On the location of zeros of the Riemann zeta-function

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Abstract

In this paper a new S-local formula is given for the Weil distribution, which is shown to be the sum of nonnegative traces of a non-positive trace class Hilbert-Schmidt operator on two orthogonal subspaces of a L^2 space. One of the two traces can be written in two different forms. This flexibility, the new S-local formula, and the two nonnegative traces all together give the desired positivity for the Weil distribution. Finally, Weil's criterion implies that all nontrivial zeros of the Riemann zeta-function lie on the critical line.

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Key Words: Fourier transform, Hilbert-Schmidt operator, trace, Weil's positivity, zeta zeros

1 Introduction

The Riemann zeta function ζ is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\Re s > 1$. It extends to an analytic function in the whole complex plane except for having a simple pole at $s = 1$. Trivially, $\zeta(-2n) = 0$ for all positive integers n. All other zeros of the Riemann zeta function are called its nontrivial zeros.

In connection with investigating the frequency of prime numbers, B. Riemann conjectured in 1859 [10] that all nontrivial zeros of ζ have real part equal to 1/2. In this paper, we confirm this conjecture.

To avoid the complication of writings, the author only considered the the Riemann zeta function in this paper. He will consider Dirichlet L-functions in a separate paper.

Meanings of all notations in this paper are given by their first appearances except for obvious exceptions. Let Q denote the field of rational numbers. For each prime p we denote by $|\,|_p$ the p-adic valuation of Q normalized so that $|\,|_p$ is the ordinary absolute value if p is the infinity prime and $|p|_p = p^{-1}$ if p is a rational prime. Q_p is the p-adic completion of Q.

For the rational number field, the Weil distribution $\Delta(h)$ [15, p. 18] is given by

$$\Delta(h) = \widehat{h}(0) + \widehat{h}(1) - \sum_p \int_{Q_p^*}' \frac{h(|u|_p^{-1})}{|1-u|_p} d^*u, \tag{1.1}$$

where the sum on p is over all primes of Q including the infinity prime. For $p \neq \infty$,

$$\int_{Q_p^*}' \frac{h(|u|_p^{-1})}{|1-u|_p} d^*u = \sum_{m=1}^{\infty} \log p \left[h(p^m) + p^{-m} h(p^{-m}) \right].$$

If p is the infinity prime of Q, then

$$\int_{\mathbb{R}^*}' \frac{h(|u|^{-1})}{|1-u|} d^*u = (\gamma + \log(2\pi))h(1) + \lim_{\epsilon \to 0} \left(\int_{|1-u| \geqslant \epsilon} \frac{h(|u|^{-1})}{|1-u|} d^*u + h(1) \log \epsilon \right)$$

with γ being Euler's constant. Weil's explicit formula reads

$$\sum_{\rho} \widehat{h}(\rho) = \widehat{h}(0) + \widehat{h}(1) - \sum_{p} \int_{Q_p^*}' \frac{h(|u|_p^{-1})}{|1-u|_p} d^*u, \tag{1.2}$$

where the sum on ρ is over all nontrivial zeros of the Riemann zeta function and

$$\widehat{h}(s) = \int_0^{\infty} h(t) t^{s-1} dt. \tag{1.3}$$

In Section 2 we will prove the following theorem.

Theorem 1.1 *Let*

$$\lambda_n = \sum_{\rho} [1 - (1 - \frac{1}{\rho})^n]$$

where the sum is over all nontrivial zeros of $\zeta(s)$ with ρ and $1-\rho$ being paired together. Then there exist a family of real-valued smooth functions $g_\epsilon(t)$ given in (2.16) on $(0, \infty)$ such that $\widehat{g}_\epsilon(0) = 0$, $g_\epsilon(t) = 0$ for $t \notin (\mu_\epsilon^{-1}, 1)$ with $\mu_\epsilon = (1+\epsilon)/\epsilon^2$ and such that

$$\lim_{\epsilon \to 0+} \Delta(h_{n,\epsilon}) = 2\lambda_n$$

where

$$h_{n,\epsilon}(x) = \int_0^{\infty} g_\epsilon(xy) g_\epsilon(y) dy. \tag{1.4}$$

In Section 3 we derive following new formula for the Weil distribution.

Theorem 1.2 *Let $\mathfrak{F}_S g$ be the Fourier transform of g on the S-local adele group $\mathbb{A}_S$, and let*

$$g(t) = t^{-1} g_\epsilon(t^{-1}) = J g_\epsilon(t), \; h(x) = \int_0^{\infty} g(xt) g(t) dt. \tag{1.5}$$

Then

$$\Delta(h) = \int_0^{\infty} |g(t)|^2 \log t \, dt + \int_0^{\infty} |\mathfrak{F}_S g(t)|^2 \log t \, dt. \tag{1.6}$$

In Section 4 we collect some preliminary results which will be used in later sections.

Let S be given as in (2.20), $O_S^* = \{\xi \in Q^* : |\xi|_p = 1, p \notin S\}$. Note that $|\xi|_S = 1$ for all $\xi \in O_S^*$. We denote $S' = S - \{\infty\}$, $\mathbb{A}_S = \mathbb{R} \times \prod_{p\in S'} Q_p$, $J_S = \mathbb{R}^\times \times \prod_{p\in S'} Q_p^*$, $O_p = \{x \in Q_p : |x|_p \leqslant 1\}$, and $C_S = J_S/O_S^*$. For $X_S = \mathbb{A}_S/O_S^*$, we define $L^2(X_S)$ as in [4, (5), p. 54] to be the Hilbert space that is the completion of the Schwartz-Bruhat space $S(\mathbb{A}_S)$ [3, 16] for the inner product given by

$$\langle f, g\rangle_{L^2(X_S)} = \int_{C_S} E_S(f)(x)\overline{E_S(g)(x)}d^\times x$$

for $f, g \in S(\mathbb{A}_S)$, where

$$E_S(f)(x) = \sqrt{|x|}\sum_{\xi\in O_S^*} f(\xi x). \tag{1.7}$$

Let $L_1^2(X_S)$ be the subspace of $L^2(X_S)$ spanned by the set $S_e(\mathbb{R}) \times \prod_{p\in S'} 1_{O_p}$, where $S_e(\mathbb{R})$ consists of all even functions in $S(\mathbb{R})$. Let Q_Λ be the subspace of all functions f in $L_1^2(X_S)$ such that $\mathfrak{F}_S f(x) = 0$ for $|x| < \Lambda$. Then

$$L_1^2(X_S) = Q_\Lambda^\perp \oplus Q_\Lambda. \tag{1.8}$$

We define

$$V_S(h)F(x) = \int_{C_S} h(x/\lambda)\sqrt{|x/\lambda|}\, F(\lambda)d^\times\lambda \tag{1.9}$$

for $F \in L^2(C_S)$. Let

$$T_\ell = V_S(h)\left(S_\Lambda - E_S\mathfrak{F}_S^t P_\Lambda \mathfrak{F}_S E_S^{-1}\right), \tag{1.10}$$

where $P_\Lambda(x) = 1$ if $|x| < \Lambda$ and 0 if $|x| \geqslant \Lambda$ and $S_\Lambda(x) = 1$ if $|x| > \Lambda^{-1}$ and 0 if $|x| \leqslant \Lambda^{-1}$.

In Section 5, we compute the trace of T_ℓ on the subspace $E_S(Q_\Lambda^\perp)$ of $L_1^2(C_S)$ in two ways and obtained the following two theorems.

Theorem 1.3 *We have*

$$trace_{E_S(Q_\Lambda^\perp)_1}(T_\ell) = \int_0^1 |\mathfrak{F}_S g(t)|^2 \log t\, dt + \int_0^1 \mathfrak{F}_S g(t)\mathfrak{F}_S\{g(u)\log|u|\}(t)dt$$
$$- \int_{\frac{1}{\mu_\epsilon}\leqslant t<1} \mathfrak{F}_S g(t)dt \int_{\mathbb{A}_S, \frac{1}{t}\leqslant|u|<\mu_\epsilon} g(u)\log|ut|\Psi_S(-ut)du.$$

Theorem 1.4 *We have*

$$trace_{E_S(Q_\Lambda^\perp)_1}(T_\ell) = 0.$$

In section 6, we compute the trace of T_ℓ on the subspace $E_S(Q_\Lambda)$ of $L_1^2(C_S)$ and prove its positivity. In particular, we proved the following two theorems.

Theorem 1.5 *We have*

$$trace_{E_S(Q_\Lambda)_1}(T_\ell) = \int_1^\infty |\mathfrak{F}_S g(t)|^2 \log t\, dt + \int_1^\infty \mathfrak{F}_S g(t)\mathfrak{F}_S\{g(u)\log|u|\}(t)dt$$
$$+ \int_{\frac{1}{\mu_\epsilon}}^1 \mathfrak{F}_S g(t)dt \int_{\mathbb{A}_S, \frac{1}{t}\leqslant|z|<\mu_\epsilon} g(z)\log|zt|\Psi_S(-zt)dz.$$

Theorem 1.6 *We have*

$$trace_{E_S(Q_\Lambda)_1}(T_\ell) \geqslant 0.$$

To bypass a technical difficulty in obtaining Theorem 1.6, we decompose T_ℓ as a sum of two trace class Hilbert-Schmidt operators. One of them has trace 0. The other one has nonnegative trace by the positivity of the convolution operator $V_S(h)$.

Finally, we prove the following Riemann hypothesis in Section 7.

Theorem 1.7 *All nontrivial zeros of the Riemann zeta function lie on the line* $\Re s = 1/2$.

2 Adjustment of $\widehat{h}(0), \widehat{h}(1)$ to 0 and proof of Theorem 1.1

Briefly speaking, we add in this section a function of the form $\sum_{n=1}^{\infty} f(nx)$ to g so that the new g remains compactly supported and the value of the new $\Delta(h)$ remains almost the same while $\widehat{h}(0)$ and $\widehat{h}(1)$ become zero for the new h.

Lemma 2.1 *([8, Theorem 1, p. 326]) A necessary and sufficient condition for the nontrivial zeros of* $\zeta(s)$ *to lie on the critical line is* $\lambda_n \geqslant 0$ *for* $n = 1, 2, \cdots$.

Lemma 2.2 *For each positive integer* n *and a sufficiently small* $\epsilon > 0$*, there exist a smooth function* $\ell_{n,\epsilon}(x)$ *on* $(0, \infty)$ *with* $\ell_{n,\epsilon}(x) = 0$ *for* $x \notin (\frac{\epsilon}{1+\epsilon}, 1)$ *and satisfying that*

$$\lim_{\epsilon \to 0+} \sum_{\rho} \widehat{\ell}_{n,\epsilon}(\rho) \widehat{\ell}_{n,\epsilon}(1-\rho) = 2\lambda_n.$$

Proof. Let

$$P_n(t) = \sum_{j=1}^{n} \binom{n}{j} \frac{t^{j-1}}{(j-1)!}$$

and

$$g_n(x) = \begin{cases} P_n(\log x) & \text{if } 0 < x < 1 \\ n & \text{if } x = 1 \\ 0 & \text{if } x > 1. \end{cases}$$

Then [1, Lemma 2, p. 282]

$$\widehat{g}_n(s) = 1 - \left(1 - \frac{1}{s}\right)^n$$

for $n = 1, 2, \cdots$.

For $0 < \epsilon < 1$ we replace $g_n(x)$ by the function

$$g_{n,\epsilon}(x) = \begin{cases} 0 & \text{if } 1-\epsilon < x < \infty \\ \frac{1}{2} g_n(1-\epsilon) & \text{if } x = 1-\epsilon \\ g_n(x) & \text{if } \epsilon < x < 1-\epsilon \\ \frac{1}{2} g_n(\epsilon) & \text{if } x = \epsilon \\ 0 & \text{if } x < \epsilon. \end{cases} \tag{2.1}$$

Let

$$\tau(x) = \begin{cases} \frac{c_0}{\epsilon} \exp\left(-1/[1-(\frac{x-1}{\epsilon})^2]\right) & \text{if } |x-1| < \epsilon, \\ 0 & \text{if } |x-1| \geqslant \epsilon. \end{cases} \tag{2.2}$$

where $c_0^{-1} = \int_{-1}^{1} e^{\frac{1}{x^2-1}} dx$ and $\int_0^\infty \tau(x)dx = 1$.

We define

$$\ell_{n,\epsilon}(x) = \int_0^\infty g_{n,\epsilon}(xy)\tau(y)dy. \tag{2.3}$$

Then $\ell_{n,\epsilon}(x)$ is a smooth function on $\mathbb{R}$ whose support is contained in the interval $(\frac{\epsilon}{1+\epsilon}, 1)$. Since

$$\widehat{\ell}_{n,\epsilon}(1-s) = \widehat{g}_{n,\epsilon}(1-s)\widehat{\tau}(s) \tag{2.4}$$

with

$$\widehat{\tau}(s) = c_0 \int_{-1}^{1} \exp(\frac{1}{u^2-1})(1+\epsilon u)^{s-1} du,$$

we have

$$\begin{aligned} &\widehat{\ell}_{n,\epsilon}(1-s)\widehat{\ell}_{n,\epsilon}(s) - \widehat{g}_{n,\epsilon}(1-s)\widehat{g}_{n,\epsilon}(s) \\ &= \widehat{g}_{n,\epsilon}(1-s)\widehat{g}_{n,\epsilon}(s)\left(\{\widehat{\tau}(s)(\widehat{\tau}(1-s)-1) + (\widehat{\tau}(s)-1)\}\right). \end{aligned} \tag{2.5}$$

By partial integration,

$$\begin{aligned} \widehat{g}_{n,\epsilon}(s) &= \int_0^1 g_n(x)x^{s-1} - \int_0^\epsilon g_n(x)x^{s-1}dx - \int_{1-\epsilon}^1 g_n(x)x^{s-1}dx \\ &= 1-(1-\frac{1}{s})^n - P_n(\log\epsilon)\frac{\epsilon^s}{s} + O\left(\frac{\epsilon^{\Re s}}{|s|^2}|\log\epsilon|^{n-2}\right) - \frac{a_\epsilon(s)}{s} \\ &= O\left(\frac{1}{|s|} + |\log\epsilon|^{n-1}\frac{\epsilon^{\Re s}}{|s|}\right) \end{aligned} \tag{2.6}$$

for $0 < \Re s < 1$ and $|s| \geqslant 1$, where

$$a_\epsilon(s) = n - P_n(\log(1-\epsilon))(1-\epsilon)^s - \int_{1-\epsilon}^1 P_n'(\log x)x^{s-1}dx.$$

The proof of [1, (3.9), p. 284] shows that

$$\max_\rho \epsilon^{\Re\rho}|\rho|^{-1/2} = O\left(e^{-c'\sqrt{|\log\epsilon|}}\right) \tag{2.7}$$

for some constant $c' > 0$. For $0 < \Re s < 1$,

$$1-\widehat{\tau}(s) = c_0\int_{-1}^1 e^{\frac{1}{t^2-1}}\left[1-(1+t\epsilon)^{s-1}\right]dt \le c_0\int_{-1}^1 e^{\frac{1}{t^2-1}}(1+\frac{1}{1-\epsilon})dt \ll 1. \tag{2.8}$$

By (2.4), (2.5), (2.6) and (2.8),

$$\sum_\rho \widehat{g}_{n,\epsilon}(1-\rho)\widehat{g}_{n,\epsilon}(\rho)\left(\{\widehat{\tau}(\rho)(\widehat{\tau}(1-\rho)-1)+(\widehat{\tau}(\rho)-1)\}\right.$$
$$\ll \sum_\rho \left(\frac{1}{|\rho|}+|\log\epsilon|^{n-1}\frac{\epsilon^{\Re\rho}}{|\rho|}\right)\left(\frac{1}{|1-\rho|}+|\log\epsilon|^{n-1}\frac{\epsilon^{1-\Re\rho}}{|1-\rho|}\right)$$
$$\times \max\left(|\int_{-1}^{1} e^{\frac{1}{t^2-1}}\left[1-(1+t\epsilon)^{\rho-1}\right]dt|, |\int_{-1}^{1} e^{\frac{1}{t^2-1}}\left[1-(1+t\epsilon)^{-\rho}\right]dt|\right).$$

From (2.7) we deduce that

$$\left(\frac{1}{|\rho|}+|\log\epsilon|^{n-1}\frac{\epsilon^{\Re\rho}}{|\rho|}\right)\left(\frac{1}{|1-\rho|}+|\log\epsilon|^{n-1}\frac{\epsilon^{1-\Re\rho}}{|1-\rho|}\right)$$
$$=\frac{1}{|\rho(1-\rho)|}\left\{1+|\log\epsilon|^{n-1}(\epsilon^{\Re\rho}+\epsilon^{1-\Re\rho})+|\log\epsilon|^{2n-2}\epsilon\right\}\ll|\rho|^{-3/2}.$$

It follows that

$$\sum_\rho \widehat{g}_{n,\epsilon}(1-\rho)\widehat{g}_{n,\epsilon}(\rho)\left(\{\widehat{\tau}(\rho)(\widehat{\tau}(1-\rho)-1)+(\widehat{\tau}(\rho)-1)\}\right.$$
$$\ll \sum_\rho \frac{1}{|\rho|^{\frac{3}{2}}}\max(|\int_{-1}^{1} e^{\frac{1}{t^2-1}}\left[1-(1+t\epsilon)^{\rho-1}\right]dt|, |\int_{-1}^{1} e^{\frac{1}{t^2-1}}\left[1-(1+t\epsilon)^{-\rho}\right]dt|).$$

For any $\epsilon_0>0$, there exists a positive k_0 such that

$$\sum_{|\rho|\geqslant k_0}|\rho|^{-3/2}<\epsilon_0/2.$$

Since

$$\lim_{\epsilon\to0}\sum_{|\rho|<k_0}\frac{1}{|\rho|^{\frac{3}{2}}}\max(|\int_{-1}^{1} e^{\frac{1}{t^2-1}}\left[1-(1+t\epsilon)^{\rho-1}\right]dt|, |\int_{-1}^{1} e^{\frac{1}{t^2-1}}\left[1-(1+t\epsilon)^{-\rho}\right]dt|)=0,$$

there exists a ϵ_1 with $0<\epsilon_1<\epsilon_0$ such that

$$|\sum_{|\rho|<k_0}\frac{1}{|\rho|^{\frac{3}{2}}}\max(|\int_{-1}^{1} e^{\frac{1}{t^2-1}}\left[1-(1+t\epsilon_1)^{\rho-1}\right]dt|, |\int_{-1}^{1} e^{\frac{1}{t^2-1}}\left[1-(1+t\epsilon_1)^{-\rho}\right]dt|)|<\frac{\epsilon_0}{2}.$$

Thus, we have proved that for any $\epsilon_0>0$ there exists a $0<\epsilon_1<\epsilon_0$ satisfying that

$$\sum_\rho\frac{1}{|\rho|^{\frac{3}{2}}}\max(|\int_{-1}^{1} e^{\frac{1}{t^2-1}}\left[1-(1+t\epsilon_1)^{\rho-1}\right]dt|, |\int_{-1}^{1} e^{\frac{1}{t^2-1}}\left[1-(1+t\epsilon_1)^{-\rho}\right]dt|)<\epsilon_0.$$

It follows that

$$\lim_{\epsilon\to0+}\sum_\rho \widehat{g}_{n,\epsilon}(1-\rho)\widehat{g}_{n,\epsilon}(\rho)\left(\{\widehat{\tau}(\rho)(\widehat{\tau}(1-\rho)-1)+(\widehat{\tau}(\rho)-1)\}=0.\right.$$

We deduce from (2.5) that

$$\lim_{\epsilon\to 0+}\sum_\rho \widehat{\ell}_{n,\epsilon}(\rho)\widehat{\ell}_{n,\epsilon}(1-\rho)=\lim_{\epsilon\to 0+}\sum_\rho \widehat{g}_{n,\epsilon}(\rho)\widehat{g}_{n,\epsilon}(1-\rho). \tag{2.9}$$

We can write

$$\begin{aligned}
&\widehat{g}_n(s)\widehat{g}_n(1-s)-\widehat{g}_{n,\epsilon}(s)\widehat{g}_{n,\epsilon}(1-s)\\
&=[\widehat{g}_n(s)-\widehat{g}_{n,\epsilon}(s)]\widehat{g}_n(1-s)+\widehat{g}_{n,\epsilon}(s)[\widehat{g}_n(1-s)-\widehat{g}_{n,\epsilon}(1-s)]\\
&=\widehat{g}_n(1-s)[P_n(\log\epsilon)\frac{\epsilon^s}{s}+O\left(\frac{\epsilon^{\Re s}}{|s|^2}|\log\epsilon|^{n-2}\right)+\frac{a_\epsilon(s)}{s}] \qquad (2.10)\\
&+\widehat{g}_{n,\epsilon}(s)[P_n(\log\epsilon)\frac{\epsilon^{1-s}}{1-s}+O\left(\frac{\epsilon^{1-\Re s}}{|1-s|^2}|\log\epsilon|^{n-2}\right)+\frac{a_\epsilon(1-s)}{1-s}]\\
&\ll\frac{1}{|s(1-s)|}[|\log\epsilon|^{n-1}\epsilon^{\Re s}+|a_\epsilon(s)|+\frac{\epsilon^{\Re s}}{|s|}|\log\epsilon|^{n-2}]\\
&+\frac{1}{|s(1-s)|}\left(1+|\log\epsilon|^{n-1}\epsilon^{\Re s}\right)[|\log\epsilon|^{n-1}\epsilon^{1-\Re s}+|a_\epsilon(1-s)|+\frac{\epsilon^{1-\Re s}}{|1-s|}|\log\epsilon|^{n-2}].
\end{aligned}$$

There exists a constant c_n such that $|a_\epsilon(s)|\leqslant c_n$ for all s inside the strip $0\leqslant\Re s\leqslant 1$. For each fixed s, we have $a_\epsilon(s)\to 0$ as $\epsilon\to 0+$. An argument similar to that made in the paragraph containing (2.9) shows that

$$\lim_{\epsilon\to 0+}\sum_\rho\frac{|a_\epsilon(\rho)+|a_\epsilon(1-\rho)|}{|\rho(1-\rho)|}=0. \tag{2.11}$$

Thus, from (2.7), (2.10) and (2.11) we derive that

$$\lim_{\epsilon\to 0+}\sum_\rho[\widehat{g}_n(\rho)\widehat{g}_n(1-\rho)-\widehat{g}_{n,\epsilon}(\rho)\widehat{g}_{n,\epsilon}(1-\rho)]=0.$$

The stated identity then follows from (2.9).

This completes the proof of the lemma. □

Proof of Theorem 1.1. Let $a(t)=1/t(t-1)$,

$$a_1=\int_0^1 e^{a(t)}dt/\{\left(\int_0^1 e^{a(t)}dt\right)^2-\left(\int_0^1\frac{1}{t}e^{a(t)}dt\right)\left(\int_0^1 te^{a(t)}dt\right)\},$$

$$a_2=-a_1\int_0^1 te^{a(t)}dt/\int_0^1 e^{a(t)}dt,$$

and

$$\alpha(t)=\begin{cases}(a_1t+a_2)e^{a(t)} & \text{if } 0<t<1,\\ 0 & \text{if } t\leqslant 0 \text{ or } 1\leqslant t.\end{cases}$$

Then

$$\int_0^\infty \alpha(t)dt = 0 \quad \text{and} \quad \int_0^\infty \alpha(t)\frac{dt}{t} = 1. \tag{2.12}$$

If we denote

$$\vartheta(t) = \sum_{n=1}^\infty (-1)^{n-1}\alpha(nt) = \sum_{n=1}^\infty \alpha(nt) - 2\sum_{n=1}^\infty \alpha(n2t), \tag{2.13}$$

by the Poisson summation formula

$$\vartheta(t) = \frac{1}{t}\sum_{n\neq 0} \mathfrak{F}\alpha(\frac{n}{t}) - \frac{1}{t}\sum_{n\neq 0} \mathfrak{F}\alpha(\frac{n}{2t}). \tag{2.14}$$

This implies that $\vartheta(t)$ is of rapid decay when $t \to 0, \infty$. It follows that $\widehat{\vartheta}(s)$ is an entire function. Since

$$\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s)$$

for $\Re s > 0$, by analytic extension we have

$$\widehat{\vartheta}(s) = (1-2^{1-s})\zeta(s)\widehat{\alpha}(s) \tag{2.15}$$

for complex s. By (2.15) and (2.12), we have

$$\widehat{\vartheta}(0) = \frac{1}{2} \text{ and } \widehat{\vartheta}(1) = 0.$$

Let

$$g_\epsilon(x) = \ell_{n,\epsilon}(x) - \frac{1}{\widehat{\vartheta}_1(0)}\int_0^\infty \ell_{n,\epsilon}(x/u)\vartheta_1(u)\frac{du}{u} \tag{2.16}$$

and

$$h_{n,\epsilon}(x) = \int_0^\infty g_\epsilon(xy)g_\epsilon(y)dy,$$

where

$$\vartheta_1(x) = \begin{cases} \vartheta(x) & \text{if } x > \epsilon, \\ \frac{1}{2}\vartheta(\epsilon) & \text{if } x = \epsilon, \\ 0 & \text{if } x < \epsilon. \end{cases}$$

Since $\widehat{\vartheta}(\rho) = 0$ for nontrivial zeros ρ of $\zeta(s)$, we have

$$\begin{aligned}\widehat{h}_{n,\epsilon}(\rho) =& \widehat{\ell}_{n,\epsilon}(\rho)\left(1 - \frac{1}{\widehat{\vartheta}_1(0)}[\widehat{\vartheta}(\rho)) - \int_0^\epsilon \vartheta(x)x^{\rho-1}dx]\right) \\ &\times \widehat{\ell}_{n,\epsilon}(1-\rho)\left(1 - \frac{1}{\widehat{\vartheta}_1(0)}[\widehat{\vartheta}(1-\rho)) - \int_0^\epsilon \vartheta(x)x^{-\rho}dx]\right) \\ =&\widehat{\ell}_{n,\epsilon}(\rho)\widehat{\ell}_{n,\epsilon}(1-\rho)\left(1 + \frac{1}{\widehat{\vartheta}_1(0)}\int_0^\epsilon \vartheta(x)x^{\rho-1}dx\right)\left(1 + \frac{1}{\widehat{\vartheta}_1(0)}\int_0^\epsilon \vartheta(x)x^{-\rho}dx\right).\end{aligned}$$

Let $h_0(x) = \int_0^\infty \ell_{n,\epsilon}(xy)\ell_{n,\epsilon}(y)dy$. Then

$$\widehat{h}_{n,\epsilon}(\rho) - \widehat{h}_0(\rho) = \widehat{\ell}_{n,\epsilon}(\rho)\widehat{\ell}_{n,\epsilon}(1-\rho)\frac{1}{\widehat{\vartheta}_1(0)}\{\int_0^\epsilon \vartheta(x)x^{\rho-1}dx + \int_0^\epsilon \vartheta(x)x^{-\rho}dx \\ + \frac{1}{\widehat{\vartheta}_1(0)}\int_0^\epsilon \vartheta(x)x^{\rho-1}dx\int_0^\epsilon \vartheta(x)x^{-\rho}dx\} \tag{2.17}$$

Since both $x\alpha'(x)$ and its Fourier transform vanish at $x=0$, by the Poisson summation

$$x\vartheta'(x) = \sum_{n=1}^\infty nx\alpha'(nx) - 2\sum_{n=1}^\infty n2x\alpha'(n2x) \\ = \frac{1}{x}\sum_{n\neq 0}\mathfrak{F}(u\alpha'(u))(\frac{n}{x}) - \frac{1}{x}\sum_{n\neq 0}\mathfrak{F}(u\alpha'(u))(\frac{n}{2x}).$$

This implies that $\vartheta'(x)$ is of rapid decay when $x\to 0$. Since $\vartheta(x)$ is also of rapid decay when $x\to 0$, we have

$$\max\{|\vartheta(x)|, |\vartheta'(x)|\} \ll |x|^n$$

for any positive integer n as $x\to 0+$. By partial integration,

$$\int_0^\epsilon \vartheta(x)x^{-s}dx = \frac{\vartheta(\epsilon)}{1-s} + \frac{1}{s-1}\int_0^\epsilon \vartheta'(x)x^{1-s}dx < \frac{c\epsilon}{|s|} \tag{2.18}$$

for $0 < \Re s < 1$ and $|s| > 2$, where c is an absolute constant independent of s.

By (2.4) and (2.6) we have

$$\widehat{\ell}_{n,\epsilon}(s) \ll \frac{1}{|s|} + |\log\epsilon|^{n-1}\frac{\epsilon^{\Re s}}{|s|} \ll \frac{|\log\epsilon|^{n-1}}{|s|} \tag{2.19}$$

for $0 < \Re s < 1$, where the implied constant depends only on n.

From (2.17), (2.18) and (2.19) we derive that

$$\sum_\rho \left(\widehat{h}_{n,\epsilon}(\rho) - \widehat{h}_0(\rho)\right) \ll \epsilon|\log\epsilon|^{2n-2}\sum_\rho \frac{1}{|\rho|^3} \to 0$$

as $\epsilon\to 0+$. That is,

$$\lim_{\epsilon\to 0+}(\Delta(h_0) - \Delta(h_{n,\epsilon})) = 0.$$

By (2.9),

$$\lim_{\epsilon\to 0+}\Delta(h_{n,\epsilon}) = 2\lambda_n.$$

This completes the proof of Theorem 1.1. □

We denote $\mu_\epsilon = \frac{1+\epsilon}{\epsilon^2}$. Since $g_\epsilon(t) = 0$ for $t \notin (\frac{\epsilon^2}{1+\epsilon}, 1)$, we have $g_\epsilon(t) = 0$ for $t \notin (\mu_\epsilon^{-1}, 1)$ and $h_{n,\epsilon}(x) = 0$ for $x \notin (\mu_\epsilon^{-1}, \mu_\epsilon)$. We also define $g_\epsilon(t) = g_\epsilon(-t)$ for $t < 0$. From now on we choose

$$S = \{\infty, \text{ all primes } p \leq X\} \tag{2.20}$$

for some $X > \mu_\epsilon$.

By (2.16), $\widehat{g}_\epsilon(0) = 0$. It follows that

$$\widehat{h}_{n,\epsilon}(0) = \widehat{h}_{n,\epsilon}(1) = 0. \tag{2.21}$$

3 A formula for the Weil distribution and proof of Theorem 1.2

In this section we derive a new formula for the Weil distribution $\Delta(h)$.

Let $\mathbb{N}_S$ be the set consisting of 1 and all positive integers which are products of powers of primes in S'. ψ_p is the character on the additive group Q_p given as in [13] and $\psi_\infty(x) = \exp(-2\pi i x)$.

The Fourier transform of $f \in L^2(\mathbb{R})$ is

$$\mathfrak{F}f(x) = \int_{-\infty}^{\infty} f(t)e^{-2\pi i x t}dt.$$

For $f \in L^2(Q_p)$ its Fourier transform is defined by

$$\mathfrak{F}_p f(\beta) = \int_{Q_p} f(\alpha)\psi_p(-\alpha\beta)d\alpha$$

for $\beta \in Q_p$.

Let $\psi_S = \prod_{p\in S}\psi_p$. When $f = \prod_{p\in S} f_p \in L^2(\mathbb{A}_S)$ we define

$$\mathfrak{F}_S f(\beta) = \int_{\mathbb{A}_S} f(\alpha)\psi_S(-\alpha\beta)d\alpha.$$

Then $\mathfrak{F}_S f = \prod_{p\in S}\mathfrak{F}_p f_p$. As $S(\mathbb{A}_S)$ is dense in $L^2(\mathbb{A}_S)$, the definition of $\mathfrak{F}_S$ can be extended to all functions in $L^2(\mathbb{A}_S)$.

Lemma 3.1 *We can write*

$$\begin{aligned}\mathfrak{F}_S g(t) &= \sum_{k,l\in\mathbb{N}_S} \frac{\mu(k)}{k}\mathfrak{F}g(\frac{lt}{k}) \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-s}\widehat{\mathfrak{F}g}(s)\prod_{p\in S'}\frac{1-p^{s-1}}{1-p^{-s}}ds \end{aligned} \tag{3.1}$$

for $c > 0$, where $\widehat{\mathfrak{F}g}(s) = \widehat{g}(1-s)\chi(1-s)$ with $\chi(s) = 2^s\pi^{s-1}\Gamma(1-s)\sin\frac{\pi s}{2}$ and

$$\mathfrak{F}g(t) = 2\int_0^\infty g(y)\cos(2\pi t y)dy.$$

We have

$$\int_0^\infty |\mathfrak{F}_S g(t)|^2 dt = \int_0^\infty |g(t)|^2 dt \tag{3.2}$$

and

$$\int_0^\infty \mathfrak{F}_S f(t)\overline{\mathfrak{F}_S g(t)}dt = \int_0^\infty f(t)\overline{g(t)}dt. \tag{3.3}$$

Moreover, if g is a real-valued function so is $\mathfrak{F}_S g$.

Proof. As $|\gamma|=1$ for $\gamma\in O_S^*$, by [6, (3.3), p. 2468] we obtain the following formula, which also shows that if g is a real-valued function so is $\mathfrak{F}_S g$,

$$\begin{aligned}
\mathfrak{F}_S g(t) &= \int_{\mathbb{A}_S} g(y)\Psi_S(-y(|t|,1,\cdots,1))dy \\
&= \sum_{\gamma\in O_S^*}\int_{\gamma I_S} g(y)\Psi_S(-y(|t|,1,\cdots,1))dy \\
&= \sum_{\gamma\in O_S^*}\varpi(\gamma)\int_0^\infty g(y)e^{2\pi i|t|y\gamma}dy \\
&= 2\sum_{0<\gamma\in O_S^*}\varpi(\gamma)\int_0^\infty g(y)\cos(2\pi y|t|\gamma)dy
\end{aligned}$$

with

$$\varpi(\gamma)=\prod_{p\in S'}\begin{cases}1-p^{-1} & \text{if } |\gamma|_p\leqslant 1,\\ -p^{-1} & \text{if } |\gamma|_p=p,\\ 0 & \text{if } |\gamma|_p>p.\end{cases}\tag{3.4}$$

That is,

$$\begin{aligned}
\mathfrak{F}_S g(t) &= \sum_{k,l\in\mathbb{N}_S,(k,l)=1}\frac{\mu(k)}{k}\prod_{p\nmid k}(1-\frac{1}{p})\mathfrak{F}g(\frac{l|t|}{k}) \\
&= \prod_{p\in\mathbb{N}_S}(1-\frac{1}{p})\sum_{k,l\in\mathbb{N}_S,(k,l)=1}\frac{\mu(k)}{k}\prod_{p|k}\frac{1}{1-p}\mathfrak{F}g(\frac{l|t|}{k}) = \sum_{k,l\in\mathbb{N}_S}\frac{\mu(k)}{k}\mathfrak{F}g(\frac{l|t|}{k}).
\end{aligned}$$

Thus, for $t>0$ we have

$$\mathfrak{F}_S g(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}t^{-s}\widehat{\mathfrak{F}g}(s)\prod_{p\in S'}\frac{1-p^{s-1}}{1-p^{-s}}ds$$

for $c>0$. By Plancherel's Theorem [12, Theorem 1.1, p. 208],

$$\int_0^\infty|\mathfrak{F}_S g(t)|^2dt=\int_{-\infty}^\infty|\widehat{\mathfrak{F}_S g}(s)|^2du=\int_{-\infty}^\infty|\widehat{\mathfrak{F}g}(s)|^2du=\int_0^\infty|\mathfrak{F}g(x)|^2dx=\int_0^\infty|g(t)|^2dt$$

where $s=1/2+2\pi iu$. It follows that

$$\int_0^\infty\mathfrak{F}_S f(t)\overline{\mathfrak{F}_S g(t)}dt=\int_0^\infty f(t)\overline{g(t)}dt.$$

This completes the proof of the lemma. □

Lemma 3.2 *([7, Theorem 3.1, p. 796]) We have*

$$\int_{\mathbb{R}}\mathfrak{F}h(u)\cos(2\pi u)\log|u|\,du=-\int_{\mathbb{R}^*}'\frac{h(|u|^{-1})}{|1-u|}d^*u.$$

Lemma 3.3 *We have*

$$\int_0^\infty |g(t)|^2 \log t\, dt + \int_0^\infty |\mathfrak{F}g(t)|^2 \log t dt = -\int_{\mathbb{R}^*}' \frac{h(|u|^{-1})}{|1-u|} d^*u$$

where $\mathfrak{F}g(u) = 2\int_0^\infty g(t)\cos(2\pi tu)dt$.

Proof. Since $h(x) = \int_0^\infty g(xt)\bar{g}(t)dt$, $1 < |xt| < \mu_\epsilon$, and $1 < |t| < \mu_\epsilon$ as g vanishes outside the interval $(1, \mu_\epsilon)$, we have

$$\mathfrak{F}h(u) = \int_{\mathbb{R}} \cos(2\pi ux)dx \int_0^\infty g(xt)\bar{g}(t)dt = \int_{\frac{1}{\mu_\epsilon}}^{\mu_\epsilon} \cos(2\pi ux)dx \int_1^{\mu_\epsilon} g(xt)\bar{g}(t)dt.$$

Because the above double integral is absolute integrable, by the Fubini Theorem we can change the order of integration to write

$$\mathfrak{F}h(u) = \int_0^\infty \bar{g}(t)dt \int_{\mathbb{R}} g(xt)\cos(2\pi ux)dx = \int_0^\infty \bar{g}(t)\frac{1}{t}\mathfrak{F}g(\frac{u}{t})dt.$$

Notice that

$$\int_{\mathbb{R}} \mathfrak{F}h(u)\cos(2\pi u)\log|u|\, du = \int_{\mathbb{R}} \cos(2\pi u)\log|u|\, du \int_0^\infty \frac{1}{t}\bar{g}(t)\mathfrak{F}g(\frac{u}{t})dt,$$

where the double integral on the right side is absolute integrable because $t \in (1, \mu_\epsilon)$ and $\mathfrak{F}g(u/t)| \ll (t/u)^2$ for large $|u|$ by using partial integration twice. Hence, we can change the order of integration and derive that

$$\begin{aligned}
\int_{\mathbb{R}} \mathfrak{F}h(u)\cos(2\pi u)\log|u|\, du &= \int_0^\infty \frac{1}{t}\bar{g}(t)dt \int_{\mathbb{R}} \cos(2\pi u)\log|u|\mathfrak{F}g(\frac{u}{t})du \\
&= \int_0^\infty \bar{g}(t)dt \int_{\mathbb{R}} \cos(2\pi ut)\log|ut|\mathfrak{F}g(u)du \\
&= \int_0^\infty \bar{g}(t)\log t dt \int_{\mathbb{R}} \cos(2\pi ut)\mathfrak{F}g(u)du + \int_0^\infty \bar{g}(t)dt \int_{\mathbb{R}} \cos(2\pi ut)\mathfrak{F}g(u)\log|u|du \\
&= \int_0^\infty |g(t)|^2 \log t\, dt + \int_0^\infty \bar{g}(t)dt \int_{\mathbb{R}} \cos(2\pi ut)\mathfrak{F}g(u)\log|u|du.
\end{aligned}$$

By Plancherel's Theorem [12, Theorem 1.1, p. 208] and the inversion formula [12, Theorem 4.2, p. 87]

$$\int_0^\infty \bar{g}(t)dt \int_{\mathbb{R}} \cos(2\pi ut)\mathfrak{F}g(u)\log|u|du = \int_0^\infty |\mathfrak{F}g(u)|^2 \log u du$$

as both $\mathfrak{F}g(u)\log|u|$ and $\int_{\mathbb{R}} \cos(2\pi ut)\mathfrak{F}g(u)\log|u|du$ are in $L^1(\mathbb{R})$ because for large $|u|$

$$\mathfrak{F}g(u) = 2\int_1^{\mu_\epsilon} g(t)\cos(2\pi tu)dt = \frac{-1}{2(\pi u)^2}\int_1^{\mu_\epsilon} g''(t)\cos(2\pi tu)dt.$$

Therefore,

$$\int_{\mathbb{R}} \mathfrak{F}h(u)\cos(2\pi u)\log|u|\,du = \int_0^\infty |g(t)|^2 \log t\,dt + \int_0^\infty |\mathfrak{F}g(t)|^2 \log t dt.$$

The stated identity then follows from Lemma 3.2.

This completes the proof of the lemma. □

Proof of Theorem 1.2. By Lemma 3.1,

$$\mathfrak{F}_S g(t) = \sum_{k,l \in \mathbb{N}_S} \frac{\mu(k)}{k}\mathfrak{F}g(\frac{lt}{k}).$$

Since the sum on k's is a finite sum as S is a finite set, we can change the order of integration and summation to write

$$\begin{aligned}
\int_0^\infty |\mathfrak{F}_S g(t)|^2 \log t dt &= \sum_{k_1,k_2 \in \mathbb{N}_S} \frac{\mu(k_1)\mu(k_2)}{k_1k_2} \int_0^\infty \sum_{l_1,l_2 \in \mathbb{N}_S} \mathfrak{F}g(\frac{l_1 t}{k_1})\overline{\mathfrak{F}g(\frac{l_2 t}{k_2})} \log t dt \\
&= \sum_{k_1,k_2 \in \mathbb{N}_S} \mu(k_1)\mu(k_2) \int_0^\infty \sum_{l_1,l_2 \in \mathbb{N}_S} \mathfrak{F}g(l_1k_2t)\overline{\mathfrak{F}g(l_2k_1t)} \log(k_1k_2t) dt \\
&= \int_0^\infty \sum_{k_1,k_2,l_1,l_2 \in \mathbb{N}_S} \mu(k_1)\mu(k_2)\mathfrak{F}g(l_1k_2t)\overline{\mathfrak{F}g(l_2k_1t)} \log(k_1k_2) dt \\
&\quad + \int_0^\infty \sum_{k_1,k_2,l_1,l_2 \in \mathbb{N}_S} \mu(k_1)\mu(k_2)\mathfrak{F}g(l_1k_2t)\overline{\mathfrak{F}g(l_2k_1t)} \log t dt.
\end{aligned}$$

For $n \in \mathbb{N}_S$, we have

$$\sum_{k|n, k \in \mathbb{N}_S} \mu(k) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

If we denote $n_1 = l_2k_1$ and $n_2 = l_1k_2$, then

$$\begin{aligned}
&\int_0^\infty \sum_{k_1,k_2,l_1,l_2 \in \mathbb{N}_S} \mu(k_1)\mu(k_2)\mathfrak{F}g(l_1k_2t)\overline{\mathfrak{F}g(l_2k_1t)} \log t dt \\
&= \int_0^\infty \sum_{n_1,n_2 \in \mathbb{N}_S} \left(\sum_{k_1|n_1, k_1 \in \mathbb{N}_S} \mu(k_1) \right) \left(\sum_{k_2|n_2, k_2 \in \mathbb{N}_S} \mu(k_2) \right) \mathfrak{F}g(n_2t)\overline{\mathfrak{F}g(n_1t)} \log t dt \\
&= \int_0^\infty |\mathfrak{F}g(t)|^2 \log t dt
\end{aligned}$$

where the rearrangement of the summation is permissible because $\mathfrak{F}g(l_ik_jt) \ll 1/l_ik_jt$ for fixed $t > 0$ by partial integration. It follows that

$$\begin{aligned}
\int_0^\infty |\mathfrak{F}_S g(t)|^2 \log t dt &= \int_0^\infty |\mathfrak{F}g(t)|^2 \log t dt \\
&\quad + \int_0^\infty \sum_{k_1,k_2,l_1,l_2 \in \mathbb{N}_S} \mu(k_1)\mu(k_2)\mathfrak{F}g(l_1k_2t)\overline{\mathfrak{F}g(l_2k_1t)} \log(k_1k_2) dt.
\end{aligned}$$

Let $\Lambda(k) = \log p$ if $k = p^a$ for some prime p and some integer $a \geqslant 1$ and $\Lambda(k) = 0$ otherwise. Then

$$\Lambda(k) = -\sum_{d|k} \mu(d) \log d$$

for all positive integers k, and $\sum_{m|k} \mu(m) = 0$ if $k > 1$. Thus,

$$\begin{aligned}
&\int_0^\infty \sum_{k_1,k_2,l_1,l_2 \in \mathbb{N}_S} \mu(k_1)\mu(k_2)\mathfrak{F}g(l_1k_2t)\overline{\mathfrak{F}g(l_2k_1t)} \log(k_1k_2)dt \\
&= 2\Re \int_0^\infty \sum_{k_1,k_2,l_1,l_2 \in \mathbb{N}_S} \mu(k_1)\mu(k_2) \log k_1 \mathfrak{F}g(l_1k_2t)\overline{\mathfrak{F}g(l_2k_1t)}dt \\
&= 2\Re \int_0^\infty \sum_{n_1,n_2 \in \mathbb{N}_S} \left(\sum_{k_1|n_1, k_1 \in \mathbb{N}_S} \mu(k_1) \log k_1 \right) \left(\sum_{k_2|n_2, k_2 \in \mathbb{N}_S} \mu(k_2) \right) \mathfrak{F}g(n_2t)\overline{\mathfrak{F}g(n_1t)}dt \\
&= -2\Re \int_0^\infty \sum_{n_1 \in \mathbb{N}_S} \Lambda(n_1)\mathfrak{F}g(t)\overline{\mathfrak{F}g(n_1t)}dt \\
&= -2\Re \sum_{n_1 \in \mathbb{N}_S} \Lambda(n_1) \int_0^\infty \mathfrak{F}g(t)\overline{\mathfrak{F}g(n_1t)}dt,
\end{aligned}$$

where changing the order of integration and summation after the 4th equality is permissible because by Hölder's inequality

$$\begin{aligned}
\int_0^\infty \sum_{n_1 \in \mathbb{N}_S} \Lambda(n_1)|\mathfrak{F}g(t)\overline{\mathfrak{F}g(n_1t)}|dt &\leq \sum_{n_1 \in \mathbb{N}_S} \Lambda(n_1) \sqrt{\int_0^\infty |\mathfrak{F}g(t)|^2 dt \int_0^\infty |\mathfrak{F}g(n_1t)|^2 dt} \\
&= \int_0^\infty |\mathfrak{F}g(t)|^2 dt \sum_{n_1 \in \mathbb{N}_S} \frac{\Lambda(n_1)}{\sqrt{n_1}} < \infty.
\end{aligned}$$

By Plancherel's Theorem [12, Theorem 1.1, p. 208],

$$\int_0^\infty \mathfrak{F}g(t)\overline{\mathfrak{F}g(n_1t)}dt = \int_0^\infty g(t)\frac{1}{n_1}\bar{g}(\frac{t}{n_1})dt = \int_0^\infty g(n_1t)\bar{g}(t)dt = h(n_1).$$

Therefore,

$$\int_0^\infty |\mathfrak{F}_S g(t)|^2 \log t dt = \int_0^\infty |\mathfrak{F}g(t)|^2 \log t dt - 2\Re \sum_{k \in \mathbb{N}_S} \Lambda(k)h(k).$$

Also,

$$2\sum_{k \in \mathbb{N}_S, k=1}^\infty \Lambda(k)h(k) = \sum_{p \in S'} \sum_{m=1}^\infty \log p \left[h(p^m) + p^{-m}h(p^{-m}) \right] = \sum_{p \in S} \int'_{Q_p^*} \frac{h(|u|^{-1})}{|1-u|_p} d^*u.$$

Hence,

$$\int_0^\infty |\mathfrak{F}_S g(t)|^2 \log t dt = \int_0^\infty |\mathfrak{F}g(t)|^2 \log t dt - \sum_{p \in S'} \int'_{Q_p^*} \frac{h(|u|^{-1})}{|1-u|_p} d^*u.$$

By Lemma 3.3, we have

$$\int_0^\infty |g(t)|^2 \log t\, dt + \int_0^\infty |\mathfrak{F}_S g(t)|^2 \log t dt = -\sum_{p\in S} \int'_{Q_p^*} \frac{h(|u|^{-1})}{|1-u|_p} d^*u.$$

Since

$$\int'_{Q_p^*} \frac{h_{n,\epsilon}(|u|_p^{-1})}{|1-u|_p} d^*u = \sum_{m=1}^{\infty} \log p \left[h_{n,\epsilon}(p^m) + p^{-m} h_{n,\epsilon}(p^{-m})\right] = 0$$

if $p \notin S$ by the choice of $h_{n,\epsilon}$ in (2.21), we have the formula

$$\Delta(h) = \int_0^\infty |g(t)|^2 \log t\, dt + \int_0^\infty |\mathfrak{F}_S g(t)|^2 \log t dt.$$

This completes the proof of Theorem 1.2. □

4 Preliminary results

In this section, we collect some preliminary results which will be used in later sections.

We denote

$$I_S = \mathbb{R}_+ \times \prod_{p\in S'} O_p^*.$$

Let $d^\times t = \frac{dt}{|t|}$ be the multiplicative measure on $\mathbb{R}^*$ and $d^\times x_p = \frac{1}{1-p^{-1}} \frac{dx_p}{|x_p|_p}$ the multiplicative measure on Q_p^*. Then $d^\times x_S = \prod_{p\in S} d^\times x_p$ is a Haar measure on J_S.

Lemma 4.1 *I_S is a fundamental domain for the action of O_S^* on J_S and $J_S = \bigcup_{\xi\in O_S^*} \xi I_S$, a disjoint union.*

Proof. Each $\alpha \in J_S$ can be written as $\alpha = t\mathfrak{b}$ with $t = |\alpha|_S \in \mathbb{R}_+$ and $\mathfrak{b} = \alpha t^{-1} \in J_S^1$, where t^{-1} also stands for the idele $(t^{-1}, 1, \cdots)$. Since $|\xi|_S = 1$ for $\xi \in O_S^*$, if $\alpha_1, \alpha_2 \in J_S$ with $|\alpha_1|_S \neq |\alpha_2|_S$, then the intersection of $\alpha_1 O_S^*$ and $\alpha_2 O_S^*$ is empty. Thus

$$C_S = \mathbb{R}_+ \times \left(J_S^1/O_S^*\right).$$

As $K = \mathbb{Q}$, for each $\mathfrak{b} \in J_S^1$ there are uniquely determined $\xi \in O_S^*$ and $\mathfrak{b}_1 \in \{1\} \times \prod_{p\in S'} O_p^*$ such that $\mathfrak{b} = \xi\mathfrak{b}_1$. Also, if $\mathfrak{b}_1, \mathfrak{b}_2$ are distinct elements in $\prod_{p\in S'} O_p^*$, then the intersection of $\mathfrak{b}_1 O_S^*$ and $\mathfrak{b}_2 O_S^*$ must be empty. Otherwise, we have $\mathfrak{b}_1\mathfrak{b}_2^{-1} \in O_S^*$. Then $\mathfrak{b}_1\mathfrak{b}_2^{-1} \in Q^*$ and $|\mathfrak{b}_1\mathfrak{b}_2^{-1}|_p = 1$ for all $p \notin S$. Since $\mathfrak{b}_1, \mathfrak{b}_2$ are elements in $\prod_{p\in S'} O_p^*$, we have $|\mathfrak{b}_1\mathfrak{b}_2^{-1}|_p = 1$ for all $p \in S'$. Hence $\mathfrak{b}_1\mathfrak{b}_2^{-1} = 1$; that is, $\mathfrak{b}_1 = \mathfrak{b}_2$. Therefore

$$J_S^1/O_S^* \cong \prod_{p\in S'} O_p^*.$$

Thus

$$C_S \cong \mathbb{R}_+ \times \prod_{p\in S'} O_p^*.$$

We have also obtained the decomposition $J_S = \bigcup_{\xi\in O_S^*} \xi I_S$, a disjoint union.

This completes the proof of the lemma. □

Lemma 4.2 *([6, Lemmas 3.13 and 3,14, pp. 2471–2477]) The operator*

$$V_S(h)\left(S_\Lambda - E_S\mathfrak{F}_S^t P_\Lambda \mathfrak{F}_S E_S^{-1}\right)$$

is a trace class Hilbert-Schmidt integral operator on $L_1^2(C_S)$.

Lemma 4.3 *([9, Theorem VI.25 (b)(c), p. 212]) trace(A^t) = trace(A), and trace(AB) = trace(BA) if A is of trace class and B is bounded.*

Lemma 4.4 *([9, Theorem VI.19(b)(a), p. 207]) Let A, B be bounded linear operators on a Hilbert space $\mathcal{H}$. If A is of trace class on $\mathcal{H}$, so are AB and BA with trace(AB) = trace(BA).*

Lemma 4.5 *(Lidskii's Theorem [5, Theorem 8.4, p. 101]) If A is of trace class on a Hilbert space, then the functional trace of A coincides with its spectral trace.*

Lemma 4.6 *([9, Theorem VI.24, p. 211]) If A is a bounded linear operator of trace class on a Hilbert space $\mathcal{H}$ and $\{\varphi_n\}_{n=1}^\infty$ is any orthonormal basis, then*

$$\sum_{n=1}^{\infty}\langle A\varphi_n, \varphi_n\rangle_{\mathcal{H}}$$

converges absolutely and the limit is independent of the choice of basis.

Lemma 4.7 *([2, Corollary 3.2, p. 237]) Let μ be a σ-finite Borel measure on a second countable space M, and let A be a trace class Hilbert-Schmidt integral operator on $L^2(M, d\mu)$. If the kernel $k(x,y)$ is continuous at (x,x) for almost every x, then*

$$\mathit{trace}(A) = \int_M k(x,x)d\mu(x).$$

The left regular representation V of C_S on $L^2(C_S)$ is given by

$$(V(g)f)(\alpha) = f(g^{-1}\alpha)$$

for $g, \alpha \in C_S$ and $f \in L^2(C_S)$. Let $C_S^1 = J_S^1/O_S^*$. Since the restriction of V to C_S^1 is unitary, we can decompose $L^2(C_S)$ as a direct sum of subspaces

$$L_\chi^2(C_S) = \{f \in L^2(C_S) : f(g^{-1}\alpha) = \chi(g)f(\alpha) \text{ for all } g \in C_S^1 \text{ and } \alpha \in C_S\} \tag{4.1}$$

for all characters χ of C_S^1.

Lemma 4.8 *Let*

$$\eta(x) = 2e^{x/2}\sum_{k\in\mathbb{N}_S}\pi k^2 e^{2x}\left(\pi k^2 e^{2x} - \frac{3}{2}\right)e^{-\pi k^2 e^{2x}}.$$

Then

$$\mathfrak{F}\eta(t) = \frac{1}{4}\xi_S(\frac{1}{2} + 2\pi i t)$$

where

$$\xi_S(s) = s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\prod_{p\in S'}\frac{1}{1-p^{-s}}.$$

Proof. By computations,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \eta(x) e^{x(s-1/2)} dx \\
&= \int_0^{\infty} 2\sqrt{u} \sum_{k\in\mathbb{N}_S} \pi(ku)^2(\pi(ku)^2 - \frac{3}{2}) e^{-\pi(ku)^2} u^{s-\frac{3}{2}} du \\
&= \frac{1}{4} s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) \prod_{p\in S'} \frac{1}{1-p^{-s}} = \frac{1}{4}\xi_S(s)
\end{aligned}$$

for $\Re s > 0$. Putting $s = 1/2 + 2\pi i t$ into above identity we get

$$\mathfrak{F}\eta(t) = \frac{1}{4}\xi_S(\frac{1}{2} + 2\pi i t).$$

This completes the proof of the lemma. □

A subspace of M of $L^2(\mathbb{R})$ is translation-invariant if $f \in M$ implies that $f_\alpha \in M$ for every real α, where $f_\alpha(x) = f(x - \alpha)$.

Let M be a closed translation-invariant subspace of $L^2(\mathbb{R})$, and let $\widehat{M}$ be the image of M under the Fourier transformation. Then $\widehat{M}$ is closed (since the Fourier transform is an L^2-isometry). By [11, Theorem 9.17, p. 190], a Lebesgue measurable set E exists in $\mathbb{R}$ such that $\widehat{M}$ consists precisely of those elements $f \in L^2(\mathbb{R})$ which vanish almost everywhere on E.

Lemma 4.9 *Let M_S be the closed translation-invariant subspace of $L^2(\mathbb{R})$ generated by η. Then $M_S = L^2(\mathbb{R})$.*

Proof. The idea of this proof is due to A. Connes in an email to the author on September 11, 2008. Let $\mathfrak{F}M_S$ be the image of M_S under the Fourier transformation. Then a Lebesgue measurable set E in $\mathbb{R}$ exists such that $\mathfrak{F}M_S$ consists precisely of those elements $f \in L^2(\mathbb{R})$ which vanish almost everywhere on E. Since

$$\mathfrak{F}\eta(t) = \frac{1}{4}\xi_S(\frac{1}{2} + 2\pi i t)$$

by Lemma 4.8 and since

$$\mathfrak{F}\eta_\alpha(t) = e^{2\pi i \alpha t}\mathfrak{F}\eta(t)$$

for every real α, E can be chosen to be the set of all real numbers t such that $1/2 - 2\pi i t$ are zeros of ξ_S. Then E has zero Lebesgue measure. Hence every function in $L^2(\mathbb{R})$ vanishes almost everywhere on E. That is, $\mathfrak{F}M_S = L^2(\mathbb{R})$. Since the mapping $f \to \mathfrak{F}f$ is a Hilbert space isomorphism of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$, we have $M_S = L^2(\mathbb{R})$.

This completes the proof of the lemma. □

Lemma 4.10 *Let $L^2_1(C_S)$ be given as in* (4.1), *and let*

$$\varrho_{S,a}(x) = \pi a x_\infty^2 \left(\pi a x_\infty^2 - \frac{3}{2}\right) e^{-\pi a x_\infty^2} \prod_{p\in S'} 1_{O_p}(x_p).$$

Then the set $\{E_S(\varrho_{S,a}) : a \in (0,\infty)\}$ is dense in $L^2_1(C_S)$.

Proof. Let g be any element in J_S^1 with $|g|_S = 1$, by Lemma 4.1 there exist elements $\gamma \in O_S^*$ and $g_1 \in \{1\} \times \prod_{p\in S'} O_p^*$ such that $g = \gamma g_1$. Thus

$$E_S(\varrho_{S,a})(g^{-1}x) = E_S(\varrho_{S,a})(g_1^{-1}x) = E_S(\varrho_{S,a})(x)$$

for all $x \in C_S$. Hence $E_S(\varrho_{S,a}) \in L_1^2(C_S)$.

Let φ be any element in $L_1^2(C_S)$. We can write

$$\varphi(x) = \varphi(|x|),$$

where $|x|$ is also meant to be the idele $(|x|, 1, 1, \cdots, 1)$. If φ is orthogonal to the image of $S_e(\mathbb{R}) \times \prod_{p\in S'} 1_{O_p}$ under E_S, then

$$\int_{C_S} E_S(f)(x)\bar{\varphi}(|x|)d^\times x = 0$$

for all $f \in S_e(\mathbb{R}) \times \prod_{p\in S'} 1_{O_p}$. In particular we have

$$\int_{\mathbb{R}} \bar{\varphi}(e^x)\eta_\alpha(x)dx = 0$$

for all real α. Since $\varphi \in L_1^2(C_S)$, $\varphi(e^x) \in L^2(\mathbb{R})$. By Lemma 4.9, $\{\eta_\alpha : \alpha \in \mathbb{R}\}$ is dense in $L^2(\mathbb{R})$. Hence $\bar{\varphi}(e^x) = 0$ as an element in $L^2(\mathbb{R})$. Therefore $\{E_S(\varrho_{S,a}) : a \in (0, \infty)\}$ is dense in $L_1^2(C_S)$, and E_S extends to a surjective isometry from $L_1^2(X_S)$ onto $L_1^2(C_S)$.

This completes the proof of the lemma. □

5 Trace of T_ℓ on $E_S(Q_\Lambda^\perp)$, its positivity, and proofs of Theorems 1.3, 1.4

In this section, we compute the trace of T_ℓ on the subspace $E_S(Q_\Lambda^\perp)$ of $L_1^2(C_S)$ in two ways.

Lemma 5.1 *Let $B_{S'} = \prod_{p\in S'} p^{-1}O_p$. If $F(y) = f(|y|)$ for some function f, then*

$$\int_{\mathbb{A}_S} F(y)\Psi_S(yx)dy = \int_{\mathbb{R}\times B_{S'}} F(y)\Psi_S(y|x|)dy.$$

Proof. Since the complement of J_S in $\mathbb{A}_S$ is negligible, as $|\xi| = 1$ for $\xi \in O_S^*$ we can write

$$\int_{\mathbb{A}_S} F(y)\Psi_S(yx)dy = \sum_{\xi\in O_S^*} \int_{I_S} F(y)\Psi_S(yx\xi)dy.$$

Since $F(y) = f(|y|)$, if we change variables $y \to y|x|/x$ with $|x| = (|x|, 1, \cdots, 1)$ then

$$\int_{\mathbb{A}_S} F(y)\Psi_S(yx)dy = \sum_{\xi\in O_S^*} \int_{I_S} f(|y|)\Psi_S(y|x|\xi)dy = \sum_{\xi\in O_S^*} \varpi(\xi) \int_0^\infty f(t)e^{-2\pi it|x|\xi}dt,$$

where $\varpi(\xi)$ is given as in (3.4).

If $\xi \in O_S^*$ satisfies $\varpi(\xi) \neq 0$, then $\xi = l/k$ for some integers $k, l \in \mathbb{N}_S$ with square free k. Thus, $\xi \prod_{p \in S'} O_p^* \subset B_{S'}$. Hence,

$$\bigcup_{\xi \in O_S^*, \varpi(\xi) \neq 0} \xi I_S \subset \mathbb{R}^* \times B_{S'}.$$

Conversely, if $x = (x_p) \in \mathbb{R}^* \times B_{S'}$ then there exists a $\xi \in O_S^*$ such that $\xi^{-1} x \in I_S$, i.e., $|\xi|_p = |x|_p \leq p$ for $p \in S'$. By (3.4), $\varpi(\xi) \neq 0$. Hence, $x \in \bigcup_{\xi \in O_S^*, \varpi(\xi) \neq 0} \xi I_S$. Thus,

$$\mathbb{R}^* \times B_{S'} \subset \bigcup_{\xi \in O_S^*, \varpi(\xi) \neq 0} \xi I_S.$$

Also if ξ, γ are distinct elements in O_S^*, then ξO_S^* and γO_S^* are disjoint sets. Therefore,

$$\mathbb{R}^* \times B_{S'} = \bigcup_{\xi \in O_S^*, \varpi(\xi) \neq 0} \xi I_S, \text{ a disjoint union.}$$

Therefore,

$$\sum_{\xi \in O_S^*, \varpi(\xi) \neq 0} \int_{I_S} F(y) \Psi_S(y|x|\xi) dy = \int_{\mathbb{R} \times B_{S'}} F(y) \Psi_S(y|x|\xi) dy.$$

That is,

$$\int_{\mathbb{A}_S} F(y) \Psi_S(yx) dy = \int_{\mathbb{R} \times B_{S'}} F(y) \Psi_S(y|x|) dy.$$

This completes the proof of the lemma. □

Lemma 5.2 *Let* $f(\lambda) = P_\Lambda(\lambda) \int_0^\infty g(v\lambda) \mathfrak{F}_S g(x/v) d^\times v$. *Then*

$$\int_{\mathbb{A}_S} \Psi_S(-uy) du \int_{\mathbb{A}_S} f(\lambda) \Psi_S(\lambda u) d\lambda = f(y).$$

Proof. We denote $x = (x_r, x_b)$ with $x_r \in \mathbb{R}$ and $x_b \in \mathbb{A}_{S'}$. By [6, Lemma 3.4 and Remark, p. 2467],

$$\begin{aligned}\left(E_S \mathfrak{F}_S E_S^{-1}\right)_u [\sqrt{|u|} \int_{\mathbb{A}_S} f(\lambda) \Psi_S(\lambda u) d\lambda](y) &= \sqrt{|y|} \left(\mathfrak{F}_S\right)_u [\int_{\mathbb{A}_S} f(\lambda) \Psi_S(\lambda u) d\lambda](y) \\ &= \sqrt{|y|} \int_{\mathbb{A}_S} \Psi_S(-uy) du \int_{\mathbb{A}_S} f(\lambda) \Psi_S(\lambda u) d\lambda.\end{aligned}$$

Since

$$\int_{\mathbb{A}_S} f(\lambda) \Psi_S(\lambda u) d\lambda = \sum_{\gamma \in O_S^*} \int_{\gamma I_S} f(\lambda) \Psi_S(\lambda u) d\lambda = \sum_{\gamma \in O_S^*} \int_{I_S} f(\lambda) \Psi_S(\lambda \gamma u) d\lambda,$$

we have

$$E_S^{-1}\{\sqrt{|u|}\int_{\mathbb{A}_S} f(\lambda)\Psi_S(\lambda u)d\lambda\} = \int_{I_S} f(\lambda)\Psi_S(\lambda u)d\lambda.$$

Hence,

$$(\mathfrak{F}_S E_S^{-1})_u\{\sqrt{|u|}\int_{\mathbb{A}_S} f(\lambda)\Psi_S(\lambda u)d\lambda\}(y) = \int_{\mathbb{A}_S} \Psi_S(-uy)du \int_{I_S} f(\lambda)\Psi_S(\lambda u)d\lambda$$
$$= \int_{-\infty}^{\infty} e^{-2\pi i u_r y_r} du_r \int_{\mathbb{A}_{S'}} \Psi_{S'}(-u_b y_b)du_b \int_{\mathbb{A}_{S'}} \phi(\lambda_b)\Psi_{S'}(\lambda_b u_b)d\lambda_b$$

where

$$\phi(\lambda_b) = \begin{cases} \int_{-\infty}^{\infty} f(|\lambda_r||\lambda_b|)e^{2\pi i\lambda_r u_r}d\lambda_r & \text{if } \lambda_b \in \prod_{p\in S'} O_p^* \\ 0 & \text{if } \lambda_b \notin \prod_{p\in S'} O_p^*. \end{cases}$$

Also,

$$\int_{I_S} f(\lambda)\Psi_S(\lambda u)d\lambda = \varpi(u)\int_0^{\infty} f(t)e^{-2\pi i t u_r}dt$$

where $\varpi(u)$ is given as in (3.4). This implies that

$$\int_{\mathbb{A}_{S'}} \phi(\lambda_b)\Psi_{S'}(\lambda_b u_b)d\lambda_b$$

as a function of u_b is supported on the compact set $B_{S'} = \prod_{p\in S'} p^{-1}O_p$. Since $\phi(\lambda_b)$ is locally constant as a function of $\lambda_b \in \mathbb{A}_{S'}$ and is supported on the compact set $\prod_{p\in S'} O_p^*$, the condition of [13, Theorem 2.2.2, p. 310] is satisfied by this function. Note that this theorem is still true if we replace k_p^+ there by $\mathbb{A}_{S'}$. Thus, by the Fourier inversion formula we have

$$\int_{\mathbb{A}_{S'}} \Psi_{S'}(-u_b y_b)du_b \int_{\mathbb{A}_{S'}} \phi(\lambda_b)\Psi_{S'}(\lambda_b u_b)d\lambda_b = \phi(y_b)$$

where $\phi(y_b) = 0$ if $y_b \notin \prod_{p\in S'} O_p^*$.

Since $f(t|y_b|)$ is a continuous and compactly supported function of $t \in [0,\infty)$, $t \neq \Lambda/|y_b|$ and is of bounded variation in an interval including y_r, by Fourier's single-integral formula [14, Theorem 12, p. 25] we have for $y_b \in \prod_{p\in S'} O_p^*$

$$(\mathfrak{F}_S E_S^{-1})_u\{\sqrt{|u|}\int_{\mathbb{A}_S} f(\lambda)\Psi_S(\lambda u)d\lambda\}(y)$$
$$= \int_{-\infty}^{\infty} e^{2\pi i u_r y_r}du_r \int_{-\infty}^{\infty} f(|\lambda_r||y_b|)e^{-2\pi i\lambda_r u_r}d\lambda_r$$
$$= \lim_{K\to\infty}\int_{-\infty}^{\infty} f(|\lambda_r||y_b|)d\lambda_r \int_{-K}^{K} e^{2\pi i u_r(y_r-\lambda_r)}du_r$$
$$= \lim_{K\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} f(|\lambda_r||y_b|)\frac{\sin 2\pi K(y_r-\lambda_r)}{y_r-\lambda_r}d\lambda_r = f(y)$$

where the 2nd equality holds because f has a compact support on $[0, \infty)$. It follows that

$$\begin{aligned}
&(E_S \mathfrak{F}_S E_S^{-1})_u \{\sqrt{|u|} \int_{\mathbb{A}_S} f(\lambda) \Psi_S(\lambda u) d\lambda\}(y)\}(y) \\
&= \sqrt{|y|} \sum_{\xi \in O_S^*} \begin{cases} f(\xi y) & \text{if } \xi y \in I_S, \\ 0 & \text{if } \xi y \notin I_S \end{cases} = \sqrt{|y|} f(y)
\end{aligned}$$

because for each $y \in \mathbb{A}_S$ with $|y| \neq 0$ there exists exactly one $\xi \in O_S^*$ such that $\xi y \in I_S$. Therefore,

$$\left(E_S \mathfrak{F}_S E_S^{-1}\right)_u [\sqrt{|u|} \int_{\mathbb{A}_S} f(\lambda) \Psi_S(\lambda z u) d\lambda](y) = \sqrt{|y|} f(y).$$

This completes the proof of the lemma. □

By Lemma 4.10,

$$L_1^2(C_S) = \{E_S(Q_\Lambda^\perp) \oplus E_S(Q_\Lambda)\} \bigcap L_1^2(C_S) =: E_S(Q_\Lambda^\perp)_1 \oplus E_S(Q_\Lambda)_1$$

where E_S is the extended map as in Lemma 4.10.

Proof of Theorem 1.3. We choose $a_i(x) \in Q_\Lambda^\perp$, $b_j(x) \in Q_\Lambda$, $i, j = 1, 2, \cdots$ such that they are invariant under the action λ_g for all $g \in \prod_{p \in S'} O_p^*$ with $\lambda_g f(x) = f(g^{-1}x)$ and such that their images under E_S form an orthonormal basis of $L_1^2(C_S)$.

By Lemma 4.2, T_ℓ is of trace class on $L_1^2(C_S)$. By Lemma 4.5

$$\text{trace}_{L_1^2(C_S)}(T_\ell) = \sum_{i,j=1}^{\infty} \{\langle T_\ell E_S(a_i), E_S(a_i)\rangle + \langle T_\ell E_S(b_j), E_S(b_j)\}.$$

Since $E_S(a_i)$, $E_S(b_j)$'s form an orthonormal basis of $L_1^2(C_S)$ and since $E_S \mathfrak{F}_S^t P_\Lambda \mathfrak{F}_S E_S^{-1}$ is the orthogonal projection of $L_1^2(C_S)$ onto $E_S(Q_\Lambda^\perp)_1$, we have

$$\langle T_\ell E_S(a_i), E_S(a_i)\rangle = \langle E_S \mathfrak{F}_S^t P_\Lambda \mathfrak{F}_S E_S^{-1} T_\ell E_S(a_i), E_S(a_i)\rangle$$

and

$$\langle E_S \mathfrak{F}_S^t P_\Lambda \mathfrak{F}_S E_S^{-1} T_\ell f, f\rangle_{L_1^2(C_S)} = 0$$

for all $f \in E_S(Q_\Lambda)_1$. Hence,

$$\begin{aligned}
\text{trace}_{E_S(Q_\Lambda^\perp)_1}(T_\ell) &= \text{trace}_{E_S(Q_\Lambda^\perp)_1}(E_S \mathfrak{F}_S^t P_\Lambda \mathfrak{F}_S E_S^{-1} T_\ell) \\
&= \text{trace}_{L_1^2(C_S)}(E_S \mathfrak{F}_S^t P_\Lambda \mathfrak{F}_S E_S^{-1} T_\ell).
\end{aligned}$$

As T_ℓ is of trace class and E_S commutes with P_Λ, by Lemma 4.4

$$\text{trace}_{L_1^2(C_S)}(E_S \mathfrak{F}_S^t P_\Lambda \mathfrak{F}_S E_S^{-1} T_\ell) = \text{trace}_{L_1^2(C_S)}(P_\Lambda E_S \mathfrak{F}_S E_S^{-1} T_\ell E_S \mathfrak{F}_S^t E_S^{-1}).$$

By Lemma 4.3,

$$\text{trace}_{L_1^2(C_S)}(P_\Lambda E_S \mathfrak{F}_S E_S^{-1} T_\ell E_S \mathfrak{F}_S^t E_S^{-1}) = \text{trace}_{L_1^2(C_S)}(E_S \mathfrak{F}_S E_S^{-1} T_\ell^t E_S \mathfrak{F}_S^t E_S^{-1} P_\Lambda).$$

So,

$$\mathrm{trace}_{E_S(Q_\Lambda^\perp)_1}(T_\ell) = \mathrm{trace}_{L_1^2(C_S)}(E_S \mathfrak{F}_S E_S^{-1} T_\ell^t E_S \mathfrak{F}_S^t E_S^{-1} P_\Lambda). \tag{5.1}$$

Let $F = E_S(f)$ with $f \in S_e(\mathbb{R}) \times \prod_{p \in S'} 1_{O_p}$. Since

$$V_S(h)F(x) = \int_{C_S} \sqrt{|x/y|} h(x/y) F(y) d^\times y = \int_{C_S} \sqrt{|xy|} \{\int_0^\infty g(xt)g(yt)dt\} F(y) d^\times y,$$

we have

$$E_S^{-1} V_S(h) F(x) = \int_{C_S} h(x/y) f(y) d^\times y.$$

Because $f \in S_e(\mathbb{R}) \times \prod_{p \in S'} 1_{O_p}$ and h has a compact support in $(0,\infty)$, we have

$$\int_0^\infty dx \int_{C_S} |h(x/y) f(y)| d^\times y = \int_{C_S} |f(y)||y| d^\times y \int_0^\infty |h(x)| dx < \infty.$$

This absolute convergence implies that we can change the order of integration to derive

$$\mathfrak{F}_S E_S^{-1} V_S(h) F(x) = \int_{C_S} |y| \mathfrak{F}_S h(yx) f(y) d^\times y.$$

Since $f \in S_e(\mathbb{R}) \times \prod_{p \in S'} 1_{O_p}$, $P_\Lambda(x) = 0$ for $|x| \geq \Lambda$ and $|\mathfrak{F}_S h(yx)| \ll 1/\sqrt{|yx|}$ by (3.1), we have

$$\int_0^\infty P_\Lambda(x) dx \int_{C_S} |y \mathfrak{F}_S h(yx) f(y)| d^\times y < \infty.$$

Hence we can change the order of integration and get

$$\mathfrak{F}_S^t P_\Lambda \mathfrak{F}_S E_S^{-1} V_S(h) F(x) = \int_{C_S} |y| \left(\int_{\mathbb{A}_S, |t| < \Lambda} \mathfrak{F}_S h(yt) \Psi_S(tx) dt \right) f(y) d^\times y.$$

That is,

$$\begin{aligned} E_S \mathfrak{F}_S^t P_\Lambda \mathfrak{F}_S E_S^{-1} V_S(h) F(x) &= \int_{C_S} \sqrt{|xy|} \left(\int_{\mathbb{A}_S, |t| < \Lambda} \mathfrak{F}_S h(yt) \Psi_S(tx) dt \right) F(y) d^\times y \\ &= \int_{C_S} \sqrt{|xy|} F(y) d^\times y \int_{\mathbb{A}_S, |t| < \Lambda} \Psi_S(tx) dt \int_{\mathbb{A}_S} \Psi_S(-uyt) du \int_0^\infty g(uv) g(v) dv. \end{aligned}$$

By changing variables $u \to u/t$, $v \to vt$ we deduce that

$$\begin{aligned} & E_S \mathfrak{F}_S^t P_\Lambda \mathfrak{F}_S E_S^{-1} V_S(h) F(x) \\ &= \int_{C_S} \sqrt{|xy|} F(y) d^\times y \int_{\mathbb{A}_S, |t| < \Lambda} \Psi_S(tx) dt \int_{\mathbb{A}_S} \Psi_S(-uy) du \int_0^\infty g(uv) g(vt) dv. \end{aligned}$$

Since $|t| < \Lambda$ and we can assume that $1 < |vt| < \mu_\epsilon$ (because $g(vt) = 0$ when v, t don't satisfy this inequality), we can assume that

$$\frac{1}{\Lambda} < \frac{1}{t} < |v| < \frac{\mu_\epsilon}{t}.$$

Thus, we can write

$$E_S\mathfrak{F}_S^t P_\Lambda \mathfrak{F}_S E_S^{-1} V_S(h)F(x)$$
$$= \int_{C_S} \sqrt{|xy|}F(y)d^\times y \int_{\mathbb{A}_S,|t|<\Lambda} \Psi_S(tx)dt \int_{\mathbb{A}_S} \Psi_S(-uy)du \int_{\frac{1}{\Lambda}}^{\frac{\mu_\epsilon}{t}} g(uv)g(vt)dv.$$

Since

$$|\int_{\mathbb{A}_S} g(uv)\Psi_S(-uy)du| = |2\sum_{k,l\in\mathbb{N}_S} \frac{\mu(k)}{k} \int_0^\infty g(uv)\cos(2\pi uy\frac{l}{k})du|$$
$$\leqslant \frac{1}{\pi|y|} \sum_{k,l\in\mathbb{N}_S} \frac{|\mu(k)|}{l} \int_1^{\mu_\epsilon} |g'(u)|du < \frac{c}{|y|}$$

for a constant c depending only on S, the integral (or series)

$$\int_{\mathbb{A}_S} g(uv)\Psi_S(-uy)du$$

converges uniformly with respect to v. So, we can change the order of integration to get

$$\int_0^\infty \mathfrak{F}_S g(\frac{y}{v})g(vt)d^\times v = \int_{\frac{1}{\Lambda}}^{\frac{\mu_\epsilon}{t}} g(vt)dv \int_{\mathbb{A}_S} g(uv)\Psi_S(-uy)du$$
$$= \int_{\mathbb{A}_S} \Psi_S(-uy)du \int_{\frac{1}{\Lambda}}^{\frac{\mu_\epsilon}{t}} g(uv)g(vt)dv = \int_{\mathbb{A}_S} \Psi_S(-uy)du \int_0^\infty g(uv)g(vt)dv.$$

Thus, we have

$$E_S\mathfrak{F}_S^t P_\Lambda \mathfrak{F}_S E_S^{-1} V_S(h)F(x) = \int_{C_S} \sqrt{|xy|}F(y)d^\times y \int_{\mathbb{A}_S,|t|<\Lambda} \Psi_S(tx)dt \int_0^\infty g(vt)\mathfrak{F}_S g(\frac{y}{v})\frac{dv}{|v|}.$$

By using (3.1) we find that $|\mathfrak{F}_S g(v)| \ll_S |v|^{-1/2}$ when $|v|<1$ if choosing $c=1/2$ and $|\mathfrak{F}_S g(v)| \ll_S |v|^{-2}$ when $|v|>1$ if choosing $c=2$. After changing variables $vt\to t$, $y/v\to v$ inside the following integral we find that

$$\int_0^\infty dt \int_0^\infty |g(vt)\mathfrak{F}_S g(\frac{y}{v})|\frac{dv}{|v|} = \frac{1}{|y|}\int_0^\infty |\mathfrak{F}_S g(v)|dv \int_1^{\mu_\epsilon} |g(t)|dt < \infty. \tag{5.2}$$

So, we can change the order of integration to write

$$\int_{\mathbb{A}_S,|t|<\Lambda} \Psi_S(tx)dt \int_0^\infty g(vt)\mathfrak{F}_S g(\frac{y}{v})\frac{dv}{|v|} = \int_0^\infty \mathfrak{F}_S g(\frac{y}{v})\frac{dv}{|v|} \int_{\mathbb{A}_S,|t|<\Lambda} g(vt)\Psi_S(tx)dt. \tag{5.3}$$

It follows from (5.3) that

$$E_S\mathfrak{F}_S^t P_\Lambda \mathfrak{F}_S E_S^{-1} V_S(h)F(x)$$
$$= \int_{C_S} \sqrt{|xy|}\{\int_0^\infty \mathfrak{F}_S g(\frac{y}{v})\frac{dv}{|v|} \int_{\mathbb{A}_S,|t|<\Lambda} g(vt)\Psi_S(tx)dt\}F(y)d^\times y$$

for $F = E_S(f)$ with $f \in S_e(\mathbb{R}) \times \prod_{p\in S'} 1_{O_p}$. Thus, we have derived that

$$T_\ell^t F(x) = \int_{C_S} \sqrt{|xy|}\{S_\Lambda(x) \int_0^\infty g(xt)g(yt)dt - \int_0^\infty \mathfrak{F}_S g(\frac{y}{v}) \frac{dv}{|v|} \int_{\mathbb{A}_S} P_\Lambda(t)g(vt)\Psi_S(tx)dt\}F(y)d^\times y \tag{5.4}$$

for $F = E_S(f)$ with $f \in S_e(\mathbb{R}) \times \prod_{p\in S'} 1_{O_p}$.

Since

$$\int_0^\infty \mathfrak{F}_S g(\frac{y}{v})d^\times v \int_{\mathbb{A}_S} g(vt)\Psi_S(tx)dt = \int_0^\infty \mathfrak{F}_S g(\frac{y}{v})\frac{d^\times v}{|v|}\int_{\mathbb{A}_S} g(t)\Psi_S(\frac{tx}{v})dt \tag{5.5}$$
$$= \int_0^\infty \mathfrak{F}_S g(yv)\overline{\mathfrak{F}_S g(xv)}dv = \int_0^\infty g(yt)\bar{g}(xt)dt = \int_0^\infty g(xt)g(yt)dt$$

as g is real-valued, we can write (5.4) as

$$T_\ell^t F(x) = \int_{C_S} \sqrt{|xy|}\{[S_\Lambda(x) - 1] \int_0^\infty g(xt)g(yt)dt - \int_0^\infty \mathfrak{F}_S g(\frac{y}{v})d^\times v \int_{\mathbb{A}_S} [P_\Lambda(t) - 1]g(vt)\Psi_S(tx)dt\}F(y)d^\times y,$$

and hence

$$E_S^{-1}T_\ell^t F(x) = \int_{C_S} |y|\{[S_\Lambda(x) - 1] \int_0^\infty g(xt)g(yt)dt - \int_0^\infty \mathfrak{F}_S g(\frac{y}{v})d^\times v \int_{\mathbb{A}_S} [P_\Lambda(t) - 1]g(vt)\Psi_S(tx)dt\}f(y)d^\times y$$

for $f \in S_e(\mathbb{R}) \times \prod_{p\in S'} 1_{O_p}$.

Note that

$$E_S^{-1}T_\ell^t F(x) = E_S^{-1}T_\ell^t F(|x|), \text{ and } S_\Lambda(x) - 1 = 0 \text{ when } |x| > \Lambda^{-1}. \tag{5.6}$$

Since g vanishes outside $(1, \mu_\epsilon)$, we have

$$[P_\Lambda(t) - 1]g(vt) = 0$$

for all t if $|v| > \mu_\epsilon/\Lambda$. Also, $(S_\Lambda(x) - 1)g(xt) = 0$ if $|t| < \Lambda$. So,

$$E_S^{-1}T_\ell^t F(x) = \int_{C_S} |y|\{[S_\Lambda(x) - 1] \int_\Lambda^{\frac{\mu_\epsilon}{|y|}} g(xt)g(yt)dt - \int_0^{\frac{\mu_\epsilon}{\Lambda}} \mathfrak{F}_S g(\frac{y}{v})d^\times v \int_{\mathbb{A}_S} [P_\Lambda(t) - 1]g(vt)\Psi_S(tx)dt\}f(y)d^\times y. \tag{5.7}$$

If we choose $c = 2$ in (3.1), we obtained that for large $|x|$

$$\left| \int_{\mathbb{A}_S} [P_\Lambda(t) - 1] g(vt) \Psi_S(tx) dt \right| \ll_S \frac{1}{|x|^2}. \tag{5.8}$$

From (5.6), (5.7), (5.8) and $f \in S_e(\mathbb{R}) \times \prod_{p \in S'} 1_{O_p}$ we find that

$$\begin{aligned} &\int_0^\infty dx \int_{C_S} |y\{[S_\Lambda(x) - 1] \int_0^\infty g(xt)g(yt)dt \\ &- \int_0^\infty \mathfrak{F}_S g(\frac{y}{v}) d^\times v \int_{\mathbb{A}_S} [P_\Lambda(t) - 1] g(vt) \Psi_S(tx) dt\} f(y)| d^\times y < \infty. \end{aligned}$$

Because of the absolute convergence of the above double integral and because of $E_S^{-1} T_\ell^t F(x) = E_S^{-1} T_\ell^t F(|x|)$ by (5.6), by using Lemma 5.1 as $B_{S'}$ is a compact set in $\mathbb{A}_{S'}$ we can change the order of integration to get

$$\begin{aligned} E_S \mathfrak{F}_S E_S^{-1} T_\ell^t F(x) = &\int_{C_S} \sqrt{|xy|} \mathfrak{F}_S \{[S_\Lambda(z) - 1] \int_0^\infty g(zt) g(yt) dt \\ &- \int_0^\infty \mathfrak{F}_S g(\frac{y}{v}) d^\times v \int_{\mathbb{A}_S} [P_\Lambda(t) - 1] g(vt) \Psi_S(tz) dt\}(x) F(y) d^\times y. \end{aligned} \tag{5.9}$$

By using (5.5) again we can rewrite (5.9) back in a form as deriving from (5.4)

$$\begin{aligned} E_S \mathfrak{F}_S E_S^{-1} T_\ell^t F(x) = &\int_{C_S} \sqrt{|xy|} \mathfrak{F}_S \{S_\Lambda(z) \int_0^{\Lambda \mu_\epsilon} g(zt) g(yt) dt \\ &- \int_0^\infty \mathfrak{F}_S g(\frac{y}{v}) d^\times v \int_{\mathbb{A}_S} P_\Lambda(t) g(vt) \Psi_S(tz) dt\}(x) F(y) d^\times y \\ = &\int_{C_S} \sqrt{|xy|} \{\int_0^\infty g(yt) dt \int_{\frac{1}{\Lambda} < |z|} g(zt) \Psi_S(-zx) dz \\ &- \int_0^\infty \mathfrak{F}_S g(\frac{y}{v}) P_\Lambda(x) g(vx) d^\times v\} F(y) d^\times y, \end{aligned} \tag{5.10}$$

where the 2nd term after above 2nd equality is obtained by using Lemma 5.2 because by (5.3) we can write

$$\int_0^\infty \mathfrak{F}_S g(\frac{y}{v}) d^\times v \int_{\mathbb{A}_S} P_\Lambda(t) g(vt) \Psi_S(tz) dt = \int_{\mathbb{A}_S} \Psi_S(tz) dt \int_0^\infty P_\Lambda(t) g(vt) \mathfrak{F}_S g(y/v) d^\times v.$$

Since $f \in S_e(\mathbb{R}) \times \prod_{p \in S'} 1_{O_p}$, both $P_\Lambda(y) f(y)$ and $\mathfrak{F}_S^t P_\Lambda f(y)$ are in $L^1(\mathbb{A}_S)$, by using both Plancherel's formula (3.3) and the Fourier inversion formula [13, Theorem 4.1.2, p. 328] to get rid of the inner most Fourier transform in front of $P_\Lambda F(y)$ on the right side after the following 1st equality, we obtain that

$$\begin{aligned}
&E_S\mathfrak{F}_S E_S^{-1} T_\ell^t E_S \mathfrak{F}_S^t E_S^{-1} P_\Lambda F(x)\\
&=\int_{C_S}\sqrt{|xy|}\{\int_0^\infty g(yt)dt\int_{\frac{1}{\Lambda}<|z|} g(zt)\Psi_S(-zx)dz\\
&\quad-\int_0^\infty \mathfrak{F}_S g(\frac{y}{v})P_\Lambda(x)g(vx)d^\times v\}E_S\mathfrak{F}_S^t E_S^{-1}P_\Lambda F(y)d^\times y\\
&=\int_{C_S}\sqrt{|xy|}\{\int_0^\infty \mathfrak{F}_S g(\frac{y}{t})\frac{dt}{t}\int_{\mathbb{A}_S,\frac{1}{\Lambda}<|z|} g(zt)\Psi_S(-xz)dz\\
&\quad-\int_0^\infty g(yv)g(vx)dvP_\Lambda(x)\}P_\Lambda(y)F(y)d^\times y
\end{aligned}\tag{5.11}$$

for $F=E_S(f)$ with $f\in S_e(\mathbb{R})\times\prod_{p\in S'}1_{O_p}$. Since $E_S\mathfrak{F}_S E_S^{-1}T_\ell^t E_S\mathfrak{F}_S^t E_S^{-1}P_\Lambda$ is bounded, (5.11) holds for all $F\in L_1^2(C_S)$ by Lemma 4.10.

Let

$$\begin{aligned}
k(x,y)=\sqrt{|xy|}\{&\int_0^\infty \mathfrak{F}_S g(\frac{y}{t})\frac{dt}{t}\int_{\mathbb{A}_S,\frac{1}{\Lambda}<|z|} g(zt)\Psi_S(-xz)dz\\
&-\int_0^\infty g(yv)g(vx)dvP_\Lambda(x)\}P_\Lambda(y).
\end{aligned}$$

Then

$$k(x,x)=|x|\{\int_0^\infty \mathfrak{F}_S g(\frac{x}{t})\frac{dt}{t}\int_{\mathbb{A}_S,\frac{1}{\Lambda}<|z|} g(zt)\Psi_S(-xz)dz-\int_0^\infty |g(xv)|^2dv\}P_\Lambda(x)$$

as g is a real-valued function. By (5.1),

$$\begin{aligned}
&\mathrm{trace}_{E_S(Q_\Lambda^\perp)_1}(T_\ell)\\
&=\int_{C_S,|x|<\Lambda}\{\int_0^\infty \mathfrak{F}_S g(\frac{x}{t})\frac{dt}{t}\int_{\mathbb{A}_S,\frac{1}{\Lambda}<|z|} g(zt)\Psi_S(-xz)dz-\int_0^\infty |g(xv)|^2dv\}|x|d^\times x.
\end{aligned}$$

Since $\mathfrak{F}_S g$ is also a real-valued function, we can write

$$\int_0^\infty \mathfrak{F}_S g(\frac{x}{t})\frac{dt}{t}\int_{\mathbb{A}_S} g(zt)\Psi_S(-xz)dz=\int_0^\infty |\mathfrak{F}_S g(\frac{x}{t})|^2\frac{dt}{t^2}=\int_0^\infty |g(xv)|^2dv$$

by (3.2) after changing $1/t\to t$. It follows that

$$\mathrm{trace}_{E_S(Q_\Lambda^\perp)_1}(T_\ell)=-\int_{C_S,|x|<\Lambda}|x|d^\times x\int_0^\infty \mathfrak{F}_S g(\frac{x}{t})\frac{dt}{t}\int_{\mathbb{A}_S,|z|<\frac{1}{\Lambda}} g(zt)\Psi_S(-xz)dz.$$

By changing variables $x\to\Lambda x$, $t\to\Lambda t$ and $z\to z/\Lambda$ we derive that

$$\mathrm{trace}_{E_S(Q_\Lambda^\perp)_1}(T_\ell)=-\int_{C_S,|x|<1}|x|d^\times x\int_0^\infty \mathfrak{F}_S g(\frac{x}{t})\frac{dt}{t}\int_{\mathbb{A}_S,|z|\leqslant 1} g(zt)\Psi_S(-xz)dz.$$

That is,

$$\operatorname{trace}_{E_S(Q_\Lambda^\perp)_1}(T_\ell) = -\int_{C_S,|x|<1} |x| d^\times x \int_0^\infty \mathfrak{F}_S g(xz) d^\times z \int_{\mathbb{A}_S,|u|\leqslant 1} g(\frac{u}{z}) \Psi_S(-xu) du \tag{5.12}$$

$$= -\int_{C_S,|x|<1} |x| d^\times x \int_0^1 \mathfrak{F}_S g(xz) dz \int_{A_S,1<|u|<\min(\mu_\epsilon,\frac{1}{|z|})} g(u) \Psi_S(-xzu) du.$$

If we choose $c = 1/4$ in (3.1), we get $|\mathfrak{F}_S g(xz)| \ll_S |xz|^{-1/4}$ and

$$|\int_{A_S,1<|u|<\min(\mu_\epsilon,\frac{1}{|z|})} g(u) \Psi_S(-xzu) du| \ll_S |xz|^{-1/4}.$$

This implies that the front double integral in (5.12) is absolute integrable. By the Fubini Theorem, we can change the order of integration to write

$$\operatorname{trace}_{E_S(Q_\Lambda^\perp)_1}(T_\ell) = -\int_0^\infty d^\times z \int_0^1 \mathfrak{F}_S g(xz) dx \int_{\mathbb{A}_S,|u|\leqslant 1} g(\frac{u}{z}) \Psi_S(-xu) du. \tag{5.13}$$

Also, by changing variables $xz \to z$, $xu \to u$ on the right side after 1st equality in (5.12) we can write

$$\operatorname{trace}_{E_S(Q_\Lambda^\perp)_1}(T_\ell) = -\int_0^\infty \mathfrak{F}_S g(z) d^\times z \int_0^1 d^\times x \int_{\mathbb{A}_S,|u|\leqslant|x|} g(\frac{u}{z}) \Psi_S(-u) du. \tag{5.14}$$

By (3.1),

$$\int_{\mathbb{A}_S,|u|\leqslant|x|} g(\frac{u}{z}) \Psi_S(-u) du = 2 \sum_{k,l\in\mathbb{N}_S} \frac{\mu(k)}{k} \int_0^{|x|} g(\frac{u}{z}) \cos(2\pi u \frac{l}{k}) du$$

$$= \sum_{k,l\in\mathbb{N}_S} \frac{\mu(k)}{\pi l} \{ g(\frac{x}{z}) \sin(2\pi |x| \frac{l}{k}) - \frac{1}{z} \int_0^{|x|} g'(\frac{u}{z}) \sin(2\pi u \frac{l}{k}) du \}$$

$$\leqslant \sum_{k,l\in\mathbb{N}_S} \frac{|\mu(k)|}{\pi l} \{ \max_u |g(u)| + \int_1^{\mu_\epsilon} |g'(u)| du \} < \infty.$$

This implies that integral (or series)

$$\int_{\mathbb{A}_S,|u|\leqslant|x|} g(\frac{u}{z}) \Psi_S(-u) du$$

converges uniformly with respect to x. So, we can change the order of integration to get

$$\int_\nu^1 d^\times x \int_{\mathbb{A}_S,|u|\leqslant|x|} g(\frac{u}{z}) \Psi_S(-u) du = \int_{\mathbb{A}_S,|u|\leqslant 1} g(\frac{u}{z}) \Psi_S(-u) du \int_{\max(\nu,|u|)}^1 \frac{dx}{x}$$

$$= -\int_{\mathbb{A}_S,|u|\leqslant 1} g(\frac{u}{z}) \log \max(\nu, |u|) \Psi_S(-u) du.$$

It follows from (5.14) that

$$\mathrm{trace}_{E_S(Q_\Lambda^\perp)_1}(T_\ell)=\int_0^\infty \mathfrak{F}_S g(z)d^\times z\lim_{\nu\to 0}\int_{\mathbb{A}_S,|u|\leqslant 1} g(\frac{u}{z})\log\max(\nu,|u|)\Psi_S(-u)du,$$

where $g(u/z)=0$ when $z>1$.

By changing variables $u\to uz$ and noticing $g(u)=0$ for $|u|<1$ we find

$$\begin{aligned}\mathrm{trace}_{E_S(Q_\Lambda^\perp)_1}(T_\ell)&=\int_{C_S}\mathfrak{F}_S g(z)|z|d^\times z\lim_{\nu\to 0}\int_{|u|\leqslant\frac{1}{|z|}} g(u)\log\max(\nu,|uz|)\Psi_S(-uz)du\\&=\int_{C_S}\mathfrak{F}_S g(z)|z|d^\times z\lim_{\nu\to 0}\int_{1<|u|} g(u)\log\max(\nu/|u|,|z|)\Psi_S(-uz)du\\&\quad+\int_{C_S}\mathfrak{F}_S g(z)|z|d^\times z\int_{\mathbb{A}_S} g(u)\log|u|\Psi_S(-uz)du\\&\quad-\int_{C_S}\mathfrak{F}_S g(z)|z|d^\times z\int_{\mathbb{A}_S,|u|>\frac{1}{|z|}} g(u)\log|uz|\Psi_S(-uz)du\end{aligned}\tag{5.15}$$

because $|uz|>1$ so that we can take off the limit in the above 4th line.

Since $\mathfrak{F}_S g(z)=O(|\log|z||^{|S'|-1})$ as $z\to 0$ which is obtained by calculating the residue at $s=0$ on the right side of (3.1), we can take off the limit in the 2nd line of (5.15) and get the following identity by using (3.3)

$$\begin{aligned}\mathrm{trace}_{E_S(Q_\Lambda^\perp)_1}(T_\ell)&=\int_0^\infty|\mathfrak{F}_S g(z)|^2\log|z|\,dz+\int_0^\infty|g(u)|^2\log|u|\,du\\&\quad-\int_{C_S}\mathfrak{F}_S g(z)|z|d^\times z\int_{\mathbb{A}_S,|u|>\frac{1}{|z|}} g(u)\log|uz|\Psi_S(-uz)du.\end{aligned}$$

Because $g(u)=0$ when $u\notin(1,\mu_\epsilon)$, for $|z|\geqslant 1$ we have

$$\begin{aligned}&\int_{z\in C_S,|z|\geqslant 1}\mathfrak{F}_S g(z)|z|d^\times z\int_{\mathbb{A}_S,|u|>\frac{1}{|z|}} g(u)\log|uz|\Psi_S(-uz)du\\&=\int_{z\in C_S,|z|\geqslant 1}\mathfrak{F}_S g(z)|z|d^\times z\int_{\mathbb{A}_S} g(u)\log|uz|\Psi_S(-uz)du\\&=\int_{z\in C_S,|z|\geqslant 1}|\mathfrak{F}_S g(z)|^2\log|z|\,|z|d^\times z+\int_{C_S,|z|\geqslant 1}\mathfrak{F}_S g(z)\mathfrak{F}_S\{g(u)\log|u|\}(z)|z|d^\times z.\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\operatorname{trace}_{E_S(Q_\Lambda^\perp)_1}(T_\ell) = \int_0^\infty |\mathfrak{F}_S g(z)|^2 \log z \, dz + \int_0^\infty |g(u)|^2 \log u \, du \\
&- \int_{C_S, \frac{1}{\mu_\epsilon} \leqslant |z| < 1} \mathfrak{F}_S g(z) |z| d^\times z \int_{\mathbb{A}_S, \frac{1}{|z|} < |u| < \mu_\epsilon} g(u) \log |uz| \Psi_S(-uz) du \\
&- \int_{z \in C_S, |z| \geqslant 1} |\mathfrak{F}_S g(z)|^2 \log |z| \, |z| d^\times z - \int_{C_S, |z| \geqslant 1} \mathfrak{F}_S g(z) \mathfrak{F}_S \{g(u) \log |u|\}(z) |z| d^\times z \qquad (5.16) \\
&= \int_0^1 |\mathfrak{F}_S g(z)|^2 \log |z| \, |z| d^\times z + \int_0^\infty |g(u)|^2 \log u \, du \\
&- \int_{C_S, \frac{1}{\mu_\epsilon} \leqslant |z| < 1} \mathfrak{F}_S g(z) |z| d^\times z \int_{\mathbb{A}_S, \frac{1}{|z|} < |u| < \mu_\epsilon} g(u) \log |uz| \Psi_S(-uz) du \\
&- \int_{C_S, |z| \geqslant 1} \mathfrak{F}_S g(z) \mathfrak{F}_S \{g(u) \log |u|\}(z) |z| d^\times z.
\end{aligned}
$$

By (3.3),

$$
\int_{C_S} \mathfrak{F}_S g(z) \mathfrak{F}_S \{g(u) \log |u|\}(z) |z| d^\times z = \int_0^\infty |g(u)|^2 \log u \, du.
$$

Thus,

$$
\begin{aligned}
&- \int_{C_S, |z| \geqslant 1} \mathfrak{F}_S g(z) \mathfrak{F}_S \{g(u) \log |u|\}(z) |z| d^\times z \\
&= \int_{C_S, |z| < 1} \mathfrak{F}_S g(z) \mathfrak{F}_S \{g(u) \log |u|\}(z) |z| d^\times z - \int_{C_S} \mathfrak{F}_S g(z) \mathfrak{F}_S \{g(u) \log |u|\}(z) |z| d^\times z \\
&= \int_{C_S, |z| < 1} \mathfrak{F}_S g(z) \mathfrak{F}_S \{g(u) \log |u|\}(z) |z| d^\times z - \int_0^\infty |g(u)|^2 \log u \, du.
\end{aligned}
$$

It follows from (5.16) that

$$
\begin{aligned}
\operatorname{trace}_{E_S(Q_\Lambda^\perp)_1}(T_\ell) &= \int_0^1 |\mathfrak{F}_S g(z)|^2 \log z \, |z| d^\times z + \int_{C_S, |z| < 1} \mathfrak{F}_S g(z) \mathfrak{F}_S \{g(u) \log |u|\}(z) |z| d^\times z \\
&\quad - \int_{C_S, \frac{1}{\mu_\epsilon} \leqslant |z| < 1} \mathfrak{F}_S g(z) |z| d^\times z \int_{\mathbb{A}_S, \frac{1}{|z|} < |u| < \mu_\epsilon} g(u) \log |uz| \Psi_S(-uz) du \\
&= \int_0^1 |\mathfrak{F}_S g(t)|^2 \log t \, dt + \int_0^1 \mathfrak{F}_S g(t) \mathfrak{F}_S \{g(u) \log |u|\}(t) dt \\
&\quad - \int_{\frac{1}{\mu_\epsilon} \leqslant t < 1} \mathfrak{F}_S g(t) dt \int_{\mathbb{A}_S, \frac{1}{t} < |u| < \mu_\epsilon} g(u) \log |ut| \Psi_S(-ut) du.
\end{aligned}
$$

This completes the proof of Theorem 1.3. $\square$

Proof of Theorem 1.4. Because $g(u/z)=0$ when $|z|>1$ and $|u|\leqslant 1$, by (5.13)

$$\operatorname{trace}_{E_S(Q_\Lambda^\perp)_1}(T_\ell)=-\int_0^1 d^\times z\int_{C_S,|x|<1}\mathfrak{F}_S g(xz)|x|d^\times x\int_{\mathbb{A}_S,|u|\leqslant 1}g(\frac{u}{z})\Psi_S(-xu)du \tag{5.17}$$

$$=-\int_0^1 d^\times z\lim_{\nu\to 0+}\int_{C_S,\nu<|x|<1}|x|d^\times x\int_{\mathbb{A}_S}g(v)\Psi_S(-vxz)dv\int_{\mathbb{A}_S,|u|\leqslant 1}g(\frac{u}{z})\Psi_S(-xu)du.$$

By (3.1),

$$\int_{\mathbb{A}_S}g(v)\Psi_S(-vxz)dv=2\sum_{k,l\in\mathbb{N}_S}\frac{\mu(k)}{k}\int_1^{\mu_\epsilon}g(v)\cos(2\pi vxz\frac{l}{k})dv$$

$$=-\sum_{k,l\in\mathbb{N}_S}\frac{\mu(k)}{\pi lxz}\int_1^{\mu_\epsilon}g'(v)\sin(2\pi vxz\frac{l}{k})du$$

and

$$\int_{\mathbb{A}_S,|u|\leqslant 1}g(\frac{u}{z})\Psi_S(-xu)du=2\sum_{k,l\in\mathbb{N}_S}\frac{\mu(k)}{k}\int_0^1 g(\frac{u}{z})\cos(2\pi ux\frac{l}{k})du$$

$$=\sum_{k,l\in\mathbb{N}_S}\frac{\mu(k)}{\pi lx}\{g(\frac{1}{z})\sin(2\pi x\frac{l}{k})-\frac{1}{z}\int_0^1 g'(\frac{u}{z})\sin(2\pi ux\frac{l}{k})du\}.$$

Because

$$|\int_{\mathbb{A}_S}g(v)\Psi_S(-vxz)dv|\leqslant\sum_{k,l\in\mathbb{N}_S}\frac{|\mu(k)|}{\pi l|xz|}\int_1^{\mu_\epsilon}|g'(v)|dv<\frac{c}{|xz|}, \tag{5.18}$$

$$|\int_{\mathbb{A}_S,|u|\leqslant 1}g(\frac{u}{z})\Psi_S(-xu)du|\leqslant\sum_{k,l\in\mathbb{N}_S}\frac{|\mu(k)|}{\pi l|x|}\{\max_u|g(u)|+\int_1^{\mu_\epsilon}|g'(u)|du\}<\frac{c}{|x|}$$

for some constant c depending only on S, the double integral (or series)

$$\int_{\mathbb{A}_S}g(v)\Psi_S(-vxz)dv\int_{\mathbb{A}_S,|u|\leqslant 1}g(\frac{u}{z})\Psi_S(-xu)du$$

converges absolutely and uniformly with respect to $x\in(\nu,1)$. So, we can change the order of integration and summation to obtain that

$$\int_\nu^1 dx\int_{\mathbb{A}_S}g(v)\Psi_S(-vxz)dv\int_{\mathbb{A}_S,|u|\leqslant 1}g(\frac{u}{z})\Psi_S(-xu)du \tag{5.19}$$

$$=\int_{\mathbb{A}_S}g(v)dv\int_{\mathbb{A}_S,|u|\leqslant 1}g(\frac{u}{z})du\int_{C_S,\nu<|x|<1}\Psi_S(-(u+vz)x)|x|d^\times x.$$

By (5.17) and (5.19),

$$-\operatorname{trace}_{E_S(Q_\Lambda^\perp)_1}(T_\ell)$$

$$=\int_0^1 d^\times z\lim_{\nu\to 0}\int_{\mathbb{A}_S}g(v)dv\int_{\mathbb{A}_S,|u|\leqslant 1}g(\frac{u}{z})du\int_{c_S,\nu<|x|<1}\Psi_S(-(u+vz)x)|x|d^\times x.$$

According to [13, Lemma 4.1.2, p. 329] we can write

$$\int_{C_S,\nu<|x|<1}\Psi_S(-(u+vz)x)|x|d^\times x=\frac{1}{|u+vz|}\int_{C_S,\nu<\frac{|x|}{|u+vz|}<1}\Psi_S(-x)|x|d^\times x$$
$$=\frac{2}{|u+vz|}\int_{\nu|u+vz|}^{|u+vz|}\cos(2\pi x)dx=\frac{\sin(2\pi|u+vz|)-\sin(2\pi\nu|u+vz|)}{\pi|u+vz|}.$$

As the measure difference between $\mathbb{A}_S$ and J_S is negligible for any finite set S, we have

$$-\operatorname{trace}_{E_S(Q_\Lambda^\perp)_1}(T_\ell)$$
$$=\int_0^1 d^\times z\lim_{\nu\to 0}\int_{J_S}g(v)dv\int_{\mathbb{A}_S,|u|\leqslant 1}g(\frac{u}{z})\frac{\sin(2\pi|u+vz|)-\sin(2\pi\nu|u+vz|)}{\pi|u+vz|}du.$$

By changing variable $u\to uv$ we derive

$$-\operatorname{trace}_{E_S(Q_\Lambda^\perp)_1}(T_\ell)$$
$$=\int_0^1 d^\times z\lim_{\nu\to 0}\int_{J_S}g(v)dv\int_{\mathbb{A}_S,|uv|\leqslant 1}g(\frac{uv}{z})\frac{\sin(2\pi|v||u+z|)-\sin(2\pi\nu|v||u+z|)}{\pi|u+z|}du$$
$$=\int_0^1 d^\times z\lim_{\nu\to 0}\int_{J_S}g(|v|)dv\int_{|uv|\leqslant 1}g(\frac{|uv|}{|z|})\frac{\sin(2\pi|v||u+z|)-\sin(2\pi\nu|v||u+z|)}{\pi|u+z|}du.$$

Since $J_S=\cup_{\xi\in O_S^*}\xi I_S$ by Lemma 4.1, we can write

$$-\operatorname{trace}_{E_S(Q_\Lambda^\perp)_1}(T_\ell)=\int_0^1 d^\times z\lim_{\nu\to 0}$$
$$\times\sum_{\xi\in O_S^*}\int_{I_S,1<|v\xi|<\mu_\epsilon}g(|v\xi|)dv\int_{\mathbb{A}_S,|uv\xi|\leqslant 1}g(\frac{|uv\xi|}{|z|})\frac{\sin(2\pi|v\xi||u+z|)-\sin(2\pi\nu|v\xi||u+z|)}{\pi|u+z|}du.$$

Since $|\xi|_S=1$ for all $\xi\in O_S^*$, we have

$$\int_{I_S,1<|v\xi|<\mu_\epsilon}g(|v\xi|)dv\int_{|uv\xi|\leqslant 1}g(\frac{|uv\xi|}{|z|})\frac{\sin(2\pi|v\xi||u+z|)-\sin(2\pi\nu|v\xi||u+z|)}{\pi|u+z|}du \tag{5.20}$$
$$=\int_{I_S}g(|v|)dv\int_{\mathbb{A}_S,|uv|\leqslant 1}g(\frac{|uv|}{|z|})\frac{\sin(2\pi|v||u+z|)-\sin(2\pi\nu|v||u+z|)}{\pi|u+z|}du$$

for all $\xi\in O_S^*$. So, the right side of (5.20) is independent of ξ. As O_S^* contains infinitely many distinct elements ξ, (5.20) implies that

$$\sum_{\xi\in O_S^*}\int_{I_S,1<|v\xi|<\mu_\epsilon}g(|v\xi|)dv\int_{\mathbb{A}_S,|uv\xi|\leqslant 1}g(\frac{|uv\xi|}{|z|})\frac{\sin(2\pi|v\xi||u+z|)-\sin(2\pi\nu|v\xi||u+z|)}{\pi|u+z|}du$$
$$=0,\ \text{or}\ \pm\infty.$$

That is,

$$\int_{C_S,\nu<|x|<1} |x|d^\times x \int_{\mathbb{A}_S} g(v)\Psi_S(-vxz)dv \int_{\mathbb{A}_S,|u|\leqslant 1} g(\frac{u}{z})\Psi_S(-xu)du = 0, \text{ or } \pm\infty. \tag{5.21}$$

By (5.18),

$$\int_{C_S,\nu<|x|<1} |x|d^\times x| \int_{\mathbb{A}_S} g(v)\Psi_S(-vxz)dv \int_{|u|\leqslant 1} g(\frac{u}{z})\Psi_S(-xu)du| < \frac{c^2}{|z|}\int_\nu^1 \frac{dx}{x^2} < \frac{c^2}{\nu|z|} < \infty. \tag{5.22}$$

From (5.21) and (5.22) we deduce that

$$\int_{C_S,\nu<|x|<1} |x|d^\times x \int_{\mathbb{A}_S} g(v)\Psi_S(-vxz)dv \int_{\mathbb{A}_S,|u|\leqslant 1} g(\frac{u}{z})\Psi_S(-xu)du = 0. \tag{5.23}$$

Combining (5.17) and (5.23) we get that

$$\text{trace}_{E_S(Q_\Lambda^\perp)_1}(T_\ell) = 0.$$

This completes the proof of Theorem 1.4. □

6 Trace of T_ℓ on $E_S(Q_\Lambda)$, its positivity, and proofs of Theorems 1.5, 1.6

In this section, we compute the trace of T_ℓ on the subspace $E_S(Q_\Lambda)$ of $L_1^2(C_S)$ and prove its positivity.

Lemma 6.1 *$V_S(h)$ is a positive operator on $L^2(C_S)$.*

Proof. Let F be any element in $L^2(C_S)$ with compact support. By definition,

$$V_S(h)F(x) = \int_0^\infty F(\lambda)\sqrt{|x/\lambda|}d^\times\lambda \int_0^\infty g(|x/\lambda|y)g(y)dy.$$

By changing variables $y \to |\lambda|y$ we can write

$$\begin{aligned}
&\int_{C_S} V_S(h)F(x)\bar{F}(x)d^\times x \\
&= \int_{C_S} \bar{F}(x)d^\times x \int_{C_S} F(\lambda)\sqrt{|x/\lambda|}d^\times\lambda \int_0^\infty g(|x/\lambda|y)g(y)dy \\
&= \int_{C_S} \bar{F}(x)\sqrt{|x|}d^\times x \int_{C_S} F(\lambda)\sqrt{|\lambda|}d^\times\lambda \int_0^\infty g(|x|y)g(|\lambda|y)dy.
\end{aligned}$$

Since the triple integral above is absolute integrable as F, g are compactly supported, we can change order of integration to derive

$$\int_{C_S} V_S(h)F(x)\bar{F}(x)d^\times x = \int_0^\infty \overline{(\int_{C_S} F(x)g(|x|y)\sqrt{|x|}d^\times x)}(\int_{C_S} F(\lambda)g(|\lambda|y)\sqrt{|\lambda|}d^\times\lambda)dy \geq 0$$

where g is a real-valued function. Since compactly supported functions are dense in $L^2(C_S)$ and $V_S(h)$ is bounded, we have

$$\langle V_S(h)F, F\rangle \geq 0$$

for all $F \in L^2(C_S)$.

This completes the proof of the lemma. $\square$

Proof of Theorem 1.5. By Lemma 4.2, T_ℓ is of trace class on $L^2_1(C_S)$. By using (5.4), (5.10) and similarly as in (5.1), (5.11) we derive both

$$\mathrm{trace}_{E_S(Q_\Lambda)_1}(T_\ell) = \mathrm{trace}_{L^2_1(C_S)}\{E_S\mathfrak{F}_S E_S^{-1}T_\ell^t E_S\mathfrak{F}_S^t E_S^{-1}(1-P_\Lambda)\}$$

and

$$\begin{aligned} &E_S\mathfrak{F}_S E_S^{-1}T_\ell^t E_S\mathfrak{F}_S^t E_S^{-1}(1-P_\Lambda)F(x) \\ &= \int_{C_S}\sqrt{|xy|}\{\int_0^\infty \mathfrak{F}_S g(\frac{y}{t})\frac{dt}{t}\int_{\mathbb{A}_S,\frac{1}{\Lambda}<|z|} g(zt)\Psi_S(-xz)dz \\ &\quad -\int_0^\infty g(yv)g(vx)dv\, P_\Lambda(x)\}(1-P_\Lambda(y))F(y)d^\times y \end{aligned} \tag{6.1}$$

for $F \in L^2_1(C_S)$. Let

$$\begin{aligned} k(x,x) = |x|\{&\int_0^\infty \mathfrak{F}_S g(\frac{x}{t})\frac{dt}{t}\int_{\mathbb{A}_S,\frac{1}{\Lambda}<|z|} g(zt)\Psi_S(-xz)dz \\ &-\int_0^\infty g(xv)g(vx)dv\, P_\Lambda(x)\}(1-P_\Lambda(x)). \end{aligned}$$

Since $P_\Lambda(x)(1-P_\Lambda(x)=0$ for all x, we have

$$k(x,x) = |x|(1-P_\Lambda(x))\int_0^\infty \mathfrak{F}_S g(\frac{x}{t})\frac{dt}{t}\int_{\mathbb{A}_S,\frac{1}{\Lambda}<|z|} g(zt)\Psi_S(-xz)dz.$$

Hence

$$\mathrm{trace}_{E_S(Q_\Lambda)_1}(T_\ell) = \int_{C_S,\Lambda<|x|}|x|d^\times x\int_0^\infty \mathfrak{F}_S g(\frac{x}{t})\frac{dt}{t}\int_{\mathbb{A}_S,\frac{1}{\Lambda}<|z|} g(zt)\Psi_S(-xz)dz.$$

By changing variables $x\to\Lambda x$, $t\to\Lambda t$ and $z\to z/\Lambda$ we deduce that

$$\mathrm{trace}_{E_S(Q_\Lambda)_1}(T_\ell) = \int_{C_S,1<|x|}|x|d^\times x\int_0^\infty \mathfrak{F}_S g(\frac{x}{t})\frac{dt}{t}\int_{\mathbb{A}_S,1<|z|} g(zt)\Psi_S(-xz)dz.$$

By changing variables $x/t\to t$, $xz\to z$ we obtain that

$$\mathrm{trace}_{E_S(Q_\Lambda)_1}(T_\ell) = \int_{C_S,1<|x|}d^\times x\int_0^\infty \mathfrak{F}_S g(t)\frac{dt}{t}\int_{\mathbb{A}_S,|x|<|z|} g(\frac{z}{t})\Psi_S(-z)dz. \tag{6.2}$$

Since $g(z/t)=0$ if $|z/t|\notin(1,\mu_\epsilon)$, we can assume that $|z|<\mu_\epsilon|t|$. Also, $|x|<|z|$ implies that $|x|<\mu_\epsilon|t|$. That is, we can assume that $|x|/\mu_\epsilon<|t|$. Hence, we can write

$$\mathrm{trace}_{E_S(Q_\Lambda)_1}(T_\ell)=\int_{C_S,1<|x|}d^\times x\int_{\frac{|x|}{\mu_\epsilon}}^\infty\mathfrak{F}_Sg(t)\frac{dt}{t}\int_{\mathbb{A}_S,|x|<|z|}g(\frac{z}{t})\Psi_S(-z)dz. \tag{6.3}$$

By (3.1),

$$\begin{aligned}&\int_{\mathbb{A}_S,|x|<|z|}g(\frac{z}{t})\Psi_S(-z)dz=2\sum_{k,l\in\mathbb{N}_S}\frac{\mu(k)}{k}\int_{|x|<|z|}g(\frac{z}{t})\cos(2\pi z\frac{l}{k})dz\\&=\sum_{k,l\in\mathbb{N}_S}\frac{\mu(k)}{\pi l}\{-g(\frac{x}{t})\sin(2\pi|x|\frac{l}{k})-\frac{1}{t}\int_{|x|<|z|}g'(\frac{z}{t})\sin(2\pi z\frac{l}{k})dz\}\\&\leqslant\sum_{k,l\in\mathbb{N}_S}\frac{|\mu(k)|}{\pi l}\{\max_u|g(u)|+\int_1^{\mu_\epsilon}|g'(u)|du\}<\infty.\end{aligned}\tag{6.4}$$

This implies that the integral (or series)

$$\int_{\mathbb{A}_S,|x|<|z|}g(\frac{z}{t})\Psi_S(-z)dz$$

converges uniformly with respect to x and t. Note that $|\mathfrak{F}_Sg(t)|\ll_S|t|^{-1}$ for large $|t|$ if choosing $c=1$ in (3.1). This implies that the front double integral in both (6.2) and (6.3) are absolute integrable. Then the Fubini Theorem implies that we can change the order of integration for the two front double integrals and write

$$\begin{aligned}\mathrm{trace}_{E_S(Q_\Lambda)_1}(T_\ell)&=\int_0^\infty\mathfrak{F}_Sg(t)\frac{dt}{t}\int_1^\infty d^\times x\int_{\mathbb{A}_S,|x|<|z|}g(\frac{z}{t})\Psi_S(-z)dz\\&=\int_0^\infty\mathfrak{F}_Sg(t)\frac{dt}{t}\int_{\mathbb{A}_S,1<|z|}g(\frac{z}{t})\Psi_S(-z)dz\int_1^{|z|}d^\times x\\&=\int_0^\infty\mathfrak{F}_Sg(t)\frac{dt}{t}\int_{\mathbb{A}_S,1<|z|}g(\frac{z}{t})\log|z|\,\Psi_S(-z)dz\end{aligned}$$

where the change of order of the inner double integral after the 1st equality is permissible by (6.4) because as g vanishes outside the interval $(1,\mu_\epsilon)$ we can write

$$\int_1^\infty d^\times x\int_{\mathbb{A}_S,|x|<|z|}g(\frac{z}{t})\Psi_S(-z)dz=\int_1^{\mu_\epsilon t}d^\times x\int_{\mathbb{A}_S,|x|<|z|<\mu_\epsilon t}g(\frac{z}{t})\Psi_S(-z)dz.$$

By changing $z\to zt$ we derive that

$$\mathrm{trace}_{E_S(Q_\Lambda)_1}(T_\ell)=\int_0^\infty\mathfrak{F}_Sg(t)dt\int_{\mathbb{A}_S,\frac{1}{t}<|z|}g(z)\log|zt|\Psi_S(-zt)dz.$$

Since $g(z)=0$ for $|z|\notin(1,\mu_\epsilon)$, we can write

$$\begin{aligned}\operatorname{trace}_{E_S(Q_\Lambda)_1}(T_\ell) &= \int_1^\infty \mathfrak{F}_S g(t)dt\int_{\mathbb{A}_S} g(z)\log|zt|\Psi_S(-zt)dz\\ &+\int_0^1 \mathfrak{F}_S g(t)dt\int_{\mathbb{A}_S,\frac{1}{t}\leqslant|z|} g(z)\log|zt|\Psi_S(-zt)dz\\ &=\int_1^\infty |\mathfrak{F}_S g(t)|^2\log t\,dt+\int_1^\infty \mathfrak{F}_S g(t)\mathfrak{F}_S\{g(z)\log|z|\}(t)dt\\ &+\int_{\frac{1}{\mu_\epsilon}}^1 \mathfrak{F}_S g(t)dt\int_{\mathbb{A}_S,\frac{1}{t}\leqslant|z|<\mu_\epsilon} g(z)\log|zt|\Psi_S(-zt)dz.\end{aligned}$$

This completes the proof of Theorem 1.5. □

Proof of Theorem 1.6. Let $F_i, i=1,2,\cdots$ be an orthnormal base of $E_S(Q_\Lambda)_1$. By Lemmas 4.6 and 4.7,

$$\operatorname{trace}_{E_S(Q_\Lambda)_1}(T_\ell)=\sum_{i=1}^\infty\langle V_S(h)\left(S_\Lambda-E_S\mathfrak{F}_S^tP_\Lambda\mathfrak{F}_SE_S^{-1}\right)F_i,F_i\rangle.$$

Since $F_i\in E_S(Q_\Lambda)_1$, we have $\mathfrak{F}_SE_S^{-1}F_i(x)=0$ for $|x|<\Lambda$. This implies that

$$P_\Lambda\mathfrak{F}_SE_S^{-1}F_i(x)=0$$

for all x, and hence

$$\operatorname{trace}_{E_S(Q_\Lambda)_1}(T_\ell)=\sum_{i=1}^\infty\langle V_S(h)S_\Lambda F_i,F_i\rangle. \tag{6.5}$$

Since T_ℓ is of trace class, so is $(1-S_\Lambda)T_\ell$ by Lemma 4.4 as $1-S_\Lambda$ is a bounded linear operator on $L^2(C_S)$. It follows from Lemma 4.6 that the series

$$\sum_{i=1}^\infty\langle(1-S_\Lambda)V_S(h)\left(S_\Lambda-E_S\mathfrak{F}_S^tP_\Lambda\mathfrak{F}_SE_S^{-1}\right)F_i,F_i\rangle=\sum_{i=1}^\infty\langle V_S(h)S_\Lambda F_i,(1-S_\Lambda)F_i\rangle$$

is absolutely convergent. As the right side of (6.5) is also absolutely convergent by Lemma 4.6 we can write

$$\begin{aligned}\operatorname{trace}_{E_S(Q_\Lambda)_1}(T_\ell)&=\sum_{i=1}^\infty\langle V_S(h)S_\Lambda F_i,S_\Lambda F_i\rangle+\sum_{i=1}^\infty\langle V_S(h)S_\Lambda F_i,(1-S_\Lambda)F_i\rangle\\ &=\sum_{i=1}^\infty\langle V_S(h)S_\Lambda F_i,S_\Lambda F_i\rangle+\sum_{i=1}^\infty\langle(1-S_\Lambda)T_\ell F_i,F_i\rangle \qquad (6.6)\\ &=\sum_{i=1}^\infty\langle V_S(h)S_\Lambda F_i,S_\Lambda F_i\rangle+\operatorname{trace}_{E_S(Q_\Lambda)_1}\{(1-S_\Lambda)T_\ell\}\\ &\geqslant\operatorname{trace}_{E_S(Q_\Lambda)_1}\{(1-S_\Lambda)T_\ell\}\end{aligned}$$

by Lemma 6.1.

By using (5.4), (5.10) and similarly as in (5.1), (5.11) we derive both

$$\mathrm{trace}_{E_S(Q_\Lambda)_1}\{(1-S_\Lambda)T_\ell\} = \mathrm{trace}_{L^2_1(C_S)}\{E_S\mathfrak{F}_S E_S^{-1}T_\ell^t(1-S_\Lambda)E_S\mathfrak{F}_S^t E_S^{-1}(1-P_\Lambda)\}$$

and

$$\begin{aligned}
&E_S\mathfrak{F}_S E_S^{-1}T_\ell^t(1-S_\Lambda)E_S\mathfrak{F}_S^t E_S^{-1}(1-P_\Lambda)F(x)\\
&=\int_{C_S}\sqrt{|xy|}\{\int_0^\infty dt\int_{|u|\leqslant\frac{1}{\Lambda}} g(ut)\Psi_S(-uy)du\int_{\frac{1}{\Lambda}<|z|} g(zt)\Psi_S(-zx)dz\\
&\quad-\int_0^\infty d^\times v\int_{|u|\leqslant\frac{1}{\Lambda}}\mathfrak{F}_S g(\frac{u}{v})\Psi_S(-uy)du P_\Lambda(x)g(vx)\}(1-P_\Lambda(y))F(y)d^\times y.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathrm{trace}_{E_S(Q_\Lambda)_1}\{(1-S_\Lambda)T_\ell\}\\
&=\int_{C_S}\{\int_0^\infty dt\int_{\mathbb{A}_S,|u|\leqslant\frac{1}{\Lambda}} g(ut)\Psi_S(-ux)du\int_{\mathbb{A}_S,\frac{1}{\Lambda}<|z|} g(zt)\Psi_S(-zx)dz\\
&\quad-\int_0^\infty d^\times v\int_{\mathbb{A}_S,|u|\leqslant\frac{1}{\Lambda}}\mathfrak{F}_S g(\frac{u}{v})\Psi_S(-ux)du P_\Lambda(x)g(vx)\}(1-P_\Lambda(x))|x|d^\times x.
\end{aligned}$$

Since $P_\Lambda(x)(1-P_\Lambda(x))=0$ for all x,

$$\begin{aligned}
&\mathrm{trace}_{E_S(Q_\Lambda)_1}\{(1-S_\Lambda)T_\ell\}\\
&=\int_{C_S,\Lambda<|x|}|x|d^\times x\int_0^\infty dt\int_{\mathbb{A}_S,|u|\leqslant\frac{1}{\Lambda}} g(ut)\Psi_S(-ux)du\int_{\mathbb{A}_S,\frac{1}{\Lambda}<|z|} g(zt)\Psi_S(-zx)dz.
\end{aligned}$$

By changing variables $x\to\Lambda x$, $t\to\Lambda t$, $u\to u/\Lambda$ and $z\to z/\Lambda$ we get

$$\begin{aligned}
&\mathrm{trace}_{E_S(Q_\Lambda)_1}\{(1-S_\Lambda)T_\ell\}\\
&=\int_{C_S,1<|x|}|x|d^\times x\int_0^\infty dt\int_{\mathbb{A}_S,|u|\leqslant 1} g(ut)\Psi_S(-ux)du\int_{\mathbb{A}_S,1<|z|} g(zt)\Psi_S(-zx)dz.
\end{aligned}$$

Since we can assume that $1<|ut|<\mu_\epsilon$ and $1<|zt|<\mu_\epsilon$ (because $g(ut)g(zt)=0$ when u,z,t don't satisfy both inequalities simultaneously), we have

$$\max(\frac{1}{|u|},\frac{1}{|z|})<|t|<\min(\frac{\mu_\epsilon}{|u|},\frac{\mu_\epsilon}{|z|}).$$

As $|u|\leqslant 1$ and $1<|z|$, we get

$$\frac{1}{|u|}<|t|<\frac{\mu_\epsilon}{|z|}.$$

In particular, we can assume that

$$1<t<\mu_\epsilon. \tag{6.7}$$

Note that g vanishes outside the interval $(1,\mu_\epsilon)$. Considering $g(ut)$ with $|u|\leqslant 1$ and $g(zt)$ with $1<|z|$, by (6.7) we can assume that

$$|z|<\mu_\epsilon \text{ and } \mu_\epsilon^{-1}<|u|. \tag{6.8}$$

From (6.7) and (6.8) we can write

$$\operatorname{trace}_{E_S(Q_\Lambda)_1}\{(1-S_\Lambda)T_\ell\} = \int_{C_S,|x|\geqslant 1} |x|d^\times x \int_1^{\mu_\epsilon} dt \tag{6.9}$$
$$\times \int_{\mathbb{A}_S,\frac{1}{\mu_\epsilon}<|u|<1} g(ut)\Psi_S(-ux)du \int_{\mathbb{A}_S,1\leqslant|z|<\mu_\epsilon} g(zt)\Psi_S(-zx)dz.$$

By (3.1),

$$|\int_{\mathbb{A}_S,\frac{1}{\mu_\epsilon}<|u|<1} g(ut)\Psi_S(-ux)du \int_{\mathbb{A}_S,1\leqslant|z|<\mu_\epsilon} g(zt)\Psi_S(-zx)dz| = \frac{1}{(\pi x)^2}$$
$$\times |\sum_{k_1,k_2;l_1,l_2\in N_S} \frac{\mu(k_1)\mu(k_2)}{l_1 l_2}\{g(t)\sin(2\pi x\frac{l_1}{k_1}) - t\int_{\frac{1}{\mu_\epsilon}<|u|<1} g'(ut)\cos(2\pi ux\frac{l_1}{k_1})du\} \tag{6.10}$$
$$\times \{g(t)\sin(2\pi x\frac{l_1}{k_1}) + t\int_{1\leqslant|z|<\mu_\epsilon} g'(zt)\cos(2\pi zx\frac{l_2}{k_2})\}dz|$$
$$\leq \frac{1}{(\pi x)^2}\left(\max_u |g(u)| + \int_1^{\mu_\epsilon} |g'(u)|du\right)^2 \sum_{k_1,k_2;l_1,l_2\in N_S} \frac{|\mu(k_1)\mu(k_2)|}{l_1 l_2} < \infty$$

for $|x|>1$ as $g(t/\mu_\epsilon)=0$ and $g(\mu_\epsilon t)=0$ for $1<|t|<\mu_\epsilon$. That is, the double integral

$$\int_{\mathbb{A}_S,\frac{1}{\mu_\epsilon}<|u|<1} g(ut)\Psi_S(-ux)du \int_{\mathbb{A}_S,1\leqslant|z|<\mu_\epsilon} g(zt)\Psi_S(-zx)dz$$

converges uniformly with respect to both t and $1<|x|$. Thus, we can change the order of integration and write (6.9) as

$$\operatorname{trace}_{E_S(Q_\Lambda)_1}\{(1-S_\Lambda)T_\ell\}$$
$$= \int_1^{\mu_\epsilon} dt \lim_{Y\to\infty} \int_{\mathbb{A}_S,\frac{1}{\mu_\epsilon}<|u|<1} g(ut)du \int_{\mathbb{A}_S,1\leqslant|z|<\mu_\epsilon} g(zt)dz \int_{C_S,1\leqslant|x|<Y} \Psi_S(-(u+z)x)|x|d^\times x$$
$$= \int_1^{\mu_\epsilon} dt \lim_{Y\to\infty} \int_{\mathbb{A}_S,\frac{1}{\mu_\epsilon}<|u|<1} g(ut)du \int_{\mathbb{A}_S,1\leqslant|z|<\mu_\epsilon} \frac{g(zt)}{|u+z|}dz \int_{C_S,1\leqslant\frac{|x|}{|u+z|}<Y} \Psi_S(-x)|x|d^\times x.$$

By computations,

$$\int_{x\in C_S,1\leqslant\frac{|x|}{|u+z|}<Y} \Psi_S(-x)|x|d^\times x = 2\int_{|u+z|}^{Y|u+z|} \cos(2\pi x)dx = \frac{\sin(2\pi Y|u+z|)-\sin(2\pi|u+z|)}{\pi}.$$

Hence,

$$\operatorname{trace}_{E_S(Q_\Lambda)_1}\{(1-S_\Lambda)T_\ell\}$$
$$= \int_1^{\mu_\epsilon} dt \lim_{Y\to\infty} \int_{\mathbb{A}_S,\frac{1}{\mu_\epsilon}<|u|<1} g(ut)du \int_{\mathbb{A}_S,1\leqslant|z|<\mu_\epsilon} g(zt)\frac{\sin(2\pi Y|u+z|)-\sin(2\pi|u+z|)}{\pi|u+z|}dz,$$

where in the right side after the 1st equality $|u+z| \neq 0$ because $|u| < 1$ and $|z| \geq 1$.

By changing variable $z \to uz$ we can write

$$\mathrm{trace}_{E_S(Q_\Lambda)_1}\{(1-S_\Lambda)T_\ell\} = \int_1^{\mu_\epsilon} dt \lim_{Y\to\infty} \int_{\mathbb{A}_S, \frac{1}{\mu_\epsilon}<|u|<1} g(|ut|)\delta_Y(t,|u|)du, \tag{6.11}$$

where

$$\delta_Y(t,|u|) = \int_{\mathbb{A}_S, 1\leqslant |uz|<\mu_\epsilon} g(|uz|t) \frac{\sin(2\pi Y|u||1+z|) - \sin(2\pi |u||1+z|)}{\pi|1+z|} dz.$$

Note that $\delta_Y(t,|\xi u|) = \delta_Y(t,|u|)$ for all $\xi \in O_S^*$.

As $J_S = \cup_{\xi\in O_S^*} \xi I_S$ by Lemma 4.1 and the measure difference between $\mathbb{A}_S$ and J_S is negligible for any finite set S, we have

$$\begin{aligned}
\int_{\mathbb{A}_S, \frac{1}{\mu_\epsilon}<|u|<1} g(|ut|)\delta_Y(t,|u|)du &= \int_{J_S, \frac{1}{\mu_\epsilon}<|u|<1} g(|u|t)\delta_Y(t,|u|)du \\
&= \sum_{\xi\in O_S^*} \int_{u\in I_S, \frac{1}{\mu_\epsilon}<|\xi u|<1} g(|\xi u|t)\delta_Y(t,|\xi u|)du \qquad (6.12)\\
&= \sum_{\xi\in O_S^*} \int_{u\in I_S, \frac{1}{\mu_\epsilon}<|u|<1} g(|u|t)\delta_Y(t,|u|)du = 0 \text{ or } \pm\infty
\end{aligned}$$

because O_S^* contains infinitely many elements and the integral

$$\int_{u\in I_S, \frac{1}{\mu_\epsilon}<|u|<1} g(|u|t)\delta_Y(t,|u|)du$$

is independent of ξ.

By (6.10) we have

$$|\int_{\mathbb{A}_S, \frac{1}{\mu_\epsilon}<|u|<1} g(ut)\Psi_S(-ux)du \int_{\mathbb{A}_S, 1\leqslant|z|<\mu_\epsilon} g(zt)\Psi_S(-zx)dz| \leqslant \frac{c}{(\pi x)^2}$$

for a constant c depending only on S. It follows that

$$\begin{aligned}
&|\int_{1\leq|x|<Y} |x|d^\times x \int_{\mathbb{A}_S, \frac{1}{\mu_\epsilon}<|u|<1} g(ut)\Psi_S(-ux)du \int_{\mathbb{A}_S, 1\leqslant|z|<\mu_\epsilon} g(zt)\Psi_S(-zx)dz| \qquad (6.13)\\
&\leqslant \frac{c}{\pi^2}\int_{1\leq|x|<Y} \frac{d^\times x}{|x|} < \frac{c}{\pi^2} < \infty.
\end{aligned}$$

Combining (6.12) and (6.13) we obtain that

$$\int_{\mathbb{A}_S, \frac{1}{\mu_\epsilon}<|u|<1} g(|ut|)\delta_Y(t,|u|)du = 0.$$

It follows from (6.11) that

$$\mathrm{trace}_{E_S(Q_\Lambda)_1}\{(1-S_\Lambda)T_\ell\} = 0. \tag{6.14}$$

By (6.6) and (6.14),

$$\mathrm{trace}_{E_S(Q_\Lambda)_1}(T_\ell) \geqslant 0.$$

This completes the proof of Theorem 1.6. □

7 Proof of Theorem 1.7

Proof of Theorem 1.7. By Theorems 1.3 and 1.5 we have

$$\begin{aligned}
&\mathrm{trace}_{E_S(Q_\Lambda^\perp)_1}(T_\ell) + \mathrm{trace}_{E_S(Q_\Lambda)_1}(T_\ell)\\
&= \int_0^1 |\mathfrak{F}_S g(t)|^2 \log t\, dt + \int_0^1 \mathfrak{F}_S g(t) \mathfrak{F}_S\{g(u)\log|u|\}(t)dt\\
&\quad - \int_{\frac{1}{\mu_\epsilon}\leqslant t<1} \mathfrak{F}_S g(t)dt \int_{\mathbb{A}_S, \frac{1}{t}\leqslant|u|<\mu_\epsilon} g(u)\log|ut|\Psi_S(-ut)du\\
&\quad + \int_1^\infty |\mathfrak{F}_S g(t)|^2 \log t\, dt + \int_1^\infty \mathfrak{F}_S g(t)\mathfrak{F}_S\{g(u)\log|u|\}(t)dt\\
&\quad + \int_{\frac{1}{\mu_\epsilon}}^1 \mathfrak{F}_S g(t)dt \int_{\mathbb{A}_S, \frac{1}{t}\leqslant|z|<\mu_\epsilon} g(z)\log|zt|\Psi_S(-zt)dz\\
&= \int_0^\infty |\mathfrak{F}_S g(t)|^2 \log t\, dt + \int_0^\infty \mathfrak{F}_S g(t)\mathfrak{F}_S\{g(u)\log|u|\}(t)dt\\
&= \int_0^\infty |\mathfrak{F}_S g(t)|^2 \log t\, dt + \int_0^\infty |g(u)|^2\log|u|du = \Delta(h)
\end{aligned}$$

by using (3.3), the fact that $\mathfrak{F}_S\{g(u)\log|u|\}$ and g are real-valued, and Theorem 1.2. Also, by Theorems 1.4 and 1.6 we have

$$\Delta(h) = \mathrm{trace}_{E_S(Q_\Lambda^\perp)_1}(T_\ell) + \mathrm{trace}_{E_S(Q_\Lambda)_1}(T_\ell) \geqslant 0.$$

Because $h(x) = x^{-1}h(x^{-1})$, we can write

$$h(x) = \int_0^\infty Jg_\epsilon(\frac{t}{x})Jg_\epsilon(t)\frac{dt}{x} = \int_0^\infty g_\epsilon(\frac{x}{t})g_\epsilon(\frac{1}{t})\frac{dt}{t^2} = h_{n,\epsilon}(x).$$

Hence,

$$\Delta(h_{n,\epsilon}) \geqslant 0.$$

By Theorem 1.1 we have

$$\lambda_n \geqslant 0$$

for $n = 1, 2, \cdots$.

Finally, the Riemann hypothesis for the Riemann zeta-function follows from Li's criterion, which states that all nontrivial zeros of $\zeta(s)$ lie on the vertical line $\Re 1 = 1/2$ if, and only if $\lambda_n \geqslant 0$ for $n = 1, 2, 3, \cdots$.

This completes the proof of Theorem 1.7. □

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