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Convolutions of harmonic convex mappings

Michael Dorff[a]*, Maria Nowak[b] and Magdalena Woloszkiewicz[b]

[a]Department of Mathematics, Brigham Young University, Provo, Utah, 84602, USA;
[b]Department of Mathematics, Maria Curie-Sklodowska University, 20-031 Lublin, Poland

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The first author proved that the harmonic convolution of a normalized right half-plane mapping with either another normalized right half-plane mapping or a normalized vertical strip mapping is convex in the direction of the real axis, provided that it is locally univalent. In this article, we prove that in general the assumption of local univalency cannot be omitted. However, we are able to show that in some cases these harmonic convolutions are locally univalent. Using this we obtain interesting examples of univalent harmonic maps one of which is a map onto the plane with two parallel slits.

Keywords: harmonic mappings; convolutions; univalence

AMS Subject Classification: 30C45

1. Introduction

Let D be the unit disc. We consider the family of complex-valued harmonic functions $f = u + iv$ defined in D, where u and v are real harmonic in D. Such functions can be expressed as $f = h + \overline{g}$, where

$$h(z) = \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n$$

are analytic in D. The harmonic function $f = h + \overline{g}$ is locally one-to-one and sense-preserving in D if and only if

$$|g'(z)| < |h'(z)| \quad \forall z \in D.$$

Let S_H^O be the class of complex-valued, harmonic, sense-preserving, univalent functions f in D, normalized so that $f(0) = 0$, $f_z(0) = 1$ and $f_{\bar{z}}(0) = 0$. Let K_H^O and C_H^O be the subclasses of S_H^O mapping D onto convex and close-to-convex domains, respectively. We will deal with C_H^O mappings that are convex in one direction.

*Corresponding author. Email: mdorff@math.byu.edu

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For analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$, their convolution (or Hadamard product) is defined as $f * F = z + \sum_{n=2}^{\infty} a_n A_n z^n$. In the harmonic case, with

$$f = h + \overline{g} = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n \quad \text{and}$$
$$F = H + \overline{G} = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B}_n \overline{z}^n,$$

define the harmonic convolution as

$$f * F = h * H + \overline{g * G} = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b_n B_n}\, \overline{z}^n.$$

There have been some results about harmonic convolutions of functions [1–4]. For the convolution of analytic functions, if $f_1, f_2 \in K$, then $f_1 * f_2 \in K$. Also, the right half-plane mapping, $\frac{z}{1-z}$, acts as the convolution identity. In the harmonic case, there are infinitely many right half-plane mappings and the harmonic convolution of one of these right half-plane mappings with a function $f \in K_H^O$ may not preserve the properties of f, such as convexity or even univalence (see [2] for an example). In [5–7], explicit descriptions are given for half-plane and strip mappings. Specifically, the collection of functions $f = h + \overline{g} \in S_H^O$ that map D onto the right half-plane, $R = \{w : \mathrm{Re}(w) > -1/2\}$, satisfy

$$h(z) + g(z) = \frac{z}{1-z} \tag{1}$$

and those that map D onto the vertical strip, $\Omega_\alpha = \left\{ w : \frac{\alpha-\pi}{2\sin\alpha} < \mathrm{Re}(w) < \frac{\alpha}{2\sin\alpha} \right\}$, where $\frac{\pi}{2} \le \alpha < \pi$, satisfy

$$h(z) + g(z) = \frac{1}{2i\sin\alpha} \log\left(\frac{1 + ze^{i\alpha}}{1 + ze^{-i\alpha}} \right). \tag{2}$$

In [2], the following results were obtained:

Theorem A Let $f_1 = h_1 + \overline{g}_1, f_2 = h_2 + \overline{g}_2 \in K_H^O$ with $h_k(z) + g_k(z) = \frac{z}{1-z}$ for $k = 1, 2$. If $f_1 * f_2$ is locally univalent and sense-preserving, then $f_1 * f_2 \in S_H^O$ and is convex in the direction of the real axis.

Theorem B Let $f_1 = h_1 + \overline{g}_1 \in K_H^O$ with $h_1(z) + g_1(z) = \frac{z}{1-z}$ and $f_2 = h_2 + \overline{g}_2 \in K_H^O$ with $h_2(z) + g_2(z) = \frac{1}{2i\sin\alpha} \log\left(\frac{1+ze^{i\alpha}}{1+ze^{-i\alpha}} \right)$. If $f_1 * f_2$ is locally univalent and sense-preserving, then $f_1 * f_2 \in S_H^O$ and is convex in the direction of the real axis.

Note that since all harmonic right half-plane mappings satisfy Equation (1) and all harmonic vertical strip mappings satisfy Equation (2), then Theorems A and B apply to harmonic right half-plane mappings and harmonic vertical strip, respectively. In Section 2, we generalize Theorem A for harmonic mappings onto slanted half-planes given by

$$H_\gamma = \left\{ z \in \mathbb{C} : \mathrm{Re}(e^{i\gamma} z) > -\frac{1}{2} \right\}, \quad \text{where } 0 \le \gamma < 2\pi.$$

Next, we deal mainly with the convolution of the canonical harmonic right half-plane mapping [1] given by

$$f_0(z) = h_0(z) + \overline{g_0(z)} = \frac{z-\frac{1}{2}z^2}{(1-z)^2} - \overline{\frac{\frac{1}{2}z^2}{(1-z)^2}} \tag{3}$$

with harmonic mappings f that are either right half-planes or strip mappings. We show that if the dilatation of f is $e^{i\theta}z^n$ ($n = 1, 2$), then $f_0 * f$ is locally univalent. However, we give examples when local univalency fails for $n \geq 3$. Also, we provide some results about univalency in the case the dilatation of f is $\frac{z+a}{1+\bar{a}z}$. Finally, we give examples of univalent harmonic maps obtained by way of convolutions.

2. The convolution of slanted half-plane mappings

We first prove a generalization of Theorem A for the slanted half-plane, H_γ, $0 \leq \gamma < 2\pi$, described in the introduction. Let $S^0(H_\gamma) \subset S_H^O$ denote the class of harmonic functions f that map D onto H_γ. In the case when $\gamma = 0$ we get the normalized class of harmonic functions that map D onto the right half-plane $\{w : \mathrm{Re}(w) > -1/2\}$.

Lemma 1 If $f = h + \bar{g} \in S^0(H_\gamma)$, then

$$h(z) + e^{-2i\gamma} g(z) = \frac{z}{1-ze^{i\gamma}}, \quad z \in D.$$

Proof If $f = h + \bar{g} \in S^0(H_\gamma)$, then $\mathrm{Re}\left(e^{i\gamma}(h(z) + \overline{g(z)})\right) > -1/2$, which means that $\mathrm{Re}\{e^{i\gamma} h(z) + e^{-i\gamma} g(z)\} > -1/2$. In other words, $\mathrm{Re}\{e^{i\gamma}(h(z) + e^{-2i\gamma} g(z))\} > -1/2$. Since f is a convex harmonic function, it follows from a convexity theorem by Clunie and Sheil-Small [1] that the function $h(z) + e^{-2i\gamma} g(z)$ is convex in the direction $\pi/2 - \gamma$, and so is univalent. It is also clear that $z \mapsto h(z) + e^{-2i\gamma} g(z)$ maps D onto H_γ which implies the result. ∎

Theorem 2 If $f_k \in S^0(H_{\gamma_k})$, $k = 1, 2$, and $f_1 * f_2$ is locally univalent in D, then $f_1 * f_2$ is convex in the direction $-(\gamma_1 + \gamma_2)$.

Proof Let

$$F_1 = (h_1 + e^{-2i\gamma_1} g_1) * (h_2 - e^{-2i\gamma_2} g_2), \text{ and}$$
$$F_2 = (h_2 + e^{-2i\gamma_2} g_2) * (h_1 - e^{-2i\gamma_1} g_1).$$

Then

$$\frac{1}{2}(F_1 + F_2) = h_1 * h_2 - e^{-2i(\gamma_1+\gamma_2)} g_1 * g_2.$$

The shearing theorem of [1] establishes that it is sufficient to show that the function $\frac{1}{2}(F_1 + F_2)$ is convex in the direction $-(\gamma_1 + \gamma_2)$, or equivalently, that $F = e^{i(\gamma_1+\gamma_2)}(F_1 + F_2)$ is convex in the direction of the real axis. A result by Royster and Ziegler [8] shows that F is convex in the real direction, if $\mathrm{Re}\{(zF'(z))/\varphi'(z)\} > 0$ $\forall z \in D$, where $\varphi(z) = \frac{ze^{i\alpha}}{(1-ze^{i\alpha})^2}$ with some $\alpha \in \mathbb{R}$. Thus, if we show this last condition, we are done.

By Lemma 1,

$$zF_1'(z) = \frac{z}{1-ze^{iy_1}} \ast [z(h_2 - e^{-2iy_2}g_2)'(z)].$$

Furthermore,

$$\begin{aligned} z(h_2 - e^{-2iy_2}g_2)'(z) &= z\,\frac{h_2'(z) - e^{-2iy_2}g_2'(z)}{h_2'(z) + e^{-2iy_2}g_2'(z)}\left(h_2'(z) + e^{-2iy_2}g_2'(z)\right) \\ &= z\,\frac{1-\tilde{\omega}(z)}{1+\tilde{\omega}(z)}\left(h_2'(z) + e^{-2iy_2}g_2'(z)\right) \\ &= \frac{zp_2(z)}{(1-e^{iy_2}z)^2}. \end{aligned}$$

Since $|\tilde{\omega}(z)| < 1$ on D and $\tilde{\omega}(0) = 0$, if we let $p_2(z) = \frac{1-\tilde{\omega}(z)}{1+\tilde{\omega}(z)}$ then we have that $\operatorname{Re}\{p_2(z)\} > 0$ for all $z \in D$. Consequently,

$$\begin{aligned} zF_1'(z) &= \frac{z}{1-ze^{iy_1}} \ast \frac{zp_2(z)}{(1-e^{iy_2}z)^2} \\ &= e^{-iy_1}\frac{ze^{iy_1}}{1-ze^{iy_1}} \ast \frac{zp_2(z)}{(1-e^{iy_2}z)^2} \\ &= \frac{zp_2(ze^{iy_1})}{(1-e^{i(y_1+y_2)}z)^2}. \end{aligned}$$

Analogously,

$$zF_2'(z) = \frac{zp_1(ze^{iy_2})}{(1-e^{i(y_1+y_2)}z)^2},$$

where $\operatorname{Re}\{p_1(z)\} > 0$ for all $z \in D$. Thus

$$\operatorname{Re}\left(\frac{e^{i(y_1+y_2)}(zF_1'(z) + zF_2'(z))}{\frac{ze^{i(y_1+y_2)}}{(1-e^{i(y_1+y_2)}z)^2}}\right) = \operatorname{Re}\left(p_1(ze^{iy_2}) + p_2(ze^{iy_1})\right) > 0.$$

This completes the proof. ∎

3. The convolution of f_0 with right half-plane mappings

In Theorems A, B, and 2, we require that the resulting convolution function is locally univalent and sense-preserving. That is,

$$|\omega(z)| = \frac{g'(z)}{h'(z)} < 1 \quad \text{with } h'(z) \neq 0 \;\; \forall z \in D.$$

When is this not a necessary assumption? In the rest of this article we establish cases of these theorems for which this assumption can be omitted.

The following result about the number of zeros of polynomials in the disc is helpful in proving the next several theorems.

Cohn's Rule ([9] or see [10, p. 375]) Given a polynomial

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n$$

of degree n, let

$$f^*(z) = z^n \overline{f(1/\bar{z})} = \overline{a_n} + \overline{a_{n-1}} z + \cdots + \overline{a_0} z^n.$$

Denote by p and s the number of zeros of f inside the unit circle and on it, respectively. If $|a_0| < |a_n|$, then

$$f_1(z) = \frac{\overline{a_n} f(z) - a_0 f^*(z)}{z}$$

is of degree $n-1$ with $p_1 = p-1$ and $s_1 = s$ the number of zeros of f_1 inside the unit circle and on it, respectively.

The main result of this section is the following.

THEOREM 3 Let $f = h + \bar{g} \in K_H^O$ with $h(z) + g(z) = \frac{z}{1-z}$ and $\omega(z) = \frac{g'(z)}{h'(z)} = e^{i\theta} z^n$ ($n \in \mathbb{Z}^+$ and $\theta \in \mathbb{R}$). If $n = 1, 2$, then $f_0 * f \in S_H^O$ and is convex in the direction of the real axis.

Proof Let the dilatation of $f_0 * f$ be given by $\tilde{\omega} = (g_0 * g)'/(h_0 * h)'$. By Theorem A and by Lewy's theorem, we just need to show that $|\tilde{\omega}(z)| < 1$ for all $z \in D$.

First, note that if F is analytic in D and $F(0) = 0$, then from Equation (3)

$$h_0(z) * F(z) = \frac{1}{2}\left[F(z) + zF'(z)\right]$$
$$g_0(z) * F(z) = \frac{1}{2}\left[F(z) - zF'(z)\right]. \qquad (4)$$

Also, since $g'(z) = \omega(z)h'(z)$, we know $g''(z) = \omega(z)h''(z) + \omega'(z)h'(z)$. Hence

$$\tilde{\omega}(z) = -\frac{zg''(z)}{2h'(z) + zh''(z)} = \frac{-z\omega'(z)h'(z) - z\omega(z)h''(z)}{2h'(z) + zh''(z)}. \qquad (5)$$

Using $h(z) + g(z) = \frac{z}{1-z}$ and $g'(z) = \omega(z)h'(z)$, we can solve for $h'(z)$ and $h''(z)$ in terms of z and $\omega(z)$:

$$h'(z) = \frac{1}{(1+\omega(z))(1-z)^2}$$

$$h''(z) = \frac{2(1+\omega(z)) - \omega'(z)(1-z)}{(1+\omega(z))^2(1-z)^3}.$$

Substituting these formulae for h' and h'' into the equation for $\tilde{\omega}$, we derive:

$$\tilde{\omega}(z) = \frac{-z\omega'(z)h'(z) - z\omega(z)h''(z)}{2h'(z) + zh''(z)}$$
$$= -z\frac{\omega^2(z) + [\omega(z) - \frac{1}{2}\omega'(z)z] + \frac{1}{2}\omega'(z)}{1 + [\omega(z) - \frac{1}{2}\omega'(z)z] + \frac{1}{2}\omega'(z)z^2}. \qquad (6)$$

Now, consider the case in which $\omega(z) = e^{i\theta}z$. Then Equation (6) yields

$$\tilde\omega(z) = -ze^{2i\theta}\frac{\left(z^2 + \frac12 e^{-i\theta}z + \frac12 e^{-i\theta}\right)}{\left(1 + \frac12 e^{i\theta}z + \frac12 e^{i\theta}z^2\right)} = -ze^{2i\theta}\frac{p(z)}{q(z)}.$$

Note that $q(z) = z^2\,\overline{p(1/\bar z)}$. In such a situation, if z_0 is a zero of p, then $\frac{1}{\bar z_0}$ is a zero of q. Hence,

$$\tilde\omega(z) = -ze^{2i\theta}\frac{(z+A)(z+B)}{(1+\bar A z)(1+\bar B z)}.$$

It suffices to show that $|A|, |B| \le 1$. We will use Cohn's rule to do this, although the results can be obtained in other ways. Note that

$$p_1(z) = \frac{\bar a_2 p(z) - a_0 p^*(z)}{z} = \frac34 z + \left(\frac12 e^{-i\theta} - \frac14\right).$$

Hence, p_1 has one zero at $z_0 = \frac13 - \frac23 e^{-i\theta} \in \bar{D}$, and so by Cohn's rule p has two zeros, namely A and B, in $\bar{D}$.

Next, consider the case in which $\omega(z) = e^{i\theta}z^2$. In this case,

$$|\tilde\omega(z)| = \left| z^2\frac{z^3 + e^{-i\theta}}{1 + e^{i\theta}z^3}\right| = |z|^2 < 1. \qquad \square$$

Remark 1 If we assume the hypotheses of the previous theorem with the exception that $n \ge 3$, then for some value of $z \in D$, $|\tilde\omega(z)| \ge 1$. To see this, suppose this is not true. Then letting $\omega(z) = -z^n$, Equation (6) yields

$$\tilde\omega(z) = -z^n\frac{z^{n+1} + \left(\frac n2 - 1\right)z - \frac n2}{1 + \left(\frac n2 - 1\right)z^n - \frac n2 z^{n+1}} = -z^n R(z).$$

The function R preserves symmetry about the unit circle, because $|R(e^{ia})| = 1$ and $1/\overline{R(1/\bar z)} = R(z)$. So, R maps the closed unit disc onto itself. Hence, R is a finite Blaschke product of order $n+1$. However, $\frac n2$ is the product of the moduli of the zeros of R in the unit disc. This is a contradiction since $n \ge 3$.

Theorem 4 Let $f = h + \bar g \in K_H^O$ with $h(z) + g(z) = \frac{z}{1-z}$ and $\omega(z) = \frac{z+a}{1+az}$ with $a \in (-1, 1)$. Then $f_0 * f \in S_H^O$ and is convex in the direction of the real axis.

Proof Using Equation (6) with $\omega = \frac{z+a}{1+az}$, where $-1 < a < 1$, we have

$$\tilde\omega(z) = -z\frac{\left(z^2 + \frac{1+3a}{2}z + \frac{1+a}{2}\right)}{\left(1 + \frac{1+3a}{2}z + \frac{1+a}{2}z^2\right)} = -z\frac{f(z)}{f^*(z)} = -z\frac{(z+A)(z+B)}{(1+\bar A z)(1+\bar B z)}.$$

Again using Cohn's rule,

$$f_1(z) = \frac{\bar a_2 f(z) - a_0 f^*(z)}{z} = \frac{(a+3)(1-a)}{4}z + \frac{(1+3a)(1-a)}{4}.$$

So f_1 has one zero at $z_0 = -\frac{1+3a}{a+3}$ which is in the unit circle since $-1 < a < 1$. Thus, $|A|, |B| < 1$. $\square$

Next, we provide some examples.

Example 1 Let $f_1 = h_1 + \overline{g_1}$, where $h_1 + g_1 = \frac{z}{1-z}$ with $\omega = z$. Then

$$h_1 = \frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{1}{2}\frac{z}{1-z}$$

$$g_1 = -\frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{1}{2}\frac{z}{1-z}.$$

Consider $F_1 = f_0 * f_1 = H_1 + \overline{G_1}$. Using Equation (4) we have

$$H_1 = h_0 * h_1 = \frac{1}{2}[h_1(z) + zh_1'(z)] = \frac{1}{8}\log\left(\frac{1+z}{1-z}\right) + \frac{\frac{3}{4}z - \frac{1}{4}z^3}{(1-z)^2(1+z)}$$

$$G_1 = g_0 * g_1 = \frac{1}{2}[g_1(z) - zg_1'(z)] = -\frac{1}{8}\log\left(\frac{1+z}{1-z}\right) + \frac{\frac{1}{4}z - \frac{1}{2}z^2 - \frac{1}{4}z^3}{(1-z)^2(1+z)},$$

and from Equation (6)

$$\omega(z) = -z\left(\frac{2z^2+z+1}{z^2+z+2}\right).$$

We show that F_1 maps the unit disc onto the domain whose boundary consists of the four half-lines given by $\{x \pm \frac{\pi}{8}i, x \le -\frac{1}{4}\}$ and $\{-\frac{1}{4} + iy, |y| \ge \frac{\pi}{8}\}$ (Figure 1). In doing so, we use a similar argument to that used by Clunie and Sheil-Small in Example 5.4 of [1]. We have

$$F_1(z) = \mathrm{Re}\left(\frac{z - \frac{1}{2}z^2 - \frac{1}{2}z^3}{(1+z)(1-z)^2}\right) + i\,\mathrm{Im}\left(\frac{1}{4}\ln\left(\frac{1+z}{1-z}\right) + \frac{1}{2}\frac{z}{(1-z)^2}\right).$$

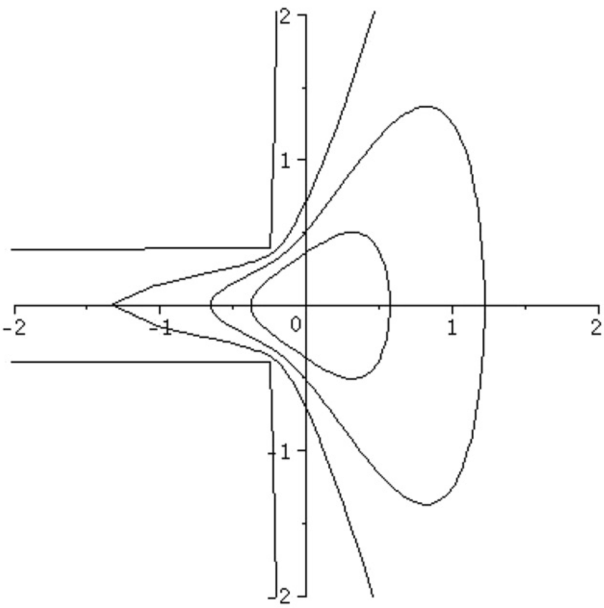

Figure 1. Image of concentric circles inside D under the convolution map $f_0 * f_1 = F_1$.

Applying the transformation $\zeta = \frac{1+z}{1-z} = \xi + i\eta$, $\xi > 0$, we get

$$F_1(z) = \mathrm{Re}\left\{\frac{1}{8}\left(3\zeta - 2 - \frac{1}{\zeta}\right)\right\} + i\,\mathrm{Im}\left\{\frac{1}{4}\left(\ln\zeta + \frac{1}{2}(\zeta^2 - 1)\right)\right\}$$
$$= \frac{1}{8}\left(3\xi - 2 - \frac{\xi}{\xi^2 + \eta^2}\right) + \frac{i}{4}\left(\arctan\frac{\eta}{\xi} + \xi\eta\right).$$

Observe first that the positive real axis $\{\zeta = \xi + i\eta : \eta = 0, \xi > 0\}$ is mapped monotonically onto the whole real axis. Next we find the images of the level curves

$$\arctan\frac{\eta}{\xi} + \xi\eta = c, \quad \xi, \eta > 0.$$

The polar coordinates equations of these level curves are

$$\theta + r^2 \sin\theta\cos\theta = c, \quad 0 < \theta < \frac{\pi}{2}. \qquad (7)$$

Hence

$$\xi = \sqrt{(c-\theta)\cot\theta}$$
$$\eta = \sqrt{(c-\theta)\tan\theta}, \quad 0 < \theta < \min\left\{c, \frac{\pi}{2}\right\}.$$

Fix $c > 0$. Then the image of the curve given in (7) under F_1 is

$$F_1(z) = \frac{1}{8}\left(3\sqrt{(c-\theta)\cot\theta} - 2 - \sqrt{\frac{\sin\theta\cos^3\theta}{c-\theta}}\right) + \frac{i}{4}c$$
$$= u(c,\theta) + \frac{i}{4}c.$$

If $0 < c < \frac{\pi}{2}$, then $\theta \in (0, c)$, and one easily finds that $\lim_{\theta\to 0+} u(c,\theta) = \infty$ and $\lim_{\theta\to c-} u(c,\theta) = -\infty$. The intermediate value property implies that in this case the image of the level curve under F_1 is the entire horizontal line $\{x + \frac{ic}{4} : -\infty < x < \infty\}$. If $c \geq \frac{\pi}{2}$, then $\lim_{\theta\to 0+} u(c,\theta) = \infty$ and $\lim_{\theta\to\pi/2-} u(c,\theta) = -\frac{1}{4}$. So in this case the images of the level curves are horizontal half-lines $\{x + \frac{ic}{4} : -\frac{1}{4} < x < \infty\}$. This means that images of the level curves under F_1 fill the domain whose boundary consists of the real axis and two half-lines $\{x + \frac{\pi}{8}i,\ x \leq -\frac{1}{4}\}$ and $\{-\frac{1}{4} + iy,\ y \geq \frac{\pi}{8}\}$. Finally, our claim follows from the fact that the range of F_1 is symmetric with respect to the real axis.

The images of concentric circles inside D under the harmonic maps f_0 and under f_1 are shown in Figure 2. The images of these concentric circles under the convolution map $f_0 * f_1 = F_1$ are shown in Figure 1.

Example 2 Let $f_2 = h_2 + \overline{g_2}$ be the harmonic mapping in the disc D such that $h_2(z) + g_2(z) = \frac{z}{1-z}$ and $\omega_2(z) = \frac{g_2'(z)}{h_2'(z)} = -z^2$. One can find that

$$h_2(z) = \frac{1}{8}\ln\left(\frac{1+z}{1-z}\right) + \frac{1}{2}\frac{z}{1-z} + \frac{1}{4}\frac{z}{(1-z)^2},$$
$$g_2(z) = -\frac{1}{8}\ln\left(\frac{1+z}{1-z}\right) + \frac{1}{2}\frac{z}{1-z} - \frac{1}{4}\frac{z}{(1-z)^2}$$

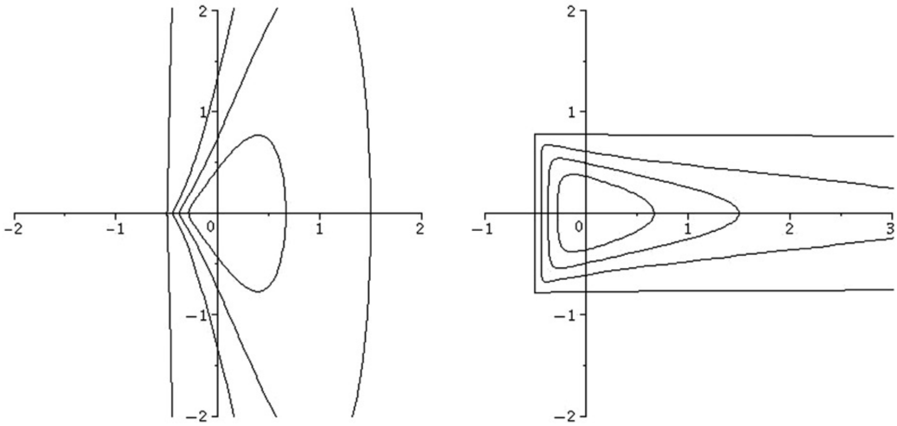

Figure 2. Image of concentric circles inside D under the maps f_0 and f_1, respectively.

and the image of D under f_2 is the right half-plane, $R = \left\{\omega : \mathrm{Re}(\omega) \ge -\frac{1}{2}\right\}$. We note here that $f_2(e^{it}) = -\frac{1}{2} + i\frac{\pi}{16}$, if $0 \le t \le \pi$ and $f_2(e^{it}) = -\frac{1}{2} - i\frac{\pi}{16}$, if $\pi \le t \le 2\pi$. Next let

$$F_2 = h_0 - h_2 + \overline{g_0 - g_2} = H_2 + \overline{G_2}.$$

By Equation (4)

$$H_2(z) = \frac{1}{2}\left\{\frac{1}{8}\ln\left(\frac{1+z}{1-z}\right) + \frac{1}{2}\frac{z}{1-z} + \frac{1}{4}\frac{z}{(1-z)^2} + \frac{z}{(1-z)^3(1+z)}\right\},$$
$$G_2(z) = \frac{1}{2}\left\{-\frac{1}{8}\ln\left(\frac{1+z}{1-z}\right) + \frac{1}{2}\frac{z}{1-z} - \frac{1}{4}\frac{z}{(1-z)^2} + \frac{z^3}{(1-z)^3(1+z)}\right\}$$

and

$$\omega_2(z) = \frac{G_2'(z)}{H_2'(z)} = z^2.$$

Analysis similar to that in Example 1 can be used to show that F_2 maps the disc onto the plane minus two half-lines given by $x \pm \frac{\pi}{16}i$, $x \le -\frac{1}{4}$. We have

$$F_2(z) = \mathrm{Re}\left\{\frac{1}{2}\frac{z(2-z+z^3)}{(1-z)^3(1+z)}\right\} + i\,\mathrm{Im}\left\{\frac{1}{8}\ln\left(\frac{1+z}{1-z}\right) + \frac{6}{8}\frac{z}{(1-z)^2}\right\},$$

which under the same transformation as in Example 1 becomes

$$F_2(z) = \frac{1}{16}\left\{\rho^3 - 3\rho\sigma^2 + 4\rho - 4 - \frac{\rho}{\rho^2+\sigma^2}\right\} + \frac{i}{8}\left\{\arctan\frac{\sigma}{\rho} + 3\rho\sigma\right\}.$$

Analogously, we find that the images of the level curves

$$\theta + \frac{3}{2}r^2\sin 2\theta = c, \quad 0 \le \theta \le \frac{\pi}{2}$$

are

$$F_2(z) = \frac{1}{16}\left[\sqrt{\frac{1}{3}(c-\theta)\cot\theta\,(c-\theta)\left(\frac{1}{3}\cot\theta-\tan\theta\right)+4-\frac{3\sin 2\theta}{2(c-\theta)}}-4\right]+\frac{i}{8}c$$

$$= u(c,\theta)+\frac{i}{8}c.$$

If $0<c<\frac{\pi}{2}$ (or $c>\frac{\pi}{2}$, respectively), then $\lim_{\theta\to c^-} u(c,\theta) = -\infty$ (or $\lim_{\theta\to\frac{\pi}{2}} u(c,\theta) = -\infty$, respectively) and $\lim_{\theta\to 0^+} u(c,\theta) = \infty$. This means that the images of the level curves are entire horizontal lines. If $c = \frac{\pi}{2}$, then $\lim_{\theta\to 0^+} u(\frac{\pi}{2},\theta) = +\infty$ and $\lim_{\theta\to\pi/2^-} u(\frac{\pi}{2},\theta) = -\frac{1}{4}$. So, F_2 maps the first quadrant onto the upper half-plane minus the half-line $\{x+i\frac{\pi}{16} : x \leq -\frac{1}{4}\}$, and the result follows from the symmetry.

4. The convolution of f_0 with vertical strip mappings

In this section we replace right half-plane maps with vertical strip maps and prove the corresponding analogues for Theorems 3 and 4.

THEOREM 5 Let $f = h + \overline{g} \in K_H^O$ with $h(z) + g(z) = \frac{1}{2i\sin\alpha}\log\left(\frac{1+ze^{i\alpha}}{1+ze^{-i\alpha}}\right)$, where $\frac{\pi}{2} \leq \alpha < \pi$ and $\omega(z) = e^{i\theta}z^n$. If $n = 1, 2$, then $f_0 * f \in S_H^O$ and is convex in the direction of the real axis.

Proof By Theorem B we need to establish that $f_0 * f = H + \overline{G}$ is locally univalent. Using $h(z) + g(z) = \frac{1}{2i\sin\alpha}\log\left(\frac{1+ze^{i\alpha}}{1+ze^{-i\alpha}}\right)$ and $g'(z) = \omega(z)h'(z)$, we get

$$h'(z) = \frac{1}{(1+\omega(z))(1+ze^{i\alpha})(1+ze^{-i\alpha})}$$

$$h''(z) = \frac{-[2(\cos\alpha+z)(1+\omega(z)) + \omega'(z)(1+2\cos\alpha z+z^2)]}{(1+\omega(z))^2(1+ze^{i\alpha})^2(1+ze^{-i\alpha})^2}.$$

Substituting these into Equation (5) yields

$$\omega(z) = -z\,\frac{\left(\frac{\omega^2(z)}{1+\cos\alpha z} + \left(\omega(z)-\frac{1}{2}\omega'(z)z\right) - \frac{1}{2}\omega'(z)\left(\frac{1+\cos\alpha z}{\cos\alpha+z}\right)\right)}{-\left(\omega(z)-\frac{1}{2}\omega'(z)z\right)\left(\frac{1+\cos\alpha z}{\cos\alpha+z}\right) + \frac{1}{2}\omega'(z)z^2}. \qquad (8)$$

$$\tilde{\omega} \quad - \frac{\quad}{\cos\alpha+z}$$

First, consider the case in which $\omega(z) = e^{i\theta}z$. We have

$$\tilde\omega(z) = ze^{2i\theta}\frac{z^3+(\cos\alpha+\frac{1}{2}e^{-i\theta})z^2-\frac{1}{2}e^{-i\theta}}{1+(\cos\alpha+\frac{1}{2}e^{i\theta})z-\frac{1}{2}e^{i\theta}z^3}$$

$$= ze^{2i\theta}\frac{f(z)}{f^*(z)}$$

$$= ze^{2i\theta}\frac{(z+A)(z+B)(z+C)}{(1+\overline{A}z)(1+\overline{B}z)(1+\overline{C}z)}.$$

We will show that $A, B, C \in \overline{\mathbb{D}}$. Let

$$\tilde\omega(z) = ze^{2i\theta}\frac{z^3+(\cos\alpha+\frac{1}{2}e^{-i\theta})z^2-\frac{1}{2}e^{-i\theta}}{1+(\cos\alpha+\frac{1}{2}e^{i\theta})z-\frac{1}{2}e^{i\theta}z^3}$$

$$= ze^{2i\theta}\frac{f(z)}{f^*(z)}$$

$$= ze^{2i\theta}\frac{(z+A)(z+B)(z+C)}{(1+\overline{A}z)(1+\overline{B}z)(1+\overline{C}z)},$$

where $\alpha \in (\frac{\pi}{2}, \pi]$, $\beta \in [-\pi, \pi]$. We apply Cohn's rule to $f(z) = z^3 + (\cos\alpha + \frac{1}{2}e^{-i\beta})z^2 - \frac{1}{2}e^{-i\beta}$. Note that $|\frac{1}{2}e^{-i\beta}| = \frac{1}{2} < 1$, thus we get

$$f_1(z) = \frac{\overline{a_3} f(z) - a_0 f^*(z)}{z} = \frac{3}{4}z^2 + \left(\cos\alpha + \frac{1}{2}e^{-i\beta}\right)z + \frac{1}{2}e^{-i\beta}\left(\cos\alpha + \frac{1}{2}e^{i\beta}\right).$$

Since $|\frac{1}{2}e^{-i\beta}(\cos\alpha + \frac{1}{2}e^{i\beta})| \le \frac{1}{2}|\cos\alpha| + \frac{1}{4} < \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ (note that $\alpha \ne \pi$), we can use Cohn's rule again; this time on f_1.

We get

$$f_2(z) = \frac{\frac{3}{4} f_1(z) - \frac{1}{2}e^{-i\beta}(\cos\alpha + \frac{1}{2}e^{i\beta}) f_1^*(z)}{z}$$
$$= \left(\frac{9}{16} - \frac{1}{4}\left|\cos\alpha + \frac{1}{2}e^{i\beta}\right|^2\right)z + \frac{3}{4}\left(\cos\alpha + \frac{1}{2}e^{-i\beta}\right) - \frac{1}{2}e^{-i\beta}\left(\cos\alpha + \frac{1}{2}e^{i\beta}\right)^2.$$

Clearly f_2 has one zero at

$$z = \frac{-\frac{3}{4}\left(\cos\alpha + \frac{1}{2}e^{-i\beta}\right) + \frac{1}{2}e^{-i\beta}\left(\cos\alpha + \frac{1}{2}e^{i\beta}\right)^2}{\frac{9}{16} - \frac{1}{4}\left|\cos\alpha + \frac{1}{2}e^{i\beta}\right|^2} = \frac{-\frac{1}{4}\cos\alpha + \frac{1}{2}e^{-i\beta}\cos^2\alpha - \frac{3}{8}e^{-i\beta} + \frac{1}{8}e^{i\beta}}{\frac{1}{2} - \frac{1}{4}\cos^2\alpha - \frac{1}{4}\cos\alpha\cos\beta}.$$

We show that $|z| \le 1$, or equivalently,

$$\left|-\frac{1}{4}\cos\alpha + \frac{1}{2}e^{-i\beta}\cos^2\alpha - \frac{3}{8}e^{-i\beta} + \frac{1}{8}e^{i\beta}\right|^2 \le \left(\frac{1}{2} - \frac{1}{4}\cos^2\alpha - \frac{1}{4}\cos\alpha\cos\beta\right)^2.$$

If we put $x = \cos\alpha$, $y = \cos\beta$, then $x \in (-1, 0]$, $y \in [-1, 1]$ and the above inequality becomes

$$-\frac{3}{16}x^4 + \frac{3}{16}x^2 + \frac{6}{16}x^3y - \frac{6}{16}xy - \frac{3}{16}x^2y^2 + \frac{3}{16}y^2 = \frac{3}{16}(1 - x^2)(x - y)^2 \ge 0.$$

Therefore, by Cohn's rule, f has all its 3 zeros in $\overline{D}$, that is $A, B, C \in \overline{D}$ and so $|\omega(z)| < 1$ for all $z \in D$.

Next, consider the case in which $\omega(z) = e^{i\beta}z^2$. In this case,

$$\omega(z) = -z^2e^{i\beta}\,\frac{\left(e^{i\beta}z^3 - \left(\frac{1+\cos\alpha z}{\cos\alpha + z}\right)\right)}{\left(-\frac{1+\cos\alpha z}{\cos\alpha + z} + e^{i\beta}z^3\right)} = -z^2e^{i\beta}.$$

Hence, $|\omega(z)| < 1$. ∎

In proving the last theorem, we will use the following corollary of the Schur–Cohn algorithm.

Corollary to the Schur–Cohn Algorithm [10, p. 383] Given a polynomial

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n$$

of degree n, let

$$M_k = \det\begin{pmatrix} B_k & A_k \\ A_k & B_k \end{pmatrix} \quad (k = 1, \ldots, n),$$

where $A^* = \overline{A}^{\top}$ is the conjugate transpose of A, and A and B are the triangular matrices

$$A = \begin{pmatrix} a_0 & a_1 & \cdots & a_{k-1} \\ & a_0 & \cdots & a_{k-2} \\ & & \ddots & \vdots \\ & & & a_0 \end{pmatrix}, \quad B = \begin{pmatrix} \bar{a}_n & \bar{a}_{n-1} & \cdots & \bar{a}_{n-k+1} \\ & \bar{a}_n & \cdots & \bar{a}_{n-k+2} \\ & & \ddots & \vdots \\ & & & \bar{a}_n \end{pmatrix}.$$

The f has all of its zeros inside the unit circle if and only if the determinants $M_1, \ldots, M_n$ are all positive.

THEOREM 6 Let $f = h + \bar{g} \in K_H^O$ with $h(z) + g(z) = \frac{1}{2i\sin\alpha}\log\left(\frac{1+ze^{i\alpha}}{1+ze^{-i\alpha}}\right)$, where $\frac{\pi}{2} \le \alpha < \pi$ and $\omega(z) = \frac{z+a}{1+az}$ with $a \in [0,1)$. Then $f_0 * f \in S_H^O$ and is convex in the direction of the real axis.

Proof Using Equation (8) with $\omega = \frac{z+a}{1+az}$ and simplifying, we have

$$\tilde{\omega}(z) = z\frac{z^3 + (\frac{1}{2} + \frac{3}{2}a + \cos\alpha)z^2 + (a + 2a\cos\alpha)z + (a\cos\alpha + \frac{1}{2}a - \frac{1}{2})}{1 + (\frac{1}{2} + \frac{3}{2}a + \cos\alpha)z + (a + 2a\cos\alpha)z^2 + (a\cos\alpha + \frac{1}{2}a - \frac{1}{2})z^3}$$

$$= -z\frac{f(z)}{f^*(z)}$$

$$= -z\frac{(z+A)(z+B)(z+C)}{(1+Az)(1+Bz)(1+Cz)}.$$

By the corollary to the Schur–Cohn algorithm, we need to show that the determinants M_1, M_2, M_3 are all positive (for convenience, let $\cos\alpha = x$; so $-1 < x \le 0$ and $-1 < a < 1$):

$$M_1 = \det\begin{pmatrix} a_3 & a_0 \\ \bar{a}_0 & \bar{a}_3 \end{pmatrix} = \det\begin{pmatrix} 1 & ax + \frac{1}{2}a - \frac{1}{2} \\ ax + \frac{1}{2}a - \frac{1}{2} & 1 \end{pmatrix}$$

$$= \frac{1}{4}(2ax + a + 1)(3 - 2ax - a) > 0,$$

$$M_2 = \det\begin{pmatrix} a_3 & 0 & a_0 & a_1 \\ a_2 & a_3 & 0 & a_0 \\ \bar{a}_0 & 0 & \bar{a}_3 & \bar{a}_2 \\ \bar{a}_1 & \bar{a}_0 & 0 & \bar{a}_3 \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & 0 & ax + \frac{1}{2}a - \frac{1}{2} & a + 2ax \\ \frac{1}{2} + \frac{3}{2}a + x & 1 & 0 & ax + \frac{1}{2}a - \frac{1}{2} \\ ax + \frac{1}{2}a - \frac{1}{2} & 0 & 1 & \frac{1}{2} + \frac{3}{2}a + x \\ a + 2ax & ax + \frac{1}{2}a - \frac{1}{2} & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{4}(1 - x)(1 - a)(1 - 2ax - a)(2 + 4ax + 4a + x - 2a^2x^2 - 5a^2x - 2a^2 - 2ax^2) > 0,$$

if $P(a, x) = 2 + 4ax + 4a + x - 2a^2x^2 - 5a^2x - 2a^2 - 2ax^2 > 0$. We will show that $P(x, a) > 0$ for $0 \le a < 1$ and $-1 < x \le 0$ (although it seems to be true for $-\frac{1}{3} < a < 1$). Now

$$\frac{\partial}{\partial x}P(a, x) = 4a + 1 - 4a^2x - 5a^2 - 4ax = \left[4a(1 - x) - 4a^2(1 + x)\right] + \left[1 - a^2\right] > 0,$$

since $0<a<1$ and $-1<x\le 0$. Assume $a=a_0>0$ is fixed. Then, $P(a_0,x)$ is increasing and attains its minimum at $x=-1$. Thus,

$$P(a_0,x)>P(a_0,-1)=(a_0-1)^2>0.$$

Note, $P(0,x)=2+x>0$.

$$M_3=\det\begin{pmatrix} a_3 & 0 & 0 & a_0 & a_1 & a_2\\ a_2 & a_3 & 0 & 0 & a_0 & a_1\\ a_1 & a_2 & a_3 & 0 & 0 & a_0\\ \overline{a}_0 & 0 & 0 & \overline{a}_3 & \overline{a}_2 & \overline{a}_1\\ \overline{a}_1 & \overline{a}_0 & 0 & 0 & \overline{a}_3 & \overline{a}_2\\ \overline{a}_2 & \overline{a}_1 & \overline{a}_0 & 0 & 0 & \overline{a}_3 \end{pmatrix}$$

$$=\det\begin{pmatrix} 1 & 0 & 0 & ax+\frac{1}{2}a-\frac{1}{2} & a+2ax & \frac{1}{2}+\frac{3}{2}a+x\\ \frac{1}{2}+\frac{3}{2}a+x & 1 & 0 & 0 & ax+\frac{1}{2}a-\frac{1}{2} & a+2ax\\ \frac{1}{2}+\frac{3}{2}a+x & a+2ax & 1 & 0 & 0 & ax+\frac{1}{2}a-\frac{1}{2}\\ ax+\frac{1}{2}a-\frac{1}{2} & 0 & 0 & 1 & \frac{1}{2}+\frac{3}{2}a+x & a+2ax\\ a+2ax & ax+\frac{1}{2}a-\frac{1}{2} & 0 & 0 & 1 & \frac{1}{2}+\frac{3}{2}a+x\\ \frac{1}{2}+\frac{3}{2}a+x & a+2ax & ax+\frac{1}{2}a-\frac{1}{2} & 0 & 0 & 1 \end{pmatrix}$$

$$=\frac{1}{4}(x+1)(1-x)^3(1-a)^3(1-2ax-a)^2(1+3a)>0.$$

Therefore, $A,B,C\in D$ and $|\omega(z)|<1$ for all $z\in D$. ∎

Remark 2 Unlike Theorem 4, this result does not hold for $-1<a<-\frac{1}{3}$ since $M_3<0$ for these values of a.

Example 3 Let $f_3=h_3+\overline{g_3}$, where $h_3+g_3=\frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right)$ (that is, $\alpha=\frac{\pi}{2}$ in Theorem 5) with $\omega=-z^2$. Then

$$h_3=\frac{1}{4}\log\left(\frac{1+z}{1-z}\right)-\frac{i}{4}\log\left(\frac{1+iz}{1-iz}\right),$$
$$g_3=-\frac{1}{4}\log\left(\frac{1+z}{1-z}\right)-\frac{i}{4}\log\left(\frac{1+iz}{1-iz}\right).$$

Consider $F_3=f_0*f_3=H_3+\overline{G_3}$. From Equation (4) we derive

$$H_3=h_0*h_3=\frac{1}{2}[h_3(z)+zh_3'(z)]=\frac{1}{8}\log\left(\frac{1+z}{1-z}\right)-\frac{i}{8}\log\left(\frac{1+iz}{1-iz}\right)+\frac{1}{2}\frac{z}{1-z^4}$$
$$G_3=g_0*g_3=\frac{1}{2}[g_3(z)-zg_3'(z)]=-\frac{1}{8}\log\left(\frac{1+z}{1-z}\right)-\frac{i}{8}\log\left(\frac{1+iz}{1-iz}\right)+\frac{1}{2}\frac{z^3}{1-z^4}.$$

From Equation (8), $\tilde{\omega}(z)=z^2$.

We now show that the image of the first quadrant of D under the mapping F_3 is the domain whose boundary consists of the positive real axis, upper imaginary axis and the lines $\{\frac{\pi}{8}+iy,\ y\in\mathbb{R}\}$, $\{x+\frac{\pi}{8}i,\ x\in\mathbb{R}\}$. We have

$$F_3(z)=\mathrm{Re}\left(-\frac{i}{4}\log\left(\frac{1+iz}{1-iz}\right)+\frac{1}{2}\frac{z}{1-z^2}\right)+i\,\mathrm{Im}\left(\frac{1}{4}\log\left(\frac{1+z}{1-z}\right)+\frac{1}{2}\frac{z}{1+z^2}\right).$$

As in the previous two examples, we use the transformation $\zeta = \frac{1+z}{1-z} = \xi + i\eta$, $\xi > 0$, This transformation maps the part of the disc in the first quadrant onto the exterior of the unit disc contained in the first quadrant, and we note that the interval $[0, i)$ is mapped onto the quarter of the unit circle. If we put $\zeta = re^{i\theta}$, $r \geq 1$, $\theta \in [0, \pi/2)$, then we get

$$\operatorname{Re} F_3(z) = \frac{1}{4}\left(\arctan \frac{r - \frac{1}{r}}{2\cos\theta} + \frac{1}{2}\left(r - \frac{1}{r}\right)\cos\theta \right)$$

$$\operatorname{Im} F_3(z) = \frac{1}{4}\left(\theta + \frac{2\sin 2\theta}{\left(r - \frac{1}{r}\right)^2 + 4\cos^2\theta} \right).$$

One can see that the image of the quarter of the unit circle in the first quadrant in the ζ-plane under F_3 is the upper imaginary axis and the image of the line $\zeta > 1$ is the positive real axis. Now we consider the level curves

$$\theta + \frac{2\sin 2\theta}{\left(r - \frac{1}{r}\right)^2 + 4\cos^2\theta} = c, \quad c > 0.$$

Since $r > 1$ and $\theta \in (0, \pi/2)$, from above we get

$$r - \frac{1}{r} = 2\cos\theta \sqrt{\frac{\tan\theta}{c - \theta} - 1}. \qquad (9)$$

Let $\theta_c \in (0, \pi/2)$ be the number satisfying the equation $\tan\theta_c = c - \theta_c$. If $0 < c < \pi/2$, we assume that $\theta_c < \theta < c$, while if $c \geq \pi/2$, we assume that $\theta_c < \theta < \pi/2$.

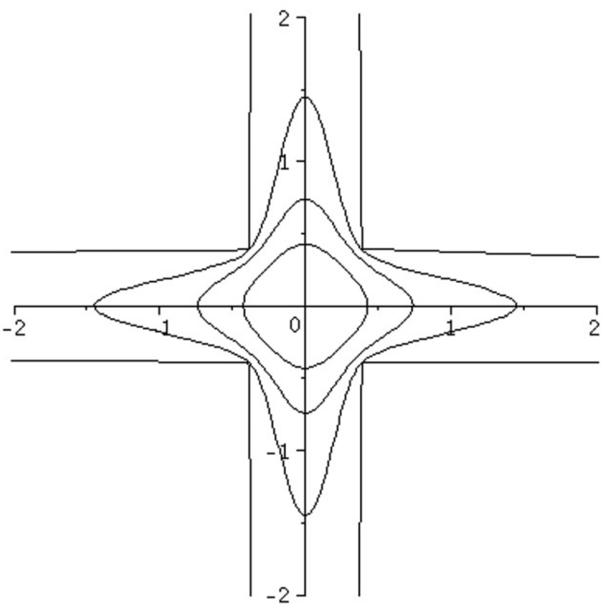

Figure 3. Image of concentric circles inside D under the convolution map $f_0 * f_3 = F_3$.

Using Equation (9) we find that on the level curve we have

$$\operatorname{Re} F_3 = \frac{1}{4}\left(\arctan\sqrt{\frac{\tan\theta}{\pi-\theta}-1} + \cos^2\theta\sqrt{\frac{\tan\theta}{\pi-\theta}-1}\right).$$

Using an analysis similar to the one in the previous examples, we get the result.

The images of concentric circles inside D under the convolution map $f_0 * f_3 = F_3$ are shown in Figure 3.

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