MASTER'S EXAMINATION IN MATHEMATICS
Saturday, 20 January 2001

INSTRUCTIONS.

Answer a total of eight questions, with at most two from Basic Analysis and at most two from Basic Algebra.

I. Basic Analysis

II. Basic Algebra

III. Analysis

IV. Advanced Algebra

V. Applied Mathematics

VI. Topology

PLEASE WRITE YOUR ANSWER TO EACH QUESTION ON A SEPARATE PAGE. USE $8\frac{1}{2} \times 11$ INCH PAPER. PLEASE WRITE YOUR NAME AT THE TOP OF EACH PAGE. THIS WILL MAKE IT EASIER TO SEPARATE AND SORT THESE PAGES FOR GRADING AND REASSEMBLING THEM AFTERWARDS.

I. BASIC ANALYSIS

1. Do one of the following:

   (a) Prove the intermediate value theorem, i.e. if $f : [a, b] \to \mathbb{R}$ is continuous and d is between $f(a)$ and $f(b)$, then $\exists\, c \in (a, b)$ with $f(c) = d$.

   (b) Let A be a compact subset of $\mathbb{R}^n$ and $\boldsymbol{v} \in \mathbb{R}$. Use the Extreme Value Theorem to prove that there is $\boldsymbol{a}_0 \in A$ such that $\|\boldsymbol{a}_0 - \boldsymbol{v}\| \leq \|\boldsymbol{a} - \boldsymbol{v}\|$ for any $\boldsymbol{a} \in A$. Is this point $\boldsymbol{a}_0$ unique?

2. Do one of the following:

   (a) Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be continuous functions with $f(x) \leq g(x)$ for all $x \in [a, b]$. Prove that $\displaystyle\int_a^b f < \int_a^b g$ if and only if there is $x \in [a, b]$ with $f(x) < g(x)$.

   (b) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ with $\|f(\boldsymbol{x}) - f(\boldsymbol{y})\| \leq \dfrac{\|\boldsymbol{x} - \boldsymbol{y}\|}{2}$. Prove that f a unique fixed point, i.e. ($\boldsymbol{x}$ with $f(\boldsymbol{x}) = \boldsymbol{x}$).

3. Do one of the following:

   (a) Let $a > 0$. Prove that $a^{\frac{1}{n}}$ converges. By considering the subsequence $\left(a^{\frac{1}{2n}}\right)$, show that $a^{\frac{1}{n}} \to 1$.

   (b) If $A \subseteq \mathbb{R}^n$ is connected, show that $\bar{A}$ is connected. Give an example where $\bar{A}$ is connected but A is not.

4. (a) State and prove the Root test for series.

   (b) Consider the equation

$$e^{2x-y} + \cos(x^2 + xy) - 2 - 2y = 0.$$

   Does the set of solutions implicitly define one of x, y as a function of the other in a neighborhood of the solution $(0, 0)$? If so, compute the derivative of this function (these functions) at the point $(0, 0)$.

II. BASIC ALGEBRA

1. For each prime p, give an example of an infinite field of characteristic p.

2. Let F and K be fields with $F \subseteq K$. Assume K is finite-dimensional as an F-vector space. Let R be a subring of K which contains F. Prove that R is a field.

3. Prove that all real matrices of the form $\begin{bmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 3 \end{bmatrix}$ are similar to each other.

4. A troop of 17 monkeys store their bananas in 11 piles of equal size with a twelfth pile of 6 left over. When they divide their bananas into 17 equal groups, none remain. What is the smallest number of bananas they can have?

III. ANALYSIS

1. Let U be an open subset of $\mathbb{R}^n$ and let $\mathbf{f} : U \to \mathbb{R}^m$. Tell what it means for $\mathbf{f}$ to have a derivative at $\mathbf{x} \in U$ and prove or disprove the assertion that $\mathbf{f}$ is continuous at $\mathbf{x}$ whenever $\mathbf{f}$ is differentiable at $\mathbf{x}$.

2. Suppose E_k is a measurable set and $\sum_{k=1}^{\infty} \mu(E_k) < \infty$. Show that the set,
$$A \equiv \{\omega \in \Omega : \omega \in E_k \text{ infinitely often}\}$$
has measure zero and is a measurable set. **Hint:** Write the set of interest in terms of countable intersections and unions of the sets, E_k.

3. Show that if f is a function in $L^1(\Omega)$ for $(\Omega, \mu, \mathcal{S})$ a measure space, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mu(E) < \delta$, then
$$\int_E |f| \, d\mu < \varepsilon.$$

4. If S is an uncountable set of irrational numbers, is it necessary that S has a rational number as a limit point? **Hint:** What about the Lebesgue measure of a countable set?

5. Let $\frac{1}{p} + \frac{1}{p'} = 1,\ p > 1,\ $ let $f \in L^p(\mathbb{R}),\ g \in L^{p'}(\mathbb{R})$. Show $f * g$ given by
$$f * g(x) \equiv \int f(x - y) \, g(y) \, dy$$
is uniformly continuous on $\mathbb{R}$ and $|(f * g)(x)| \leq ||f||_{L^p} ||g||_{L^{p'}}$. The measure is Lebesgue measure.

IV. ADVANCED ALGEBRA

1. Show that every vector space has a basis.

2. Give an example (with proof) of a field extension which is not separable.

3. Show that $\mathbb{Q}(\sqrt{2})$ is not isomorphic to $\mathbb{Q}(\sqrt{3})$.

4. Calculate (with proof) the Galois group over $\mathbb{Q}$ of $x^3 - 2$.

V. APPLIED MATHEMATICS

1. Suppose that f is a continuous real-valued function defined on the real line satisfying: $f(0) = 4, f(2) = 0$, and $f(3) = 2$. Prove that f has an orbit of prime period three.

2. Here is a reasonable dynamical systems question. Find the equilibria and determine their stability for the dynamical system

$$\frac{du}{dt} = Au - Buv$$
$$\frac{dv}{dt} = -Cu + Duv$$

   where A, B, C, D are positive constants. There may be several cases to consider. In each case draw the phase portrait for the system.

3. Let X be the space of continuous real-valued functions on $[0, 1]$. Define a bilinear form on X by

$$\langle f, g\rangle = \int_0^1 xf(x)g(x)dx.$$

   (a) Prove that this defines an inner product on X and give the associated norm.

   (b) Let $Y = \text{span}\{1, x, x^2\} \subset X$. Find the element of Y which is closest to x^3 with respect to this norm.

4. Find all the eigenvalues and eigenfunctions of the differential operator

$$Lu \equiv \frac{d^2u}{dx^2} - 2\frac{du}{dx}$$

   with boundary conditions $u(0) = 0, u(1) = 0$.

VI. TOPOLOGY

1. Let$R$ be the real numbers. Show that the following properties are equivalent.

   (a) R is connected.

   (b) Every nonempty set which is bounded above has a least upper bound.

   (c) Every bounded increasing sequence converges.

2. Let $\{U_\alpha\}$ be a collection of open sets in the real line which covers a set A. Show that a countable subcollection of $\{U_\alpha\}$ covers A.

3. Let $f_n : X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If (f_n) converges uniformly to the function f, show f is continuous.

4. Let X be a locally compact Hausdorff space which is not compact.

   (a) Define the *one-point compactification* of X.

   (b) Show that X is a subspace of its one-point compactification.

   (c) Show that the one-point compactification is a compact Hausdorff space.