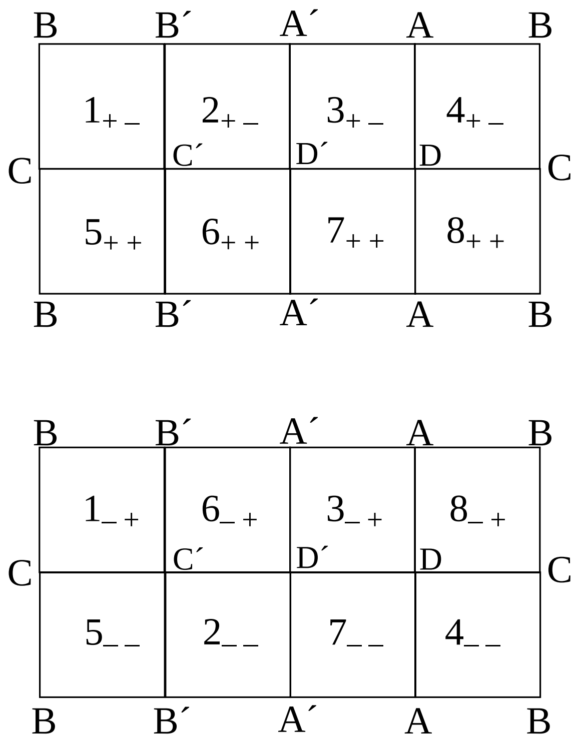

Figure 4.

One checks that all of $A_{++}, A_{+-}, A_{-+}, A_{--}$ are identified and similarly for B, C, D and A', B', C', D'. The edges $AB_{++}, AB_{+-}, AB_{-+}, AB_{--}$ are similarly all identified. One also checks that at each edge all four cubes come together as if they were stacked in $\mathbb{R}^3$. The resulting space is a manifold except at the 8 vertices (the points of $\mathcal{S}_4$), where one can check that the link is an $\mathbb{RP}^2$.

Now let T^{n-1} be the $(n-1)$-torus $S^1 \times \cdots \times S^1$ $(n-1$ times). Here we represent S^1 as $\mathbb{R}$ mod 2π. Thus $S^1 = I_1 \cup I_2$ where $I_1 = [0, \pi], I_2 = [\pi, 2\pi]$. Let $\alpha : T^{n-1} \to T^{n-1}$ be the antipodal map $\alpha(x) = -x$ for all $x \in T^{n-1}$. We will think of α as acting on all the T^{n-1}. It is clear that α respects adjacencies. Further, the fixed point set of α for this action is $\{0, \pi\}^{n-1}$ and this set is contained in each cube.

Theorem 7.6. *For all $n > 3$ the orbifold quotient $T^{n-1}/ < \alpha >$ is homeomorphic to $V_{n2}^{(2)}$. The 2^{n-1} singular points of $V_{n2}^{(2)}$ are non-manifold points whose links are all homeomorphic to $\mathbb{RP}^{n-2}$. The open manifold $V_{n2}^{(2)} \setminus \mathcal{S}_n$ is acted upon by B_n.*

Proof. We will exhibit T^{n-1} as a cubical complex which is invariant under the map α and then show that the cubes in the quotient complex have the same face identifications as we obtained for $V_{n2}^{(2)}$ above.

Now we naturally have

$$T^{n-1} = \bigcup_{i_1, i_2, \ldots, i_{n-1} \in \{1,2\}} I_{i_1} \times I_{i_2} \times \cdots \times I_{i_{n-1}}.$$

Note that there are 2^{n-1} such cubes in this decomposition and we may denote each cube by $C(i_1, i_2, \ldots, i_{n-1})$. Now $\alpha(I_1) = I_2, \alpha(I_2) = I_1$. Thus in the quotient $T^{n-1}/ < \alpha >$

each such cube can be represented by a cube of the form $I_1 \times \dots$. Now two distinct cubes $C(i_1, i_2, \dots, i_{n-1})$ and $C(i'_1, i'_2, \dots, i'_{n-1})$ have a face in common if and only if the sequences $(i_1, i_2, \dots, i_{n-1})$ and $(i'_1, i'_2, \dots, i'_{n-1})$ differ in exactly one position.

To each cube $C(i_1, i_2, \dots, i_{n-1})$ we associate a sequence $\epsilon(i_1, i_2, \dots, i_{n-1}) = (\epsilon_2, \dots, \epsilon_{n-1})$ as follows:

$$\epsilon_j = \begin{cases} +1 & \text{if} \quad i_j = 1, i_1 = 1; \\ -1 & \text{if} \quad i_j = 2, i_1 = 1. \end{cases} \quad \epsilon_j = \begin{cases} -1 & \text{if} \quad i_j = 1, i_1 = 2; \\ +1 & \text{if} \quad i_j = 2, i_1 = 2. \end{cases}$$

Then $C(i_1, i_2, \dots, i_{n-1}) \mapsto Im(f_{\epsilon(i_1, i_2, \dots, i_{n-1})})$ gives a one-to-one correspondence between the classes of cubes in $T^{n-1}/ < \alpha >$ and the cubes $Im(f_\epsilon)$. We now show that if $C(i_1, i_2, \dots, i_{n-1})$ and $C(i'_1, i'_2, \dots, i'_{n-1})$ share a face, then so do $Im(f_{\epsilon(i_1, i_2, \dots, i_{n-1})})$ and $Im(f_{\epsilon(i'_1, i'_2, \dots, i'_{n-1})})$.

Now $C(i_1, i_2, \dots, i_{n-1})$ and $C(i'_1, i'_2, \dots, i'_{n-1})$ share a face if and only if the sequences $(i_1, i_2, \dots, i_{n-1})$ and $(i'_1, i'_2, \dots, i'_{n-1})$ differ in exactly one entry. First, if $(i_1, i_2, \dots, i_{n-1})$ and $(i'_1, i'_2, \dots, i'_{n-1})$ differ only in the jth entry where $j > 1$, then $Im(f_{\epsilon(i_1, i_2, \dots, i_{n-1})})$ and $Im(f_{\epsilon(i'_1, i'_2, \dots, i'_{n-1})})$ share a face by (7.i) above. Otherwise if $(i_1, i_2, \dots, i_{n-1})$ and $(i'_1, i'_2, \dots, i'_{n-1})$ differ only in the first entry, then $Im(f_{\epsilon(i_1, i_2, \dots, i_{n-1})})$ and $Im(f_{\epsilon(i'_1, i'_2, \dots, i'_{n-1})})$ share a face by (7.ii) above.

Lastly, the link of a vertex of one of the $C(i_1, i_2, \dots, i_{n-1})$ in T^{n-1} is an S^{n-2} and so the link of a vertex of $Im(f_{\epsilon(i_1, i_2, \dots, i_{n-1})})$ in $T^{n-1}/ < \alpha >$ is an $\mathbb{RP}^{n-2}$. $\square$.

Returning to the $n = 4$ case we wish to explicitly find the fundamental group $\pi_1(V_{42}^{(2)})$. We will find $\pi_1(V_{42}^{(2)})$ by first finding the fundamental groupoid $\pi_1(V_{42}^{(2)}, P)$, where $P = \cup_{(\pm,\pm)} p_{\pm\pm}$. For information on fundamental groupoids see [Br].

For $1 \le i < n, \mu \in \{\pm 1\}$ we let $g_{i,\epsilon,\mu}$ be an arc from p_ϵ to the centre of $F_{i,\epsilon,\mu}$. Thus for $i > 1$ and ϵ, ϵ' differing only in the ith entry we have an arc $g_{i,\epsilon,\mu} g_{i,\epsilon',\mu}^{-1}$ from p_ϵ to $p_{\epsilon'}$. Similarly $g_{1,\epsilon,\mu} g_{1,-\epsilon,\mu}^{-1}$ goes from p_ϵ to $p_{-\epsilon}$.

Proposition 7.7. *The fundamental groupoid $\pi_1(V_{42}^{(2)}, P)$ has the following generators and relations:*

$$\begin{aligned} < a_1, b_1, c_1, d_1, e_1, f_1, a_2, b_2, c_2, d_2, e_2, f_2 | \\ & e_1 b_1 f_1 a_1^{-1}, e_1 b_2 f_1 a_2^{-1}, e_2 b_1 f_2 a_1^{-1}, e_2 b_2 f_2 a_2^{-1}, \\ & c_1 b_1^{-1} d_1 a_1^{-1}, c_1 b_2^{-1} d_1 a_2^{-1}, c_2 b_1^{-1} d_2 a_1^{-1}, c_2 b_2^{-1} d_2 a_2^{-1}, \\ & c_1 f_1 d_1^{-1} e_1^{-1}, c_1 f_2 d_1^{-1} e_2^{-1}, c_2 f_1 d_2^{-1} e_1^{-1}, c_2 f_2 d_2^{-1} e_2^{-1} > . \end{aligned}$$

The fundamental group $\pi_1(V_{42}^{(2)})$ has the following generators and relations:

$$< b_1, f_2, b_2, d_2 | b_1^2, f_2^2, b_2^2, d_2^2, (f_2 b_1 b_2)^2, (d_2 b_1 b_2)^2, (d_2 b_1 f_2)^2 > .$$

The group $\pi_1(V_{42}^{(2)})$ has a normal subgroup of index 2 which is isomorphic to $\mathbb{Z}^3$.

Proof. First define some generators as follows:

$$\begin{aligned} a_1 &= g_{1+++} g_{1--+}^{-1}; b_1 = g_{1+-+} g_{1-++}^{-1}; c_1 = g_{2+++} g_{21-++}^{-1}; \\ d_1 &= g_{2+-+} g_{2--+}^{-1}; e_1 = g_{3+++} g_{3+-+}^{-1}; f_1 = g_{3-++} g_{3--+}^{-1}; \\ a_2 &= g_{1++-} g_{1---}^{-1}; b_2 = g_{1+--} g_{1-+-}^{-1}; c_2 = g_{2++-} g_{2-+-}^{-1}; \\ d_2 &= g_{2+--} g_{2---}^{-1}; e_2 = g_{3++-} g_{3+--}^{-1}; f_2 = g_{3-+-} g_{3---}^{-1}. \end{aligned}$$

These are arcs which connect the points of P. They correspond to the identifications of the faces given above; for example a_1 corresponds to the identification $BB'C'C_{++} \equiv BB'C'C_{--}$. They generate $\pi_1(V_{42}^{(2)}, P)$. Now each edge of each cube determines a relation. The set of such generators and relations suffices for a presentation of $\pi_1(V_{42}^{(2)}, P)$. Since each edge is adjacent to each of the four cubes we obtain relators of length 4 in the above generators. One calculates that they are as indicated in Proposition 7.7. For example the edge CC' corresponds to the first relator given in the presentation.

To obtain a presentation for the fundamental group from the given presentation for the fundamental groupoid we just need to "collapse a maximal tree" in the generator graph; this collapsed tree will give the base point for the fundamental group. We choose the tree determined by a_1, c_1, e_1. A calculation now gives the presentation indicated in the proposition. The last statement follows from the given presentation for $\pi_1(V_{42}^{(2)})$, since one can easily show that the subgroup generated by f_2b_1, b_2b_1, d_2b_1 is isomorphic to $\mathbb{Z}^3$ and has index 2. $\square$

Now B_4 acts on $V_{42}^{(2)}$ and at any point of $V_{42}^{(2)}$ the determinant of U_4 is zero. This however leaves the possibility that not all of the invariants $c'_{4i}, i = 1, \ldots, 4$ are constant on $V_{42}^{(2)}$. In fact one checks that

$$\begin{aligned} c'_{43} = -a_{21}a_{31}a_{32} + a_{21}a_{32}a_{41}a_{43} - a_{21}a_{41}a_{42} - a_{32}a_{42}a_{43} - a_{31}a_{41}a_{43} \\ + a_{21}^2 + a_{31}^2 + a_{32}^2 + a_{41}^2 + a_{42}^2 + a_{43}^2 - 8 \end{aligned}$$

is not constant on $V_{42}^{(2)}$. Thus $V_{42}^{(2)}$ is a union of the level sets of c'_{43}, each such level set also being invariant under the action of B_4. We now describe these level sets, first noting that one can show that on $V_{42}^{(2)}$ the function c'_{43} only takes on values in the range: $[-4, 0]$.

Proposition 7.8. *For $0 < t < 4$ the set $V_{42}^{(2)} \cap V(c'_{43} + t)$ is a union of four singular tori. For $t = 0, 4$ the set $V_{42}^{(2)} \cap V(c'_{43} + t)$ is a union of two singular tori.*

Proof. Assume first that $0 < t < 4$. Then solving for $a_{ij}(\epsilon), 1 \le j < i < 4$ (for each of the 4 possible ϵ) and substituting into c'_{43} we obtain a degree 4 equation in a_{43}. Solving this equation gives 16 solutions and so we get 16 discs. One calculates the identifications on the boundaries of these discs and sees that four of these form a torus and that this happens 4 times. The cases $t = 0, 4$ are similar. $\square$

§8 The rank $n - 1$ case

In this section we assume that $R = \mathbb{R}$. We have already investigated this case for $n = 3$ in §4.

Theorem 8.1. *For all $n \ge 3$ there is a one-parameter family of semialgebraic subsets of $\mathbb{R}^{\binom{n}{2}}$ each of which is homeomorphic to a smooth sphere of dimension $\binom{n}{2} - 1$. Each point x of one of these spheres corresponds to a matrix $U_n(x)$ which is positive definite. These spheres are invariant under the action of B_n. Moreover the kernel of the action of B_n on these spheres is the cyclic centre of B_n for a dense set of values of the parameter.*

Proof. We will be considering the level sets of the function $det(U_n)$. First note that the matrix $U_n((0, 0, \ldots, 0))$ corresponding to the origin of $\mathbb{R}^{\binom{n}{2}}$ is twice the identity matrix, which is positive definite. Since positive-definiteness is an open condition all matrices corresponding to points in a sufficiently small neighbourhood of the origin of $\mathbb{R}^{\binom{n}{2}}$ will be positive definite.

We will use Morse theory applied to the function $det(U_n)$ and will need:

Lemma 8.2. *The origin is a singular point of* $det(U_n)$*. The Hessian of the function* $det(U_n)$ *at the origin is non-degenerate and has index* $\binom{n}{2}$*.*

Proof. One easily sees that $det(U_n)$ has constant term 2^n and that all other monomials have degree at least 2. Thus the origin is a singular point of $det(U_n)$. In fact the only monomials of $det(U_n)$ having degree 2 are those of the form $-2^{n-2}a_{ij}^2$. Thus the Hessian of $det(U_n)$ at the origin is a diagonal matrix with -2^{n-1}s on the diagonal. It is thus non-degenerate and has index $\binom{n}{2}$. □

Thus Morse theory [GG, II, §6] says that the origin is an isolated singular point and that near the origin $det(U_n)$ looks like (after a change of variables):

$$2^n - \sum_{1\le j<i\le n} a_{ij}^2,$$

and so the level sets of $det(U_n)$ near the origin are all smooth topological spheres of dimension $\binom{n}{2} - 1$.

Now a dense set of these spheres contain points whose coordinates are algebraically independent. At any such point the kernel of the action of B_n is just the cyclic centre of B_n [H1]. This proves Theorem 8.1 and a part of Theorem 2 in §1. □

Remark. We note that the level sets of the invariants c'_{ni} intersect these spheres and give in general smaller sets on which the braid groups act.

The case $n = 4$ can be considered in more detail: What we proved in §4 about the $n = 3$ case can be summarised as follows: solving $det(U_3) = -2(c'_{31} + 1) = 0$ for a_{21} gives two diffeomorphisms $a_{21}^{\pm}(a_{31}, a_{32})$ with domain $[-2, 2]^2$ such that the two closed discs $a_{21}^{+}([-2, 2]^2), a_{21}^{-}([-2, 2]^2)$ meet along a piece-wise linear circle which is $a_{21}^{+}(\partial[-2, 2]^2) = a_{21}^{-}(\partial[-2, 2]^2)$, where ∂ denotes the boundary. The union $a_{21}^{+}([-2, 2]^2) \cup a_{21}^{-}([-2, 2]^2)$ is a 2-sphere.

Now consider the case $n = 4$. Here we use Proposition 5.1 to solve the quadratic equation $det(U_n) = 0$ for a_{21}. We obtain two solutions $a_{21}^{\pm}(a_{31}, a_{32}, a_{41}, a_{42}, a_{43})$. The same result shows that the discriminant of this equation is $4U_{4[1]}U_{4[2]}$. Now by the $n = 3$ case the equations $U_{4[1]}(a_{32}, a_{42}, a_{43}) = 0, U_{4[2]}(a_{31}, a_{41}, a_{43}) = 0$ define two 2-spheres in their respective 3-dimensional spaces. Further we have that

$$a_{21}^{+}(a_{31}, a_{32}, a_{41}, a_{42}, a_{43}) = a_{21}^{-}(a_{31}, a_{32}, a_{41}, a_{42}, a_{43})$$

at all points where $U_{4[1]}(a_{32}, a_{42}, a_{43})U_{4[2]}(a_{31}, a_{41}, a_{43}) = 0$. We wish to restrict the domain of $a_{21}^{\pm}(a_{31}, a_{32}, a_{41}, a_{42}, a_{43})$ to the closure E_4 of the component of $[-2, 2]^5 \setminus (U_{4[1]}U_{4[2]})^{-1}(0)$ containing 0. We will show that this is a 5-ball and that $U_{4[1]}(a_{32}, a_{42}, a_{43})U_{4[2]}(a_{31}, a_{41}, a_{43})$ is equal to zero only on the boundary of this 5-ball, which is a 4-sphere. Thus

$$a_{21}^{+}(a_{31}, a_{32}, a_{41}, a_{42}, a_{43})(E_4) \cup a_{21}^{-}m(a_{31}, a_{32}, a_{41}, a_{42}, a_{43})(E_4)$$

will be a union of two 5-balls along a 4-sphere i.e. a 5-sphere.

Now we can also consider $U_{4[1]}(a_{32}, a_{42}, a_{43})$ as a function $U_{4[1]}(a_{31}, a_{32}, a_{41}, a_{42}, a_{43})$ and as such $U_{4[1]}(a_{31}, a_{32}, a_{41}, a_{42}, a_{43}) = 0$ has solution set $S^2 \times D^2$, where the D^2 corresponds to $-2 \le a_{31}, a_{41} \le 2$ and the S^2 is in the cube $-2 \le a_{32}, a_{42}, a_{43} \le 2$. A similar thing happens for $U_{4[2]}(a_{31}, a_{32}, a_{41}, a_{42}, a_{43}) = 0$. By Theorem 4.1 the S^2s in these $S^2 \times D^2$s bound convex 3-balls which contain the origin. These two $D^3 \times D^2$s are now both convex and so intersect in a convex 5-ball with four-sphere boundary as required. This proves a part of Theorem 2.

Theorem 8.3. *For all $1 \le k \le n$ there is a compact subset of $V_{n,k}$ which is B_n-invariant.*

Proof. For the moment let us consider solving $det(U_n) = 0$ for the variable a_{21} as given in Proposition 5.1. Then a_{21} is a quadratic function of the remaining coordinates $a_{ij} \ne a_{21}$. The discriminant of this quadratic function is $4U_{n[1]}U_{n[2]}$ (see Proposition 5.1 again) and so a_{21} is not defined at points where $4U_{n[1]}U_{n[2]} < 0$.

Consider a ray $\rho = \rho(t), t \ge 0$, starting at the origin in the Euclidean space $\mathbb{R}^{\binom{n}{2}-1}$ with coordinates $(a_{31}, a_{32}, \dots, a_{n\,n-1})$. Let $\eta_1, \eta_2 : \mathbb{R}^{\binom{n}{2}-1} \to \mathbb{R}^{\binom{n-1}{2}-1}$ denote the projections with codomains having coordinates $(a_{ij}), i, j \ne 1$ and $(a_{ij}), i, j \ne 2$ respectively. At the origin the discriminant $4U_{n[1]}U_{n[2]}$ is positive, in fact each of $U_{n[1]}, U_{n[2]}$ is positive. Let $\epsilon > 0, i = 1, 2$. As this ray passes through the cube $[-2-\epsilon, 2+\epsilon]^{\binom{n}{2}-1} \subset \mathbb{R}^{\binom{n}{2}-1}$ the sign of $U_{n[i]}(\eta_i(\rho(t)))$ will change or at least become 0. This is true for $n = 2, 3$ and is proved in general by induction on n. Let $t_{\rho,i} > 0$ denote the first value of t when this happens. Let

$$D_n = \{\rho(t) | t \le min(t_{\rho,1}, t_{\rho,2}), t_{\rho,1} \ne t_{\rho,2}\} \subset \mathbb{R}^{\binom{n}{2}-1}.$$

Now the set of rays ρ where $t_{\rho,1} = t_{\rho,2}$ is in the closure of the set of rays with $t_{\rho,1} \ne t_{\rho,2}$. Thus the closure $\overline{D_n}$ is a compact set of dimension $\binom{n}{2} - 1$. However $\overline{D_n}$ is a subset of the domain of each of the diffeomorphisms $a_{21}^{\pm}(a_{31}, a_{32}, \dots, a_{n\,n-1})$. The $(\binom{n}{2} - 1)$-discs which are the images of these two functions meet exactly over points where $U_{n[1]}U_{n[2]} = 0$, namely over points which are in the image of ∂D_n. Thus $a_{21}^{+}(D_n) \cup a_{21}^{-}(D_n)$ is the union of two compact sets of dimensions $\binom{n}{2} - 1$ meeting along a common subset of one dimension less.

Now the action of B_n on $\mathbb{R}^{\binom{n}{2}}$ fixes the origin. Further each ray $\rho \subset \mathbb{R}^{\binom{n}{2}}$ starting at the origin must cross $a_{21}^{+}(\overline{D_n}) \cup a_{21}^{-}(\overline{D_n})$, which is the union of the graphs of the functions a_{21}^{+}, a_{21}^{-} with domain $\overline{D_n}$, since the projection of this ray to the $\mathbb{R}^{\binom{n}{2}-1}$ with coordinates $(a_{31}, a_{32}, \dots, a_{n\,n-1})$ gives either a point or a ray which eventually passes out of $\overline{D_n}$. The point at which ρ hits $a_{21}^{+}(\overline{D_n}) \cup a_{21}^{-}(\overline{D_n})$ is the first point along ρ at which $det(U_n + I_n) = 0$. Thus this compact set of dimension $\binom{n}{2} - 1$ is clearly invariant under the action of B_n. $\square$

§9 Faithfulness of the action of B_n on quotients of R'_n

Recall that we showed in [H1] that the kernel of $B_n \to Aut(R_n)$ or of $B_n \to Aut(R'_n)$ is the centre $Z(B_n)$. The main result of this section is the following:

Theorem 9.1. *Let $n > 2$ and let $c \in R'_n, c \notin R$, be invariant under the action of B_n. Assume further that either*
(i) c has $a_{21}a_{32}a_{43} \dots a_{n\,n-1}a_{n1}$ as a factor of a monomial summand of highest degree; or
(ii) c has more than one monomial summand of highest degree.

Then B_n acts on the quotient $R'_n/ < c >$, where $< c >$ is the ideal of R'_n generated by c. The kernel of this action is $Z(B_n)$.

Suppose that $\mathcal{I}$ is a B_n-invariant ideal in R'_n such that for all $x \in \mathcal{I}, x \neq 0$, the number of monomial summands of x of highest degree is greater than 1, then B_n acts on the quotient $R'_n/\mathcal{I}$ and the kernel of this action is $Z(B_n)$

Proof. Clearly $< c >$ is invariant under B_n and so B_n acts on $R'_n/ < c >$ as required. Now suppose that $\alpha \in B_n, \alpha \neq id$, is in the kernel of this action. Then there are $1 \leq j < i \leq n$ and $k_{ij} \in R'_n, k_{ij} \neq 0$, such that

$$\alpha(a_{ij}) = a_{ij} + k_{ij}c.$$

However Lemma 2.3 shows that $\alpha(a_{ij})$ has a unique monomial of highest degree and leading coefficient ± 1 . This would contradict hypothesis (ii) in Theorem 9.1 which would imply that $\alpha(a_{ij}) = a_{ij} + k_{ij}c$ has more than one monomial of highest degree.

Now assume (i). We will need:

Lemma 9.2. *For all $1 \leq i, j \leq n$ the monomial $a_{21}a_{32}a_{43}\dots a_{n\,n-1}a_{n1}$ is never a factor of the unique monomial of highest degree in $\alpha(a_{ij})$.*

Proof. Let $c_1, c_2, \dots, c_n$ be a cut system for the generators $x_1, x_2, \dots, x_n$. Thus c_i is a vertical arc joining π_i to the boundary of D_n (above π_i) and c_i only intersects x_i (see Figure 1). Now suppose that $a_{21}a_{32}a_{43}\dots a_{n\,n-1}a_{n1}$ is a factor of a monomial of highest degree in $\alpha(a_{ij})$. Then by Lemma 2.2 we see that each of

$$(x_1^{\pm1}x_2^{\pm1})^{\pm1}, (x_2^{\pm1}x_3^{\pm1})^{\pm1}, \dots, (x_{n-1}^{\pm1}x_n^{\pm1})^{\pm1}, (x_n^{\pm1}x_1^{\pm1})^{\pm1}$$

is a subword of the cyclically reduced form of $\alpha(x_ix_j)$ for some choice of ±1s. We will in fact show that in this case each of

$$(x_1x_2)^{\pm1}, (x_2x_3)^{\pm1}, \dots, (x_{n-1}x_n)^{\pm1}, (x_nx_1)^{\pm1}$$

is a subword of the cyclically reduced form of $\alpha(x_ix_j)$. For suppose that $x_kx_{k+1}^{-1}$ is a subword of $\alpha(x_ix_j)$. Let γ_{ij} denote the simple closed curve containing π_i and π_j in its interior and representing the conjugacy class x_ix_j; we assume that $\alpha(\gamma_{ij})$ meets the cut-arcs c_m minimally. Then we see that $\alpha(\gamma_{ij})$ has an oriented subarc δ going from c_k on the right to c_{k+1} on the right and such that $c_k \cup \delta \cup c_{k+1}$ cuts off a punctured disc. Now since $\alpha(\gamma_{ij})$ is simple and meets the c_m minimally we see that the next c_m crossed by $\alpha(\gamma_{ij})$ is c_k. Thus $x_{k+1}^{-1}x_k^{-1}$ is a subword of $\alpha(x_ix_j)$.

The case where $x_k^{-1}x_{k+1}$ is a subword of $\alpha(x_ix_j)$ is similar, as are the cases $x_n^{-1}x_1$ and $x_nx_1^{-1}$.

We thus see that the simple closed curve $\alpha(\gamma_{ij})$ has subarcs joining each c_k (on the right) to c_{k+1} (on the left) for $k = 1, \dots, n-1$ together with subarcs joining c_n (on the right) to c_1 (on the left). Let ζ_k be such an arc for each $k = 1, \dots, n$ and assume further that among all such subarcs the ζ_k that we choose is the one closest to the boundary of D_n. Then it easily follows that the endpoint of ζ_k on c_{k+1} is the end of ζ_{k+1} on c_{k+1}. Thus the ζ_ks join up to form a simple closed curve parallel to the boundary. Since $n > 2$ we see that this cannot be $\alpha(\gamma_{ij})$, a contradiction.

The proof of the statement in the last paragraph of Theorem 9.1 is similar to the proof of (ii) in the above. $\square$

Corollary 9.3. *The group $B_n/Z(B_n)$ acts faithfully on the quotient $R'_n/ < c >$ in each of the following cases:*
(i) $c = c_{ni}$, for $0 < i < n$;
(ii) $c = det(U_n)$, for $n \geq 3$.

Proof. (i) By [H2, Theorem 2.8] we see that the c'_{ni} are invariant under the action of B_n. In equation (3.1) a specific matrix is given whose determinant is the characteristic polynomial of $T_1T_2 \dots T_n$ with variable λ. It is clear from the nature of this matrix that for $i \neq 1, n-1$ the coefficient of λ^i has more than one monomial of highest degree. The result follows in this case from Theorem 9.1 (ii). If $i = 1, n-1$, then we first note that $c_{n1} = \pm c_{n\,n-1}$ and that in this case c_{n1} has a single monomial of highest degree, namely $a_{21}a_{32}a_{43} \dots a_{n\,n-1}a_{n1}$. Theorem 9.1 (i) now concludes this case as well.

(ii) If $n > 3$, then again $c = det(U_n)$ has more than one monomial of highest degree. However if $n = 3$, then $det(U_3) = -2(c_{31} + 1)$ and we are similarly done by Theorem 9.1 (i). □.

This proves Theorem 1 (iii).

We will say that a ring has a *Gröbner basis algorithm* if the fundamental theorem of Gröbner bases is satisfied for any polynomial ring over that ring. See [AL; Theorem 1.9.1 and Ch. 4] where it is shown that for example $\mathbb{Z}$ or any field has a Gröbner basis algorithm.

Corollary 9.4. *Suppose that R is a ring containing an ideal $\mathcal{K}$ such that $R/\mathcal{K}$ has characteristic 2 and has a Gröbner basis algorithm. Then for all $1 < r < n$ the group $B_n/Z(B_n)$ acts faithfully on the quotient $R'_n/\mathcal{I}_{n\,r}$.*

Proof. We first note that we need only consider the case where the coefficients are in $R/\mathcal{K}$ since we have natural maps:

$$B_n/Z(B_n) \to Aut(R'_n/\mathcal{I}_{nr}) \to Aut((R/\mathcal{K})'_n/\mathcal{I}_{nr})$$

and if the composition is injective, then so is the first map.

Let $\alpha \in B_n \setminus Z(B_n)$. If r is even, then $\mathcal{I}_{nr} = \{0\}$ over $R/\mathcal{K}$, but by Lemma 2.3 there are i, j such that $\alpha(a_{ij})$ has a unique monomial of highest degree greater than one, with coefficient ± 1 and so $\alpha(a_{ij}) \neq a_{ij}$ over $R/\mathcal{K}$.

Now assume that r is odd so that $\mathcal{I}_{nr}$ is not trivial. First note that since $R/\mathcal{K}$ has characteristic 2 the $(r+1)$-minors of U_n are homogeneous polynomials of degree $r+1$. Thus the ideals $\mathcal{I}_{nr}$ are all homogeneous and have a Gröbner basis algorithm. Thus if $\omega \in \mathcal{I}_{nr}$, then the homogeneous components of ω are in $\mathcal{I}_{nr}$ also. Thus if we have a non-central element in the kernel of the action on $(R/\mathcal{K})'_n/\mathcal{I}_{nr}$, then by Lemma 2.3 the ideal $\mathcal{I}_{nr}$ must contain a monomial. Thus there cannot be a symmetric matrix over $(R/\mathcal{K})'_n$ with 0s on the diagonal and all of whose off-diagonal entries are non-zero, contradicting Proposition 5.2. □

This proves Theorem 1 (iv).

Theorem 9.5. *(i) For $n \geq 3$ the group $B_n/Z(B_n)$ does not act faithfully on the quotient $R'_n/\mathcal{I}_{n1}$.*

(ii) Suppose that R is a ring in which 2 is invertible. Then for $n > 3$ the group $B_n/Z(B_n)$ does not act faithfully on the quotient $R'_n/\mathcal{I}_{n2}$. In fact if we denote the image of B_n in $Aut(R'_n/\mathcal{I}_{n2})$ by B_{n2}, then B_{42} fits into a split short exact sequence

$$0 \to \mathbb{Z}^2 \to B_{42} \to B_3/ < (\sigma_1\sigma_2)^6 > \to 1,$$

and so B_{42} has presentation

$$B_{42} =< \sigma_1, \sigma_2, \sigma_3 | \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \sigma_1\sigma_3 = \sigma_3\sigma_1, (\sigma_1\sigma_2)^6, (\sigma_1\sigma_3^{-1}, \sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}) > .$$

Proof. (i) One checks that σ_1^4 acts trivially on $R'_n/\mathcal{I}_{n1}$. Generators for the ideal $\mathcal{I}_{n1}$ were given in Proposition 3.3. One needs to check the two cases where the characteristic of R is or is not 2. This is straightforward. This, together with the results of §3, proves Theorem 1 (i).

(ii) It is easy to check that if α is the Dehn twist about the curve surrounding the first three punctures, so that $\alpha = (\sigma_1\sigma_2)^3$, then α^2 is in the kernel of this representation. For example for $3 < r \leq n$ we have

$$\begin{aligned}\alpha^2(a_{r1}) = a_{r1} + a_{r1}U_n([1,2,3])^2/4 - 5a_{r1}U_n([1,2,3],[2,3,r])/2 \\ + (a_{r1}a_{32}^2 - a_{r1}a_{31}a_{32} + a_{r2}a_{21} + a_{r3}a_{31})U_n([1,2,3])/2.\end{aligned}$$

Since $n > 3$ we see that $\alpha^2 \notin Z(B_n)$. This proves Theorem 1 (ii).

For the $n = 4$ case we need to prove (a) that the image of the rank 2 free subgroup $< \beta_1 = \sigma_1\sigma_3^{-1}, \beta_2 = \sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1} >\subset B_4$ [Bi] in B_{42} is isomorphic to $\mathbb{Z}^2$; and (b) that there are no other relations in the image of the subgroup $< \sigma_1, \sigma_2 >$ in B_{42} other than the ones we have already listed.

Now [GL] there is a split short exact sequence $1 \to F_2 \to B_4 \to B_3 \to 1$, where F_2 is the free group of rank 2 freely generated by β_1, β_2. Now one also checks that $\beta_1\beta_2\beta_1^{-1}\beta_2^{-1}$ has trivial action on $R'_4/\mathcal{I}_{42}$ (but not on $R'_n/\mathcal{I}_{n2}$ for $n > 4$). To proceed we need to show that the action of $\beta_1^r\beta_2^s$ on $R'_4/\mathcal{I}_{42}$ is non-trivial for all $r, s \in \mathbb{Z}$ with $(r, s) \neq (0, 0)$. It will suffice to find a homomorphism $\phi : R'_4 \to \mathbb{R}$ such that $\phi(U_4)$ has rank 2 and such that $\phi(\beta_1^r(U_4)) \neq \phi(\beta_2^s(U_4))$ for all such r, s. In fact we will show that the 12 entries of $\phi(\beta_1^r(U_4))$ and $\phi(\beta_2^s(U_4))$ are different. Let ϕ be determined by:

$$\phi(U_4) = \begin{pmatrix} 2 & 3/2 & 3/2 & \sqrt{14}/2 \\ 3/2 & 2 & 1/4 & \sqrt{14}/4 \\ 3/2 & 1/4 & 2 & \sqrt{14}/2 \\ \sqrt{14}/2 & \sqrt{14}/4 & \sqrt{14}/2 & 2 \end{pmatrix}.$$

Note also that we may without loss assume that r, s are both even. Now we note that using (1.2) one can show that β_1^2 acts on U_4 as follows: $\beta_1^2(U_4) = EU_4E^t$ where E is the matrix

$$E = \begin{pmatrix} 1 - a_{21}^2 & a_{21} & 0 & 0 \\ -a_{21} & 1 & 0 & 0 \\ 0 & 0 & 1 & -a_{43} \\ 0 & 0 & a_{43} & 1 - a_{43}^2 \end{pmatrix}.$$

It follows that the 12 entry of $\beta_1^2(U_4) = EU_4E^t$ is a_{21} and that similarly the 43 entry is a_{43}. Since β_1 fixes a_{21} and a_{43} we see that the action of any even power of β_1 is given by: $\beta_1^{2r}(U_4) = E^rU_4(E^r)^t$ for all $r \in \mathbb{Z}$. Now

$$\beta_2^s(a_{21}) = \sigma_2\beta_1^s\sigma_2^{-1}(a_{21}) = \sigma_2\beta_1^s(a_{31})$$

and so if $\beta_2^s(a_{21}) = a_{21}$, then we would have $\beta_1^s(a_{31}) = a_{31}$ and so $\phi(\beta_1^s(a_{31})) = \phi(a_{31})$. Now write $E = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$, where E_1, E_2 are 2×2 matrices. Then putting $U_4 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ we have (where $r = 2w$):

$$\beta_1^r(U_4) = E^w U_4 (E^w)^t = \begin{pmatrix} E_1^w A (E_1^w)^t & E_1^w B (E_2^w)^t \\ E_2^w C (E_1^w)^t & E_2^w D (E_2^w)^t \end{pmatrix}. \tag{9.1}$$

Thus we need to show that the 11 entry of $E_1^w B (E_2^w)^t$ is never $3/2$ for $w \neq 0$. We assume that there is a $w \neq 0$ such that the 11 entry of $E_1^w B (E_2^w)^t$ is $3/2$. Now let us order the entries of the matrix $B = \begin{pmatrix} a_{31} & a_{41} \\ a_{32} & a_{42} \end{pmatrix}$ as $(a_{31}, a_{41}, a_{32}, a_{42})$. Then relative to this ordering the action of E on B given by the 12 block of (9.1) can be described by the following matrix:

$$H = \begin{pmatrix} -5/4 & 5/8\sqrt{14} & 3/2 & -3\sqrt{14}/4 \\ -5/8\sqrt{14} & \frac{25}{8} & 3\sqrt{14}/4 & -\frac{15}{4} \\ -3/2 & 3\sqrt{14}/4 & 1 & -\sqrt{14}/2 \\ -3\sqrt{14}/4 & \frac{15}{4} & \sqrt{14}/2 & -5/2 \end{pmatrix}$$

so that the entries of $E_1^w B (E_2^w)^t$ are given by $H^w (a_{31}, a_{41}, a_{32}, a_{42})^t$. Now H has 4 distinct eigenvalues and expressing the vector $(3/2, \sqrt{14}/2, 1/4, \sqrt{14}/4)$ as a linear combination of these eigenvectors we see that the 11 entry of $E_1^w B (E_2^w)^t$ is

$$\begin{aligned} &\frac{(3-i)}{4}\left(-\frac{9}{16} + \frac{5}{16} i\sqrt{7}\right)^w + \frac{(3+i\sqrt{7})}{4}\left(-\frac{9}{16} - \frac{5}{16} i\sqrt{7}\right)^w \\ &= 3/2\,\cos\left(w\left(\arctan(5/9\sqrt{7}) - \pi\right)\right) - \sqrt{7}/2 \sin\left(w\left(\arctan(5/9\sqrt{7}) - \pi\right)\right), \end{aligned} \tag{9.2}$$

where $i^2 = -1$. Now let $x = \cos(w\left(\arctan(5/9\sqrt{7}) - \pi\right))$ and solve for the expression (9.2) equal to $3/2$. This gives $x = 1, \frac{1}{8}$. Now if we have $\cos(w\left(\arctan(5/9\sqrt{7}) - \pi\right)) = 1$, then $\arctan(5/9\sqrt{7}) - \pi = 2n\pi/w$ for some $n \in \mathbb{Z}$. Thus $tan(2n\pi/w) = 5\sqrt{7}/9$, from which we get that $cos(2n\pi/w) = \pm 9/16$. Now from [CJ, Theorem 7] we see that there is no rational multiple of π whose cosine is $\pm 9/16$, a contradiction.

Now assume that $\cos(w\left(\arctan(5/9\sqrt{7}) - \pi\right)) = 1/8$. Then a similar computation shows that $cos(arccos(\frac{1}{8})/w) = \pm 9/16$. Now we may assume that $w \geq 1$ and if we think of w as a real variable, then $g(w) = cos(arccos(\frac{1}{8})/w)$ is a real-valued function and as such is strictly increasing on the domain $[1, \infty)$. One checks that on this domain $g(w)$ is positive and that $1 < g^{-1}(9/16) < 2$; thus there is no integral w with $g(w) = \pm 9/16$. This shows that the subgroup $< \beta_1, \beta_2 >$ of B_4 is represented in $Aut(R_4'/\mathcal{I}_{42})$ as $\mathbb{Z}^2$ as required.

The remainder of the proof of Theorem 9.5 follows from the case $n = 4$ of the following result, requiring the calculation and use of a Gröbner basis, for which we use the computer algebra system MAGMA.

Proposition 9.6. *Let $n = 4, 5, 6, 7, 8$. Then the image of the group $B_3/ < (\sigma_1\sigma_2)^6 >$ in $B_n/ < (\sigma_1\sigma_2)^6 >$ contains no elements in the kernel of the action on $R_n'/\mathcal{I}_{n2}$.*

Proof. Let $\alpha \in B_3 < B_n$ be in the kernel. Now one can check that the generator $\beta = (\sigma_1\sigma_2)^3$ of the cyclic centre of B_3 has non-trivial action on $R_n'/\mathcal{I}_{n2}, n = 4, 5, 6, 7, 8$, using MAGMA

[MA] (one calculates the ideal $\mathcal{I}_{n2}, n = 4,5,6,7,8$ and shows that $\beta(a_{41}) - a_{41} \notin \mathcal{I}_{n2}$ by computing a Gröbner basis for $\mathcal{I}_{n2}$ for each given value of n, and then showing that the normal form of $\beta(a_{41}) - a_{41}$ is non-zero relative to this Gröbner basis). However, if $\alpha \in B_3 \setminus Z(B_3)$, then α acts non-trivially on R'_3 i.e. there are $1 \leq i \neq j \leq 3$ such that $d = \alpha(a_{ij}) - a_{ij} \neq 0$. Since α is in the kernel of the action on $R'_n/\mathcal{I}_{n2}$, then we must have $d \in \mathcal{I}_{n2} \cap R'_3$. But $\mathcal{I}_{n2} \cap R'_3$ is an elimination ideal [AL] and a calculation using the elimination ideal routine algorithm in MAGMA shows that $\mathcal{I}_{n2} \cap R'_3 =< c'_{31} >$ for $n = 4,5,6,7,8$. Thus we would have $d = kc'_{31}$ for some $k \in R'_3, k \neq 0$, contradicting Corollary 9.3. The result follows. □

Remark 9.7. Of course we conjecture that $\mathcal{I}_{n2} \cap R'_3 =< c'_{32} >$ for all $n \geq 4$, which would in turn show that Proposition 9.6 was true for all $n \geq 4$.

Remark 9.8. We note that $B_3/ < (\sigma_1\sigma_2)^6 > \cong SL(2,\mathbb{Z})$ and so it should not be surprising to find that one can prove that $B_{42} \cong < m_1, m_2, m_3 >$ is a subgroup of $SL(3,\mathbb{C})$ where

$$m_1 = \begin{pmatrix} 1 & 1 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad m_2 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & f \\ 0 & 0 & 1 \end{pmatrix}, \quad m_3 = \begin{pmatrix} 1 & 1 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Here $a, c, f \in \mathbb{C}$ must satisfy $a \neq c$ for this linear representation to be faithful.

§10 Action of $\mathcal{B}_n$

In this section we will want to think of the $exp(tD(\alpha))$ as acting as automorphisms of power series rings (the power series rings associated to R_n and R'_n) and also as acting on the Euclidean space $\mathbb{R}^{\binom{n}{2}}$. We now explain the first of these.

Let $R = \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial algebra and let $R^* = \mathbb{C}[[x_1, \ldots, x_n]]$ be the corresponding ring of formal power series. Let W_n be the *general Lie algebra of Cartan type*, i.e. W_n is linearly generated by all $\mathbb{C}$-derivations of the form

$$f\frac{\partial}{\partial x_i}, i = 1, \ldots, n$$

where $f \in R^*$ [Ca, SS, Ka]. Now W_n (and so any of its subalgebras) can be given a filtration

$$W_n = W_{n,-1} \supset W_{n,0} \supset W_{n,1} \supset \cdots \supset W_{n,i} \supset \ldots$$

Here an element of $W_{n,i}$ is a linear sum of $f\frac{\partial}{\partial x_j}$'s where f has degree at least $i+1$. There is a corresponding graded algebra with graded pieces $W_{n,i}/W_{n,i+1}$. Using this grading we can define a valuation on W_n as follows (c.f. [J, p.171]): $|0| = 0$, and if $a \neq 0$, then $|a| = 2^{-i}$ where $a \in W_{n,i}$ and $a \notin W_{n,i+1}$. Then we have:

$$|a| \geq 0, \qquad |a| = 0 \text{ if and only if } a = 0,$$
$$|[a,b]| \leq |a||b|, \quad \text{and} \quad |a+b| \leq \max\{|a|,|b|\}.$$

This makes W_n into a topological algebra where $a_1 + a_2 + \ldots$ converges if and only if $|a_i| \to 0$ as $i \to \infty$.

For any derivation $D \in W_n$ and any $x \in R^*$ we define

$$\exp(D)(x) = x + D(x) + \frac{1}{2!}D^2(x) + \frac{1}{3!}D^3(x) + \ldots.$$

In [H3] we proved that each $D(\alpha)$ converges in the above topology for any $\alpha \in P_n$. Thus the groups $\mathcal{P}_n$ and $\mathcal{B}_n$ are also well-defined.

Next we note that the action of $\alpha \in P_n$ on the ring R'_n is also given as $exp(D(\alpha))$, where we can just ignore all the $\frac{\partial}{\partial a_{ij}}$ for $i < j$ and replace a_{ij} by $-a_{ji}$ for $i < j$ in $D(\alpha)$.

We remarked in §1 that each such $D(\alpha)$ has the property that $D(\alpha)(c_{ni}) = 0$ for all $\alpha \in P_n$ and all $i \leq n$. This easily implies that each $exp(tD(\alpha)), \alpha \in P_n$ fixes each of the c_{ni}.

In order to study the effect of $exp(tD(\alpha)), \alpha \in P_n$, on the real varieties V_{nk} we need to see that the derivation $D(\alpha)$ with the a_{ij} replaced by real numbers converges. Since for $|x| < 1$ we have

$$\begin{aligned} arcsinh(x) =& x - \frac{1}{2\cdot 3}x^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7 + \\ & \cdots + (-1)^n \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)\cdot (2n+1)}x^{2n+1} + \ldots \end{aligned}$$

(with a similar expression in the case $|x| > 1$) we see that $arcsinh(x)/x$ converges everywhere in $\mathbb{R}$. Thus the exponential $exp(tD(\sigma_1^2))$ given in §1 converges for all values of $t \in \mathbb{R}$. It follows similarly that each $exp(tD(\alpha))$ is well-defined for any $\alpha \in P_n$ and at any point in $\mathbb{R}^{\binom{n}{2}}$. Thus the groups $\mathcal{P}_n$ and $\mathcal{B}_n$ are also well-defined as sets of invertible functions acting on $\mathbb{R}^{\binom{n}{2}}$.

Now each $exp(tD(\alpha)), \alpha \in P_n, t \in \mathbb{R}$, acts as an automorphism of the power series ring $\bar{R}'_n$ and so acts on the matrix U_n in such a way as to preserve rank. Thus the real algebraic subvarieties $V_{nk}, V_{nk}^{(2)}$ are fixed by the action of $\mathcal{P}_n$ and so by $\mathcal{B}_n$. This proves Theorem 3.

Proposition 10.1. *The action of* $\exp(tD(\sigma_1))$ *is given as follows:*

$$\begin{aligned} & a_{21} \mapsto a_{21}; \\ & a_{r1} \mapsto a_{r1}cosh(t\alpha_{21}) - (2a_{r2} - a_{r1}a_{21})sinh(t\alpha_{21})/\sqrt{a_{21}^2 - 4}; \\ & a_{r2} \mapsto a_{r2}cosh(t\alpha_{21}) + (2a_{r1} - a_{r2}a_{21})sinh(t\alpha_{21})/\sqrt{a_{21}^2 - 4}. \end{aligned}$$

Here $r > 2$ *and* $\alpha_{21} = arcsinh(\sqrt{a_{21}^4/4 - a_{21}^2})$.

Proof. From the expression for $D(\sigma_1^2)$ given in §1 we can prove by induction that

$$\begin{aligned} & D(\sigma_1^2)^i(a_{r1}) = a_{r1}t^i a_{21}^i \alpha_{21}^i (a_{21}^2 - 4)^{i/2}/2^i \quad \text{for} \quad i \in 2\mathbb{N}; \\ & D(\sigma_1^2)^i(a_{r1}) = (2a_{r2} - a_{r1}a_{21})t^i a_{21}^i \alpha_{21}^i (a_{21}^2 - 4)^{(i-1)/2}/2^i \quad \text{for} \quad i \notin 2\mathbb{N}. \end{aligned}$$

From this we deduce the image of a_{r1}. There is a similar expression for $D(\sigma_1^2)(a_{r2})$ giving the image of a_{r2}. □

For $1 \leq i < n$ the action of σ_i^2 can easily be deduced from this.

We recall Thurston's earthquake theorem, namely that earthquakes act transitively on the Teichmüller $\mathcal{T}_{g,0,0}$; for definitions and results about earthquakes see [Ke1, Ke2, Th]. In our situation we see that an earthquake does not act transitively on $\mathcal{T}_{0,n,1}$ since an earthquake can't change the length of a boundary geodesic.

Conjecture 10.2. *The action of* $exp(D(\sigma_1^2)$ *given in 10.1 is the action on Teichmüller space of the earthquake along the curve* x_1x_2. *The action of* $\mathcal{B}_n$ *is transitive on level sets.*

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§11 Teichmüller space

In this section we have $R = \mathbb{R}$. Basic facts about Teichmüller spaces and Fuchsian groups can be found in [IT]. The punctured disc D_n can be represented as the quotient $\mathbb{H}^2/G$ where G is the Fuchsian group generated by elements $s_1, s_2, \ldots, s_n$ where $s_2, \ldots, s_n$ are hyperbolic and $p_1 = s_1^{-1}, p_2 = s_2^{-1}s_1, p_3 = s_3^{-1}s_2, \ldots, p_n = s_n^{-1}s_{n-1}$ are all parabolic. See Figure 5 to see how the identifications occur.

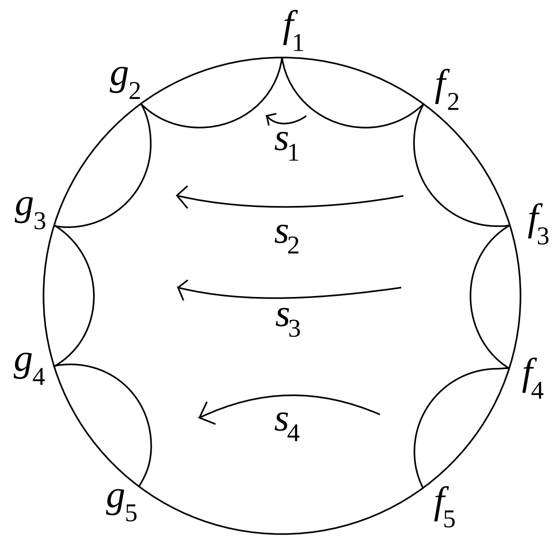

Figure 5.

In this diagram (for the case $n = 4$) the parabolic matrices $p_i, i \leq n$, have fixed points f_i as shown. The hyperbolic element $s_i, i > 1$, takes the geodesic joining f_i to f_{i+1} to a geodesic joining g_i to g_{i+1} as indicated (here $g_1 = f_1$). Now up to conjugation in $PSL_2(\mathbb{R})$ we may assume that

$$p_1 = s_1^{-1} = \begin{pmatrix} -1 & 0 \\ u & -1 \end{pmatrix}, \quad p_2 = s_2^{-1}s_1 = \begin{pmatrix} -1-v & v \\ -v & v-1 \end{pmatrix},$$
$$p_3 = s_3^{-1}s_2 = \begin{pmatrix} -1 & w \\ 0 & -1 \end{pmatrix}, \tag{11.1}$$

so that $p_1(0) = s_1(0) = 0 = f_1, p_2(1) = 1 = f_2$ and $p_3(\infty) = \infty = f_3$. These three conditions are the only control that we have on the parabolics listed above (since an element of $PSL_2(\mathbb{R})$ is completely determined by its action on any ordered triple of elements of $\mathbb{R} \cup \{\infty\}$) and so determine a 'normalised' set of generators. The rest of the parabolics must then have the form:

$$p_i = s_i^{-1}s_{i-1} = \begin{pmatrix} x_{i1} & x_{i2} \\ -(1+x_{i1})^2/x_{i2} & -2-x_{i1} \end{pmatrix}, \tag{11.2}$$

for $3 < i \leq n$; here $x_{i2} \neq 0$ since p_i does not fix 0. This gives $2n-3$ unknowns u, v, w, x_{ij}. Note that the real numbers f_i, g_i must satisfy:

$$f_4 < f_5 < \cdots < f_n < f_{n+1} < g_{n+1} < g_n < \cdots < g_3 < g_2 < g_1 = f_1 = 0.$$

In fact given any n parabolics as in (11.1) and (11.2) with the first three fixing $0, 1, \infty$, then we obtain a Fuchsian group G such that $\mathbb{H}^2/G \cong D_n$ if the above inequalities are satisfied. This thus gives a set of coordinates $u, v, w, x_{41}, x_{42}, \ldots, x_{n1}, x_{n2}$ for the Teichmüller space $\mathcal{T}_{0,n,1}$.

Example 11.1. In the case $n = 3$ we need $f_4 < g_4 < g_3 < g_2 < g_1 = f_1 = 0$. But one can verify that

$$g_3 = \frac{v-1}{uv+v-u}, \quad g_4 = \frac{-2\,v - 2\,uwv - wv + uv + uw + uwv^2 + wv^2 - uv^2}{(-u+uv+v)\,(uwv+wv-uv-uw-2)},$$
$$g_2 = \frac{1}{u+1}; \quad f_4 = -1/2\,\frac{-2\,v + uwv + wv - uv - uw}{-u+uv+v}.$$

Thus we get the inequalities

$$u < -1, \quad uv + v - u < 0, \quad uvw + vw - uv - uw < 0. \tag{11.3}$$

Of course the matrices p_i, s_i are only defined up to a sign, however Keen [K1 pp. 210-211, K2] notes that when choosing representative matrices from $SL_2(\mathbb{R})$ for the elements of $PSL_2(\mathbb{R})$, there is a canonical choice for the signs of the traces of the generators, namely we may take them all to be negative, so that in our case we choose $trace(p_i) = -2$. We will also need to note that $s_i^{-1} = p_i p_{i-1} \dots p_2 p_1$ for $i > 1$ and that these elements are hyperbolic i.e. they have squared trace greater than 4.

Now it follows from [O, Theorem 4.1] that $2n-3$ of the traces $trace(p_i p_j)$ completely determine the point of Teichmüller space corresponding to this representation. The $2n-3$ traces given in [O] are those of $p_1 p_i, i = 2, \dots, n$ and $p_2 p_i, i = 3, \dots, n$.

As indicated in equation (1.4), we would like to solve $trace(p_i p_j) = 2 - a_{ij}^2$. Now we see that there are $2n-3$ of the u, v, w, x_{ij}, but $\binom{n}{2}$ of the a_{ij} and so in general there is no hope of solving for the u, v, w, x_{ij} as functions of the a_{ij}, except when $n = 3$. Also, since $s_2, \dots, s_n$ and the $p_i p_j, i \neq j$ are all hyperbolic we are interested in having

$$|trace(p_1 p_2)|, |trace(p_1 p_3)|, |trace(p_2 p_3)|, \dots, |trace(p_1 p_2 p_3)|, \dots > 2.$$

Consider the case $n = 3$ again. Solving $trace(p_i p_j) = 2 - a_{ij}^2$ for u, v, w we get two solutions:

$$\left\{w = \frac{a_{31}\,a_{32}}{a_{21}}, v = \frac{a_{32}\,a_{21}}{a_{31}}, u = -\frac{a_{31}\,a_{21}}{a_{32}}\right\}, \quad \left\{w = -\frac{a_{31}\,a_{32}}{a_{21}}, v = -\frac{a_{32}\,a_{21}}{a_{31}}, u = \frac{a_{31}\,a_{21}}{a_{32}}\right\}.$$

Substituting into the above equalities we get:

$$\mp\frac{a_{31}\,a_{21} \mp a_{32}}{a_{32}} < 0; \quad \pm\frac{a_{21}\,\left({a_{31}}^2 \mp a_{21}\,a_{31}\,a_{32} + {a_{32}}^2\right)}{a_{31}\,a_{32}} < 0;$$
$$\mp a_{21}\,a_{31}\,a_{32} + {a_{32}}^2 + {a_{21}}^2 + {a_{31}}^2 < 0.$$

Here we consistently take the top or bottom choice of signs in these equations, corresponding to the two solutions given in the previous paragraph. We note that this shows that points of $\bar{\mathcal{T}}_{0,3,1}$ lie in V_t with $t < 0$.

From the above we see that (when $n = 3$) there are points of Teichmüller space with $|trace(p_1 p_2)| > 2$; thus with $|a_{21}|, |a_{31}|, |a_{32}| > 2$ (in fact they can be arbitrarily large). Thus in each component of $\bar{\mathcal{T}}_{0,3,1} \subset \mathbb{R}^3$ that covers $\mathcal{T}_{0,3,1}$ we must have points (a_{21}, a_{31}, a_{32}) satisfying:

$$|a_{21}|, |a_{31}|, |a_{32}| > 2 \quad \text{and} \quad trace(p_3 p_2 p_1) = a_{21}^2 + a_{31}^2 + a_{32}^2 - a_{21} a_{31} a_{32} - 2 \neq 2.$$

Now let $\mathcal{P}$ be the component of the complement of the surface defined by $a_{21}^2 + a_{31}^2 + a_{32}^2 - a_{21}a_{31}a_{32} - 4 = 0$ which contains the point $(3, 3, 3)$. Note that $\mathcal{P}$ is an open 3-ball. Then one can also check that $\mathcal{P}$ contains a part of $\bar{\mathcal{T}}_{0,3,1}$. Not every point of $\mathcal{P}$ will represent a point of Teichmüller space; in fact only those satisfying the conditions corresponding to (11.3) will do so. We will use these facts later.

In order to prove Theorem 6 and the first part of Theorem 4 we need to show that the actions of B_n on both sides of the equation $2 - trace(p_ip_j) = a_{ij}^2, i > j$, are compatible at least for some set of lifts of the a_{ij} which are permuted by the B_n action. More precisely, for $\alpha \in B_n$ suppose that $\alpha(a_{ij}) = f_{\alpha,i,j}(a_{21}, a_{31}, \ldots, a_{nn-1})$. Then what we need to do amounts to finding all $\epsilon_{ij} \in \{\pm 1\}, i > j$, such that for all $\beta \in B_n$ there are $\delta_{ij} \in \{\pm 1\}, n \geq i > j \geq 1$, so that if $a_{ij} = \epsilon_{ij}\sqrt{2 - trace(p_ip_j)}$ then

$$f_{\beta,i,j}(\epsilon_{21}a_{21}, \epsilon_{31}a_{31}, \ldots, \epsilon_{nn-1}a_{nn-1}) = \delta_{ij}\sqrt{2 - trace(\beta(p_i)\beta(p_j))}. \tag{11.5}$$

To proceed we will need to note the following trace identity for three non-pair-wise commuting parabolics p_1, p_2, p_3 with trace -2:

$$\sqrt{2 - trace(p_1p_2p_1^{-1}p_3)} = \sqrt{(2 - trace(p_1p_2))(2 - trace(p_1p_3))} - \sqrt{2 - trace(p_2p_3)}. \tag{11.6}$$

This is easily checked since we can conjugate p_1, p_2, p_3 so that they are as in (11.1) and then easily check this identity (recall that parabolics commute if and only if they have the same fixed point).

We reduce immediately to the case where $\beta \in B_n$ is a generator: $\beta = \sigma_i$; and the action of β is given by (1.2) only where we have $a_{ij} = -a_{ji}$. Now $\beta(a_{i+1i}) = -a_{i+1i}$ and so (11.5) gives

$$-\epsilon_{i+1i}\sqrt{2 - trace(p_{i+1}p_i)} = \delta_{i+1i}\sqrt{2 - trace(p_{i+1}p_i)},$$

so that we have $-\epsilon_{i+1i} = \delta_{i+1i}$ for all $1 \leq i < n$.

Next for $j < i$ we have $\beta(a_{i+1j}) = a_{ij}$ and so (11.5) gives $\epsilon_{ij}\sqrt{2 - trace(p_ip_j)} = \delta_{i+1j}\sqrt{2 - trace(p_ip_j)}$, which gives $\epsilon_{ij} = \delta_{i+1j}$.

Keeping $j < i$ we also have $\beta(a_{ij}) = a_{i+1j} - a_{i+1i}a_{ij}$, so that (11.5) and (11.6) give

$$\begin{aligned}
\epsilon_{i+1j}&\sqrt{2 - trace(p_{i+1}p_j)} - \epsilon_{i+1i}\epsilon_{ij}\sqrt{(2 - trace(p_{i+1}p_i))(2 - trace(p_ip_j))} \\
&= \delta_{ij}\sqrt{2 - trace(p_ip_{i+1}p_i^{-1}p_j)} \\
&= \delta_{ij}\left(\sqrt{(2 - trace(p_{i+1}p_i))(2 - trace(p_ip_j))} - \sqrt{2 - trace(p_{i+1}p_j)}\right),
\end{aligned}$$

showing that $\epsilon_{i+1j} = -\delta_{ij}$ and $\epsilon_{i+1i}\epsilon_{ij} = -\delta_{ij}$.

Continuing this analysis for $\beta(a_{ji}) = a_{ji+1} - a_{ji}a_{i+1i}$ and $\beta(a_{ji+1}) - a_{ji}$ we obtain the following equations relating the ϵ_{ij} and δ_{ij}:

$$\begin{aligned}
&(i) \quad \epsilon_{i+1i} = -\delta_{i+1i} \quad \text{for all} \quad 1 \le i < n;\\
&(ii) \quad \epsilon_{ij} = \delta_{i+1j} \quad \text{for all} \quad 1 \le j < i \le n;\\
&(iii) \quad \epsilon_{i+1j} = -\delta_{ij} \quad \text{for all} \quad 1 \le j < i \le n;\\
&(iv) \quad \epsilon_{i+1i}\epsilon_{ij} = -\delta_{ij} \quad \text{for all} \quad 1 \le j < i \le n;\\
&(v) \quad \epsilon_{ji} = \delta_{ji+1} \quad \text{for all} \quad 1 < i+1 < j \le n;\\
&(vi) \quad \epsilon_{ji+1} = -\delta_{ji} \quad \text{for all} \quad 1 < i+1 < j \le n;\\
&(vii) \quad \epsilon_{ji}\epsilon_{i+1i} = -\delta_{ji} \quad \text{for all} \quad 1 < i+1 < j \le n.
\end{aligned}$$

From (iii) and (iv) and from (vi) and (vii) we get

$$\epsilon_{ji+1}\epsilon_{ji}\epsilon_{i+1i} = 1, \quad \epsilon_{i+1j}\epsilon_{ij}\epsilon_{i+1i} = 1.$$

In fact from the above relations it is apparent that these are the only relations (for fixed i) satisfied by the ϵ_{rs}. Further, from $\epsilon_{i+1j}\epsilon_{ij}\epsilon_{i+1i} = 1$ and (i), (ii) and (iii) we see that we must have $\delta_{i+1j}\delta_{ij}\delta_{i+1i} = 1$, and that similarly we must have $\delta_{ji+1}\delta_{ji}\delta_{i+1i} = 1$ following from (i), (v) and (vi).

Since we are interested in having these relations for all $i < n$ one checks that for any ϵ_{rs} satisfying $\epsilon_{ij}\epsilon_{jk}\epsilon_{ik} = 1$ for all $i > j > k$ there are δ_{rs} satisfying $\delta_{ij}\delta_{jk}\delta_{ik} = 1$ for all $i > j > k$ which solve (i)-(vii). But any such ϵ_{rs} are clearly determined by $n-1$ of them, namely $\epsilon_{n1}, \epsilon_{n2}, \dots, \epsilon_{nn-1}$ and that we must have

$$\epsilon_{rs} = \epsilon_{nr}\epsilon_{ns} \quad \text{for all} \quad n > r > s \ge 1.$$

Similar conditions hold for the δ_{rs}. Thus we see that we get $\bar{\mathcal{T}}_{0,n,1}$, a cover of $\mathcal{T}_{0,n,1}$ (in the broad sense) consisting of points (a_{ij}) over $\mathcal{T}_{0,n,1}$ where $sign(a_{ij}a_{jk}a_{ik}) = 1$ for all $i > j > k$. This gives 2^{n-1} disjoint copies of $\mathcal{T}_{0,n,1}$ embedded smoothly in $\mathbb{R}^{\binom{n}{2}}$ with coordinate functions $a_{21}, a_{31}, \dots, a_{nn-1}$ such that the action of B_n on these 2^{n-1} copies of $\mathcal{T}_{0,n,1}$ is by polynomial automorphisms, as in (1.2). This completes the proof of Theorem 6 and part of Theorem 4. □

We note that using the natural coordinates u, v, w, x_{41}, x_{42} for $\mathcal{T}_{0,4,1}$ we have the following action of B_4 as rational automorphisms (the action of B_3 on the natural coordinates for $\mathcal{T}_{0,3,1}$ is obtained by restricting σ_1 and σ_2 to u, v, w):

$$\begin{aligned}
\sigma_1(u,v,w,x_{41},x_{42}) &= (v(1-u), \frac{u}{1-u}, w(u-1),\\
&\qquad \frac{x_{41}x_{42}(u-1) - (1+x_{41})^2}{x_{42}(u-1)}, \frac{(1+x_{41}+x_{42}-ux_{42})^2}{x_{42}(u-1)}),\\
\sigma_2(u,v,w,x_{41},x_{42}) &= (-u(1+v), -w(1+v), \frac{-v}{1+v}, x_{41}+x_{42}, -x_{42}/(1+v)),\\
\sigma_3(u,v,w,x_{41},x_{42}) &= (\frac{u(w+wx_{41}+x_{42})}{1+w+wx_{41}+x_{41}+x_{42}}, \frac{v(1+w+wx_{41}+x_{41}+x_{42})}{w+wx_{41}+x_{42}},\\
&\quad \frac{(w+wx_{41}+x_{41}+x_{42}+1)(w+wx_{41}+x_{42})}{x_{42}}, \frac{-(2wx_{41}+2w+x_{42})}{w+wx_{41}+x_{42}},\\
&\quad \frac{w(1+w+wx_{41}+x_{42}+x_{41})}{w+wx_{41}+x_{42}}) \qquad (11.4)
\end{aligned}$$

This is a non-polynomial action, whereas the action of B_n on the a_{ij}-coordinates for $\bar{\mathcal{T}}_{0,n,1}$ gives a polynomial action, as we have just indicated.

In what follows we will identify points on all of the 2^{n-1} components of $\bar{\mathcal{T}}_{0,n,1}$ by making all the signs positive and so identify $\mathcal{T}_{0,n,1}$ with the positive component $\bar{\mathcal{T}}_{0,n,1} \cap \mathbb{R}^{\binom{n}{2}}_{>2}$ of $\bar{\mathcal{T}}_{0,n,1}$.

For the case $n = 4$ we have 6 of the a_{ij} whereas $\mathcal{T}_{0,4,1}$ and $\bar{\mathcal{T}}_{0,4,1}$ have dimension 5. Thus the a_{ij} must satisfy some relation on the components of $\bar{\mathcal{T}}_{0,4,1}$. This relation will depend on the choice of signs of the generators p_i. We indicate one such relation for the canonical choice of signs in:

Proposition 11.2. *Let $n = 4$ and suppose that p_1, p_2, p_3, p_4 are all as given in (11.1) and (11.2) with variables u, v, w, x_{41}, x_{42}. Then in the ideal generated by the $trace(p_ip_j)-(2-a_{ij}^2)$ there is a single relation satisfied by the a_{ij}:*

$$\begin{aligned}&(a_{21}a_{43} - a_{31}a_{42} - a_{32}a_{41})(a_{21}a_{43} - a_{31}a_{42} + a_{32}a_{41})\\ \times&(a_{21}a_{43} + a_{31}a_{42} - a_{32}a_{41})(a_{21}a_{43} + a_{31}a_{42} + a_{32}a_{41}).\end{aligned}$$

The factor $a_{21}a_{43} - a_{31}a_{42} + a_{32}a_{41} \in R'_4$ is invariant under the action of $kerB_4 \to \mathbb{Z}_2, \sigma_i \mapsto 1$.

Proof. One first calculates the elements $trace(p_ip_j) - (2 - a_{ij}^2)$ and considers the ideal of $\mathbb{Q}[a_{21}, a_{31}, a_{41}, a_{32}, a_{42}, a_{43}, u, v, w, x_{41}, x_{42}]$ generated by them. One then uses the elimination algorithm [AL, Theorem 2.3.4], as implemented in MAGMA [MA], to eliminate the variables u, v, w, x_{41}, x_{42} and so get the above relation as the only relation satisfied by the a_{ij} (actually, one gets the above relation multiplied by a_{41}^2, but this extra factor can be ignored in our case, since we are considering real coefficients). It is easy to check that the generators $\sigma_i^{\pm1}\sigma_j^{\pm1}$ of $kerB_4 \to \mathbb{Z}_2$ fix the factor $a_{21}a_{43} - a_{31}a_{42} + a_{32}a_{41}$. □

The fact that the relation in the above result factors nicely is another indication that Keen's choice of signs for the generating parabolics really is natural (one can check that other choices of sign do not give relations which so factor).

Of course one can do the above calculation for any $n > 3$ and obtain an elimination ideal that gives relations that the a_{ij} have to satisfy on the components of $\bar{\mathcal{T}}_{0,n,1}$. For $n = 5$ one obtains an ideal with 15 generators, five of which look like the one given in Proposition 11.2.

Note that the equation $x^2 + y^2 + z^2 - xyz = t$ is related to the Markoff equation $x^2 + y^2 + z^2 = 3xyz$, since if one substitutes $3x, 3y, 3z$ for x, y, z, then it gives $x^2 + y^2 + z^2 - 3xyz = t/9$. The general theory of the Markoff equation [CF] tells us that all positive integral solutions are related to each other via the Markoff tree and an action of a group on the set of solutions.

Theorem 11.3. *Let $t > 4$ and let V_t denote the surface $a_{21}^2 + a_{31}^2 + a_{32}^2 - a_{21}a_{31}a_{32} - t = 0$ in $\mathbb{R}^3$. Then $B_3\backslash V_t$ is compact. Further on the level sets $V_t, t > 4$, there are points with infinite stabilisers.*

Proof. First note that we have already described V_t for $t = 4$ (in §4) and have indicated that for $t < 4$, V_t has 4 unbounded components, each a topological disc, while for $t > 4$, V_t is homeomorphic to a sphere with 4 holes. Let $x = a_{21}$, $y = a_{31}$, $z = a_{32}$. The first statement in this result will follow from the next result which is based on ideas of [Mo, pp. 106-110]; in fact the next result is, for our equation, a stronger version of [Mo; Theorem 8 p. 107].

Lemma 11.4. *For $t > 4$ and $(x, y, z) \in V_t \subset \mathbb{R}^3$ there is $\alpha \in B_3$ such that $\alpha(x, y, z) = (u, v, w)$ where $u^2 + v^2 + w^2 \leq t$.*

Proof. One first checks that the elements μ_1, μ_2 where $\mu_1(x, y, z) = (-x, -y, z), \mu_2(x, y, z) = (x, -y, -z)$ generate a subgroup M_3 of $Diff(\mathbb{R}^3)$, the group of diffeomorphisms of $\mathbb{R}^3$, having order 4. We note that M_3 fixes each V_t. Regarding B_3 as giving a subgroup of $Diff(\mathbb{R}^3)$ one sees easily that B_3 normalises M_3. Thus we may without loss assume in what follows that any $(x, y, z) \in V_t$ has $x, y \geq 0$.

Note that if we have a solution (x, y, z) with $xyz = 0$, then $x^2 + y^2 + z^2 = t$ in this case and so we are done. Thus in what follows we may assume $xyz \neq 0$, so that with the above we may assume $x, y > 0$.

Now we have

$$\sigma_1(x, y, z) = (-x, z - xy, y), \quad \sigma_2(x, y, z) = (y - xz, x, -z)$$
$$\sigma_1^{-1}(x, y, z) = (-x, z, y - xz), \quad \sigma_2^{-1}(x, y, z) = (y, x - yz, -z).$$

We let $||(x, y, z)|| = x^2 + y^2 + z^2$. Now given $p = (x, y, z) \in \mathbb{R}^3$ we replace p by $\sigma_k^\epsilon(p)$ (for any choice of $k = 1, 2, \epsilon = \pm 1$) if $||\sigma_k^\epsilon(p)|| < ||p||$ and continue doing this until we have $||\sigma_k^\epsilon(p)|| \geq ||p||$ for all $k = 1, 2, \epsilon = \pm 1$. Clearly this is a finite process. We claim that for such a p we must have $||p|| \leq t$. Suppose not for some t and $p = (x, y, z)$. Then $x^2 + y^2 + z^2 - xyz = t$ and $||p|| = x^2 + y^2 + z^2 > t$ yields $xyz > 0$, which, together with $x, y > 0$ gives $z > 0$.

Now $||\sigma_k^\epsilon(p)|| \geq ||p||$, for all $k = 1, 2, \epsilon = \pm 1$, yields the conditions

$$x^2y^2 - 2xyz \geq 0, \quad x^2z^2 - 2xyz \geq 0, \quad y^2z^2 - 2xyz \geq 0.$$

Since $x, y, z > 0$ this gives

$$xy - 2z \geq 0, \quad xz - 2y \geq 0, \quad yz - 2x \geq 0.$$

Now multiplying $xy \geq 2z$ and $yz \geq 2x$ gives $y^2 \geq 4$ and we similarly see that $x^2, z^2 \geq 4$. Next we note that the equations $xy - 2z \geq 0, xz - 2y \geq 0, yz - 2x \geq 0$ determine an open convex region $R \subset \mathbb{R}^3_{\geq 2}$. This is indicated in Figure 6, where we have drawn the part of the surfaces $xy - 2z = 0, \quad xz - 2y = 0, \quad yz - 2x = 0$ where $x, y, z \geq 2$. The z coordinate is in the vertical direction and so the top component is the one determined by $z = xy/2$. The region R is that 'inside' this cone.

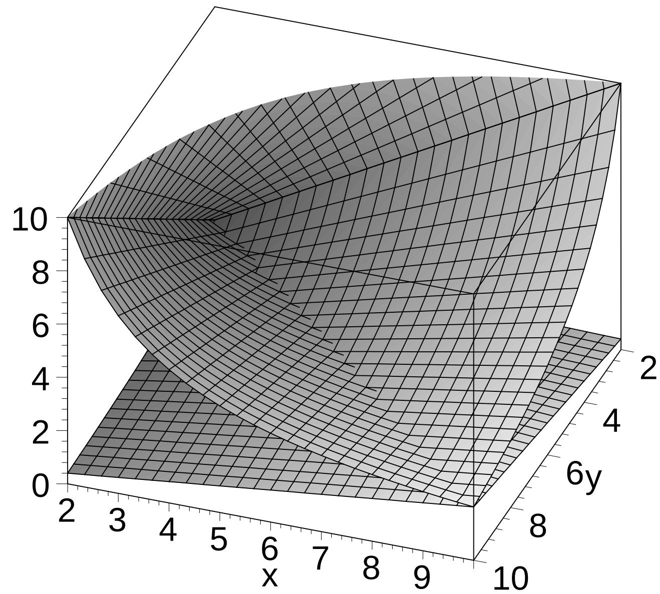

Figure 6.

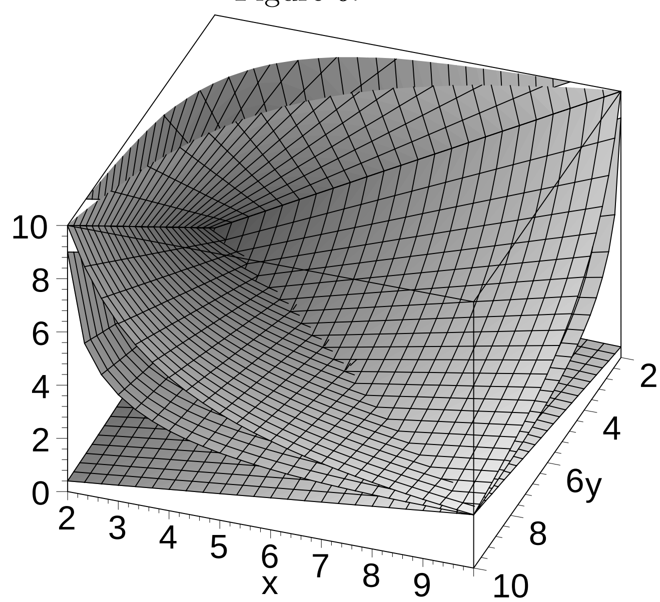

Figure 7.

In Figure 7 we have added to Figure 6 a part of the surface $x^2+y^2+z^2-xyz=t$ (for $t=5$) to indicate how this relates to the region R. We will show that for $t>4$ the surface V_t does not meets R, a contradiction.

Now the surfaces $xz-2y=0,\ yz-2x=0$ meet along the line where $x=\pm y, z^2=4$, while the surfaces $xy-2z=0, xz-2y=0$ meet where $x^2=4$ and the surfaces $xy-2z=0, yz-2x=0$ meet where $y^2=4$. The surface V_t is determined by

$$z_{\pm}=(xy\pm\sqrt{(x^2-4)(y^2-4)+4(t-4)})/2.$$

Now we need to check for example that if $x,y\geq 2$, then $z_+(x,y)\geq xy/2$, but this is clear. Another case is that when $x>y$, then we would like to show that $z_-(x,y)\leq 2x/y$. To see this note that solving the equation $2x/y-z_-(x,y)=0$ for y gives

$$y_{\pm,\pm}(x,t)=\left(\pm\sqrt{x^2+t\pm\sqrt{(x^2+t)^2-16\,x^2}}\right)/\sqrt{2},$$

where all 4 values of the square roots are allowed. Now y_{++} is above and asymptotic to the line $y = x$. Thus since $x > y$ in this case we see that $2x/y - z_-(x, y) \geq 0$ as required. The $y_{-\pm}$ cases aren't allowed, since then $y < 0$

For the case y_{+-} we see that solving $y_{+-}(x, t) = 2$ gives $t = 4$, a contradiction. Thus $y_{+-}(x, t) \neq 2$. It follows that $y_{+-}(x, t) < 2$ for all $x \geq 2, t > 4$ and so this case does not occur either. This shows that $z_-(x, y) \leq 2x/y$ when $x > y$. Since everything is symmetric in the variables x, y, the last case ($z_-(x, y) \leq 2y/x$ when $y > x$) follows by interchanging x, y in the above. This proves Lemma 11.4 and so the first part of Theorem 11.3.

We now show that there are points of the level sets $V_t, t > 4$, with infinite stabilisers. For example the element $\sigma_1^2 \in B_3$ fixes all $(a_{21}, a_{31}, a_{32}) = (0, \sqrt{t}cos(\theta), \sqrt{t}sin(\theta)) \in V_t$. $\square$

Let G_3 denote the group of homeomorphisms of $\mathbb{R}^3$ generated by $\mu_1, \mu_2, \sigma_1, \sigma_2$. Theorem 11.3 and Lemma 11.4 imply the following related facts:

Corollary 11.5. *Suppose that $x^2 + y^2 + z^2 - xyz = t > 4$ has an integral solution. Then any integral solution of $x^2 + y^2 + z^2 - xyz = t$ is in the G_3-orbit of some (x, y, z), where $x, y, z \in \mathbb{Z}$ and $|x|, |y|, |z| \leq \sqrt{t}$.*

There are no integral solutions of $x^2 + y^2 + z^2 - xyz = t$ for any $t \in \mathbb{Z}$, with $t \equiv 3$ mod 4 or $t \equiv 3, 6$ mod 9.

For $t = 5$ every integral solution of $x^2 + y^2 + z^2 - xyz = t$ is in the G_3-orbit of $(0, 1, 2)$.

For $t = 8$ every integral solution of $x^2 + y^2 + z^2 - xyz = t$ is in the G_3-orbit of $(1, 1, -2)$ or $(0, 2, 2)$. These orbits are distinct.

For $t = 9$ every integral solution of $x^2 + y^2 + z^2 - xyz = t$ is in the G_3-orbit of $(3, 0, 0)$. The orbit in this case is finite.

Proof. The first statement follows directly from Lemma 11.4. Thus for a given integral $t > 4$ it is easy to check whether there are any integral solutions. Now if $x, y, z \in \mathbb{Z}$, then one easily checks that $x^2 + y^2 + z^2 - xyz$ mod $4 \in \{0, 1, 2\}$ and that $x^2 + y^2 + z^2 - xyz$ mod $9 \in \{0, 1, 2, 4, 5, 7, 8\}$

When $t = 5$ a computer calculation shows that all integral solutions are in the orbit of $(0, 1, 2)$. The proof for $t = 8, 9$ is similar. We also need to note that the action of G_3 does not change the *gcd* of the entries; thus when $t = 8$ the orbits of $(1, 1, -2)$ and $(0, 2, 2)$ are distinct. $\square$

Remark 11.6. The integral values of $t > 4$ for which $x^2 + y^2 + z^2 - xyz = t$ has no integral solutions given in the above result do not exhaust such values of t. For example this set also includes

$$\{46, 56, 86, 124, 126, 142, 161, 198, 206, 216, 217\}.$$

This is easily proved using Lemma 11.4.

Lemma 11.7. *Any point of V_t is in the G_3-orbit of some $(x, y, z) \in \mathbb{R}^3$ where*

$$|x - yz| \geq |x|, \quad |y - xz| \geq |y|, \quad \text{and} \quad |z - xy| \geq |z|.$$

Proof. This follows from the argument in the proof of Theorem 11.3. $\square$

Lemma 11.7 will help us determine a fundamental domain for the action of G_3 on $V_t, t < 0$. As usual we may assume that $x, y > 0$. Now if $z \leq 0$, then $0 > t = x^2 + y^2 + z^2 - xyz > 0$.

Thus we may restrict attention to the octant $x, y, z > 2$. Now if we have $yz \leq x$, then we would have

$$t = x^2 + y^2 + z^2 - xyz \geq x^2 + y^2 + z^2 - x^2 = y^2 + z^2 > 4,$$

a contradiction. Thus we must have $yz > x$ and similarly $xy > z$ and $xz > y$. Putting these together with Lemma 11.7 we see that any point of $V_t, t < 0$, is in the G_3-orbit of some (x, y, z) where

$$xy > 2z, \quad yz > 2x \quad \text{and} \quad xz > 2y.$$

Now the equations $xy = 2z, yz = 2x, xz = 2y$ determine real algebraic varieties which cut out a region of V_t. An example of this is shown in Figure 8, which, for $t = -7$, is the projection onto the xy-plane of these three curves in V_t. Here we have used each of the above three equations to determine the variable z and substituted this value of z into $x^2+y^2+z^2-xyz-t = 0$. This gives an equation which we have solved for y as a function of x and t. One gets the following values for y (since we are only interested in having $x, y > 2$):

$$2\sqrt{\frac{x^2-t}{x^2-4}}, \quad x\sqrt{\frac{x^2-t}{x^2-4}}, \quad \frac{4x}{\sqrt{2\,x^2+2\,t+2\,\sqrt{x^4+2\,x^2t+t^2-16\,x^2}}},$$
$$\frac{4x}{\sqrt{2\,x^2+2\,t-2\,\sqrt{x^4+2\,x^2t+t^2-16\,x^2}}}.$$

This gives curves which in this projection are asymptotic to the $x = 2, y = 2$ and $x = y$ lines. The last two equations correspond to solutions of the case $yz = 2x$, while the first corresponds to $xy = 2z$ and the second to $xz = 2y$. More generally, the three 'spokes' shown in Figure 8 are each asymptotic (in projection) to one of the lines (i) $z = 2$, $x = y$; (ii) $y = 2$, $x = z$; (iii) $x = 2$, $y = z$.

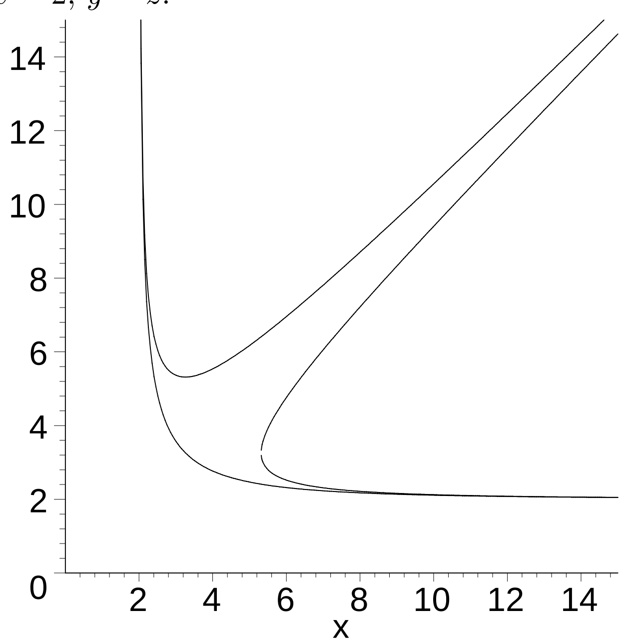

Figure 8

In Figure 8 the component which is above the $y = x$ line corresponds to the equation $xz = 2y$ (we will denote it by γ_y), while the component which is below the $y = x$ line corresponds to the equation $2x = yz$ (we will denote it by γ_x). The component which is asymptotic to the $x = 2, y = 2$ lines corresponds to the equation $2z = xy$ (we will denote it by γ_z). Note that the latter curve is symmetric relative to the line $y = x$, while the other two are interchanged by reflection in this line.

Let $\mathcal{F}_t$ denote the closed region of V_t determined in this way. Let $\alpha = \sigma_1\sigma_2\sigma_1, \beta = \sigma_1\sigma_2$.

Theorem 11.8. *The region $\mathcal{F}_t$ is a fundamental domain for the action of the subgroup $J_3 =< \alpha, \beta\alpha\beta^{-1}, \beta^{-1}\alpha\beta, \mu_1, \mu_2 >$ of index 3 in G_3 on $V_t, t < 0$. The subgroup $< \mu_1, \mu_2 >$ is normal in J_3 and in G_3 with $G_3/ < \mu_1, \mu_2 >\cong B_3/Z(B_3)$. The region $\mathcal{F}_t$ contains three copies of a fundamental region for the action of G_3.*

Proof. First we note that α and β generate B_3, that J_3 has index 3 in G_3 and that $\alpha^2 = 1, \beta^3 = 1$. The normality of $< \mu_1, \mu_2 >$ has already been noted. Further, using the Reidemeister-Schreier method [MKS] as implemented in MAGMA [MA] one can show that $J_3/ < \mu_1, \mu_2 >\cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$. The second statement is clear and the last follows from the above and the first statement, which we now prove.

Next we see that

$$\beta(x, y, z) = (z, -x, -y), \quad \alpha(x, y, z) = (-z, -y + xz, -x).$$

For $(x, y, z), (u, v, w) \in \mathbb{R}^3$ we define $(x, y, z) \sim (u, v, w)$ if (x, y, z) and (u, v, w) are in the same orbit under the action of J_3. Now, if $(x, y, z) \in \gamma_y$, so that $xz = 2y$, then

$$(x, y, z) \sim (-z, xz - y, -x) = (-z, y, -x) \sim (y, x, z),$$

showing that points of the curves in Figure 8 which are asymptotic to the line $y = x$ are identified under the J_3 action in the same way as a reflection across this line would identify them. Since everything is symmetric in x, y, z we similarly obtain two other such identifications.

Standard arguments for the action of Schottky groups (see for example [Ly; p. 197]) now show that $\mathcal{F}_t$ is a fundamental domain for the action of J_3 on V_t. $\square$

One can check that the three curves $\gamma_x, \gamma_y, \gamma_z$ are permuted by the action of β: $\beta(\gamma_y) = \gamma_z, \beta(\gamma_z) = \gamma_x, \beta(\gamma_x) = \gamma_y$. Further $\alpha(\gamma_y) = \gamma_y$ (since $\sigma_1\sigma_2\sigma_1(2a_{31} - a_{21}a_{32}) = -(2a_{31} - a_{21}a_{32})$) and $\alpha(\mathcal{F}_t) \cap \mathcal{F}_t = \gamma_y$.

We can also check that β has exactly one fixed point in $\mathcal{F}_t$, namely the point (x_0, x_0, x_0) where x_0 is the real solution of $3x^2 - x^3 - t = 0$, namely

$$x_0 = \frac{(8 - 4t + 4\sqrt{t(t-4)})^{2/3} + 4 + 2(8 - 4t + 4\sqrt{t(t-4)})^{1/3}}{2(8 - 4t + 4\sqrt{t(t-4)})^{1/3}}.$$

Since $\beta(x, y, z) \sim (z, x, y)$ we see that a fundamental domain for $< \sigma_1, \sigma_2, \mu_1, \mu_2 >$ is obtained by dividing $\mathcal{F}_t$ into three pieces, all meeting at the point (x_0, x_0, x_0). Further these three pieces are permuted by β and can be chosen so that they contain part of the curves where V_t meets the planes $x = y, y = z, z = x$. This proves Theorem 5.

We now note two consequences. We have already seen that the curves $\gamma_x, \gamma_y, \gamma_z$ are asymptotic to two of the lines $\{x = 2, y = z\}, \{y = 2, x = z\}, \{z = 2, y = x\}$. Further, one easily sees from the equations for these curves that all of these curves are contained in the positive octant cut out by the planes $x = 2, y = 2, z = 2$. Thus if we have an integer point of V_t, then it can't be very far up one of the 'spokes' of $\mathcal{F}_t$ Thus we have:

Corollary 11.9. *For $t < 0$ there are only finitely many B_3-orbits of integer solutions to the equation $x^2 + y^2 + z^2 - xyz = t$. In particular, on each level set $V_t \cap (\bar{\mathcal{T}}_{0,3,1} \cap \mathbb{R}^3_{>2})$ of Teichmüller space there are only finitely many B_3-orbits of integer points.* $\square$

Now $B_3/Z(B_3) \cong PSL_2(\mathbb{Z}) =< a, b|a^2, b^3 >$ and the action of $B_3/Z(B_3)$ on the quotient $< \mu_1, \mu_2 > \backslash V_t$ looks exactly like the action of the index 3 subgroup $J_3' =< a, bab^{-1}, b^{-1}ab >$ of $PSL_2(\mathbb{Z})$ on the fundamental domain $\mathcal{F}_t'$. Here we refer to Figure 9 where $A \cup B$ is a standard fundamental domain for the action of $PSL_2(\mathbb{Z})$ on the upper half plane $\mathbb{H}^2$. Then $G \cup A$ is also a fundamental domain. We let $\mathcal{F}_t' = A \cup C \cup D \cup E \cup F \cup G$. One easily sees that $\mathcal{F}_t'$ is a fundamental domain for J_3'. Note that $\mathcal{F}_t$ and $\mathcal{F}_t'$ are homeomorphic and so there is a homeomorphism $f_t : V_t \cap \mathbb{R}^3_{>2} \to \mathbb{H}^2$ such that $f_t(\mathcal{F}_t) = \mathcal{F}_t'$.

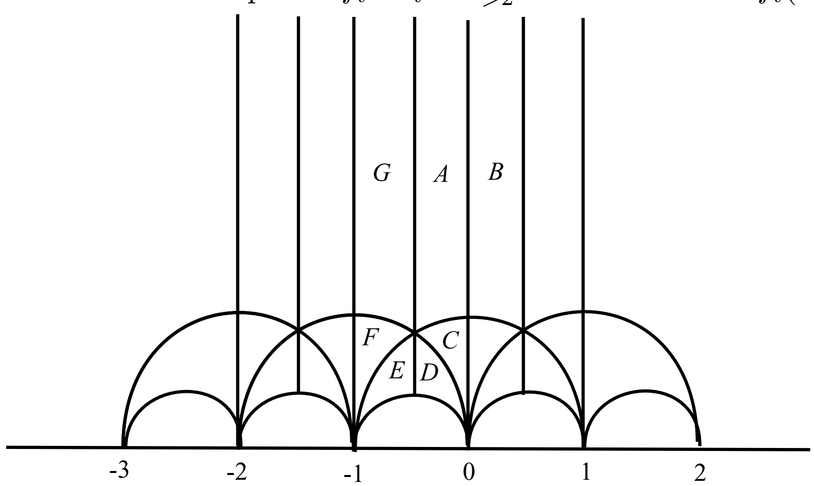

Figure 9

Using f_t we can pull back the hyperbolic metric from $\mathbb{H}^2$ to $V_t \cap \mathbb{R}^3_{>2}$ so as to get:

Corollary 11.10. *There is a hyperbolic metric on $V_t \cap \mathbb{R}^3_{>2}, t < 0$, such that the action of J_3 on $V_t \cap \mathbb{R}^3_{>2}$ is by hyperbolic isometries. Further, the curves defining $\mathcal{F}_t$ are geodesics in this metric.* $\square$

Remark 11.11. The strata referred to in Theorem 4 are more easily understood for $n = 3, 4$: Consider $\bar{\mathcal{T}}_{0,3,1} \subseteq \mathbb{R}^3$. This has dimension 3 and is a union of 2-dimensional strata coming from the B_3-invariant level sets of $c_{31}' = a_{21}^2 + a_{31}^2 + a_{32}^2 - a_{21}a_{31}a_{32} + 3$.

For $n = 4$ we note that $\bar{\mathcal{T}}_{0,4,1} \subseteq \mathbb{R}^6$ has dimension 5, however we have two independent B_4-invariant functions

$$\begin{aligned}
c_{41}' &= a_{21}a_{32}a_{41}a_{43} - a_{21}a_{31}a_{32} - a_{21}a_{41}a_{42} - a_{31}a_{41}a_{43} - a_{32}a_{42}a_{43} \\
&\quad + a_{21}^2 + a_{31}^2 + a_{32}^2 + a_{41}^2 + a_{42}^2 + a_{43}^2 - 4, \\
c_{42}' &= a_{21}^2a_{43}^2 - 2a_{21}a_{31}a_{42}a_{43} + a_{31}^2a_{42}^2 - 2a_{31}a_{32}a_{41}a_{42} + a_{32}^2a_{41}^2 + 2a_{21}a_{31}a_{32} \\
&\quad + 2a_{21}a_{41}a_{42} + 2a_{31}a_{41}a_{43} + 2a_{32}a_{42}a_{43} - 2(a_{21}^2 + a_{31}^2 + a_{32}^2 + a_{41}^2 + a_{42}^2 + a_{43}^2) + 6,
\end{aligned}$$

which thus shows that $\bar{\mathcal{T}}_{0,4,1}$ is a union of at most 4-dimensional B_4-invariant strata.

We now consider the cases $n > 4$; we use the notation of Figure 5 at the beginning of this section. We will show that there is a 1-parameter family of surfaces which give points of $\mathcal{T}_{0,n,1}$ where $c_{n\,n-1}'$ is not constant. This will prove the last part of Theorem 4.

For $n > 4$ let us define the following matrices:

$$p_1 = \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}, \quad p_2 = \begin{pmatrix} -4 & 3 \\ -3 & 2 \end{pmatrix},$$
$$p_3 = \begin{pmatrix} -1 & 2n+3 \\ 0 & -1 \end{pmatrix}, \quad p_4 = \begin{pmatrix} 6n-1 & 12n^2 \\ -3 & -6n-1 \end{pmatrix}$$

Now for $4 < m < n$ we let

$$p_m = \begin{pmatrix} 8n-4m+15 & 4(2n-m+4)^2 \\ -4 & 4m-8n-17 \end{pmatrix}.$$

Finally we let

$$p_n = \begin{pmatrix} \frac{(3n+11)x_1+4n+15}{x_1+1} & \frac{(3x_1+4)(n+4)^2}{x_1+1} \\ \frac{-3x_1-4}{x_1+1} & \frac{-(3n+13)x_1-4n-17}{x_1+1} \end{pmatrix}.$$

One checks that $p_1, \dots, p_n$ are all parabolics which (respectively) fix the points

$$f_1 = 0, f_2 = 1, f_3 = \infty, f_4 = -2n, f_5 = -2n+1,$$
$$f_6 = -2n+2, \dots, f_{n-1} = -(n+5), f_n = -(n+4) + x_1.$$

We will also let $f_{n+1} = -(n+3)$. The matrices s_i are as follows:

$$s_1 = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} -2 & 3 \\ 1 & -2 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 2 & 4n+3 \\ -1 & -2n-1 \end{pmatrix},$$
$$s_4 = \begin{pmatrix} 7 & 14n-3 \\ -2 & 1-4n \end{pmatrix},$$
$$s_m = \begin{pmatrix} 2m-1 & (4m-2)n-(2m^2-8m+3) \\ -2 & (2m-7)-4n \end{pmatrix}, \text{ for } \quad 4 < m < n,$$
$$s_n = \begin{pmatrix} \frac{nx_1+2n-1}{x_1+1} & \frac{(n^2+3n+1)x_1+2n^2+6n-3}{x_1+1} \\ \frac{-(x_1+2)}{x_1+1} & \frac{(n+3)x_1-2n-7}{x_1+1} \end{pmatrix}.$$

Now $s_i(f_i) = g_i$ for $i = 1, \dots, n$ where we put $g_i = -(i-1)$ for $i = 1, \dots, n+1$. We also have $s_n(f_{n+1}) = g_{n+1}$. Thus following the argument at the beginning of this section we see that for sufficiently small values of x_1 these matrices do give a point of $\mathcal{T}_{0,n,1}$. Now to consider the corresponding points in $\bar{\mathcal{T}}_{0,n,1}$ one solves the equations $trace(p_ip_j) - (2 - a_{ij}^2)$. We now list the values of $2 - trace(p_ip_j)$; here we assume that $4 < m, m' < n$:

$$\begin{aligned}
&2 - trace(p_1p_2) = 6, \quad 2 - trace(p_1p_3) = 2(2n+3), \quad 2 - trace(p_1p_4) = 24n^2, \\
&2 - trace(p_1p_m) = 8(m-22)^2, \quad 2 - trace(p_1p_n) = \frac{6(n+4)^2(x_1+4/3)}{x_1+1}, \\
&2 - trace(p_2p_3) = 3(2n+3), \quad 2 - trace(p_2p_4) = (6n+3)^2, \\
&2 - trace(p_2p_m) = 12(m-23)^2, \quad 2 - trace(p_2p_n) = \frac{9(n+5)^2(x_1+4/3)}{x_1+1}, \\
&2 - trace(p_3p_4) = 3(2n+3), \quad 2 - trace(p_3p_m) = 4(2n+3), \\
&2 - trace(p_3p_n) = \frac{3(2n+3)(x_1+4/3)}{x_1+1}, \quad 2 - trace(p_4p_m) = 12(2n+m-22)^2, \\
&2 - trace(p_4p_n) = \frac{9(n-4)^2(x_1+4/3)}{x_1+1}, \quad 2 - trace(p_mp_{m'}) = 16(m-m')^2, \\
&2 - trace(p_mp_n) = \frac{12(m+n-16)^2(x_1+4/3)}{x_1+1}. \qquad (11.7)
\end{aligned}$$

Lemma 11.12. *For $n > 1$ we have*

$$\begin{aligned}
-c'_{n\,n-1} =& Trace(T_1T_2\dots T_n) = n - \sum_{i_1<i_2} a_{i_2i_1}^2 + \sum_{i_1<i_2<i_3} a_{i_2i_1}a_{i_3i_1}a_{i_3i_2} \\
&- \sum_{i_1<i_2<i_3<i_4} a_{i_2i_1}a_{i_3i_2}a_{i_4i_1}a_{i_4i_3} + \sum_{i_1<i_2<i_3<i_4<i_5} a_{i_2i_1}a_{i_3i_2}a_{i_4i_3}a_{i_5i_1}a_{i_5i_4} - \dots
\end{aligned}$$

Proof. This is proved by induction on n, or one can use Lemma 3.1. □

Lemma 11.13. *If we solve the equations $2 - trace(p_ip_j) = a_{ij}^2$ for the a_{ij} and substitute into the cycle $c = a_{i_2i_1}a_{i_3i_2}a_{i_4i_3}\dots a_{i_ri_{r-1}}a_{i_ri_1}$ with $i_1 < i_2 < \dots < i_r < n$, then we get an integer. If we substitute into the cycle $c = a_{i_2i_1}a_{i_3i_2}a_{i_4i_3}\dots a_{i_ri_{r-1}}a_{ni_r}a_{ni_1}$ with $i_1 < i_2 < \dots < i_r < n$, then we get an integral multiple of $\frac{x_1+4/3}{x_1+1}$.*

Proof. Since for $4 < m \neq m' < n$ we have that $2 - trace(p_mp_{m'})$ is a perfect square we may assume that in any such cycle we have at most one index m with $4 < m < n$. This reduces the proof to checking a finite number of the remaining cases, which one does. □

We will use the above two results to show that when one substitutes any solution to the equations $2 - trace(p_ip_j) = a_{ij}^2$ into $c'_{n\,n-1}$, then the result is a non-constant function of x_1. In fact:

Lemma 11.14. *If we solve the equations $2 - trace(p_ip_j) = a_{ij}^2$ for the a_{ij} and substitute into $c'_{n\,n-1}$, then we get a function of the form*

$$q_1\frac{x_1+4/3}{x_1+1} + q_2,$$

where q_1, q_2 are integers with q_1 being odd.

Proof. Of course there are many solutions to the equations $2 - trace(p_ip_j) = a_{ij}^2$, however, they all differ by various signs and since we are only interested in the parity of the integer

q_1, the specific choice of signs will not concern us. Now given Lemmas 11.12 and 11.13 the only thing we need to do is to show that q_1 is odd. There are two cases: n even or odd. We will do the n odd case, the other being similar. Assume that n is odd. Then $c'_{n\,n-1}$ is given as a sum of sums by Lemma 11.12. We look at each of these sums.

First for $\sum_{i_1<i_2} a^2_{i_2 i_1}$. Note that here we are only interested in summing over those $i_1 < i_2$ with $i_2 = n$. Now by (11.7) we see that a^2_{n1} is always even; that a^2_{n2} is even since n is odd; that a^2_{n3} is always odd; that a^2_{n4} is odd since n is odd; that a^2_{nm} is even for all $4 < m < n$. Thus $\sum_{i_1<i_2} a^2_{i_2 i_1}$ is even.

Next for $\sum_{i_1<i_2<i_3} a_{i_2 i_1} a_{i_3 i_1} a_{i_3 i_2}$ we again need only consider the cases where $i_3 = n$. Next note that if $i_1 = 1$, then $a_{i_2 i_1} a_{n i_1} a_{n i_2}$ is always even. Thus we may assume that $i_1 > 1$. Similarly, if $i_1 = 2$, then $a_{i_2 i_1} a_{n i_1} a_{n i_2}$ is even. Further, if $4 < m < n$ and $i_2 = m$, then $a_{i_2 i_1} a_{n i_1} a_{n i_2}$ is even. Thus we have reduced to the case $i_1 = 3, i_2 = 4$, and we find that this is odd. Thus $\sum_{i_1<i_2<n} a_{i_2 i_1} a_{n i_1} a_{n i_2}$ is odd.

For the case $\sum_{i_1<i_2<i_3<i_4} a_{i_2 i_1} a_{i_3 i_2} a_{i_4 i_1} a_{i_4 i_3}$ we again have $i_4 = n$ and as above we must have $i_1 > 2$. But this forces $i_3 > 4$ which gives an even number also.

The rest of the cases are similar to the last one. This proves the Lemma and the last part of Theorem 4. $\square$

Let $\pi : B_3/Z(B_3) \to PSL_2(\mathbb{Z})$ be the isomorphism so that

$$\pi(\sigma_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \pi(\sigma_2) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

with these matrices acting as linear fractional transformations of the upper half plane $\mathbb{H}^2$. We now return to the situation of Theorem 11.8 and Corollary 11.10. These show that for all $t < 0$ there is a diffeomorphism $f_t : V_t \cap \mathbb{R}^3_{>2} \to \mathbb{H}^2$ such that for all $\alpha \in B_3/Z(B_3)$ we have $f_t(\mathcal{F}_t) = \mathcal{F}'_t$ and

$$f_t(\alpha(x, y, z)) = \pi(\alpha) f_t(x, y, z), \quad \text{for all} \quad (x, y, z) \in V_t \cap \mathbb{R}^3_{>2}.$$

Now let $g : \mathbb{H}^2 \to \mathbb{C}$ be a modular form of weight k, so that $g(\beta(z)) = (cz + d)^k g(z)$ for all $z \in \mathbb{H}^2$ and $\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{Z})$ [Ko]. Then we can define a *modular form* on $\bigcup_{t<0} V_t \cap \mathbb{R}^3_{>2}$ by

$$\bar{g}(x, y, z) = g(f_t(x, y, z)), \quad \text{for} \quad (x, y, z) \in V_t \cap \mathbb{R}^3_{>2}.$$

Then we have

Theorem 11.15. *For $\alpha \in B_3/Z(B_3)$, $(x, y, z) \in V_t \cap \mathbb{R}^3_{>2}$ and a modular form g of weight k we have*

$$\bar{g}(\alpha(x, y, z)) = (cz + d)^k \bar{g}(x, y, z), \quad \textit{where} \quad \pi(\alpha) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Proof. For $(x, y, z) \in V_t \cap \mathbb{R}^3_{>2}$ we have:

$$\begin{aligned} \bar{g}(\alpha(x, y, z)) &= g(f_t \alpha(x, y, z)) \\ &= g(\pi(\alpha) f_t(x, y, z)) \\ &= (cz + d)^k g(f_t(x, y, z)) \\ &= (cz + d)^k \bar{g}(x, y, z), \end{aligned}$$

where $\pi(\alpha) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. $\square$

Since $\mathcal{T}_{0,3,1}$ can be thought of as a subset of $\bigcup_{t<0} V_t \cap \mathbb{R}^3_{>2}$ we see that the above result gives a way of defining modular forms on $\mathcal{T}_{0,3,1}$. It would be nice to have an explicit formula for the functions f_t.

Acknowledgement. The author wishes to thank Jim Cannon, Tyler Jarvis, Ken Johnson and Eric Swenson for useful conversations regarding this paper.

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