

MASTER'S EXAMINATION IN MATHEMATICS EDUCATION  
Saturday, 12 January 2002

**INSTRUCTIONS.**

Answer a total of eight questions, with the restriction of exactly two questions from Mathematics Education and at most two questions from any other category.

- I. Basic Analysis
- II. Basic Algebra
- III. Analysis
- IV. Advanced Algebra
- V. Applied Mathematics
- VI. Topology
- VII. Mathematics Education

PLEASE WRITE YOUR ANSWER TO EACH QUESTION ON A SEPARATE PAGE. USE  $8\frac{1}{2} \times 11$  INCH PAPER. PLEASE WRITE YOUR NAME AT THE TOP OF EACH PAGE. THIS WILL MAKE IT EASIER TO SEPARATE AND SORT THESE PAGES FOR GRADING AND REASSEMBLING THEM AFTERWARDS.

## I. BASIC ANALYSIS

1. Prove that Cauchy sequences converge. You may use the fact that every bounded sequence has a convergent subsequence.
2. Discuss the convergence of

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left( \frac{x+1}{x-3} \right)^n.$$

3. Show that a non-singleton metric space with a countable number of points is not connected.
4. Prove that if  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  has the property that for some  $0 \leq c < 1$

$$\|f(x) - f(y)\| \leq c\|x - y\| \quad \forall x, y \in \mathbb{R}^n$$

then  $\exists x_0 \in \mathbb{R}^n$  with  $f(x_0) = x_0$ .

## II. GROUPS AND LINEAR ALGEBRA

1. If  $G$  is a group, show that if  $H$  is a normal subgroup of  $G$ , then the multiplication  $Hg_1Hg_2 = Hg_1g_2$  is well defined.

Find an example of a group  $G$  with a subgroup  $H$  for which this multiplication would not be well defined.

2. Show that every finite group is isomorphic to a subgroup of  $S_n$ , the group of permutations of  $\{1, 2, \dots, n\}$ .

Show that  $A_n$ , the group of even permutations is a normal subgroup of  $S_n$ .

3. Let  $A$  be an  $m \times n$  matrix. Show that if  $\mathbf{x} \in \mathbb{R}^n$  and  $\mathbf{b} \in \mathbb{R}^m$ , then each solution of  $A\mathbf{x} = \mathbf{b}$  is of the form  $\mathbf{x}_0 + \mathbf{y}$  where  $\mathbf{x}_0$  is fixed and  $\mathbf{y}$  is a solution of  $A\mathbf{y} = \mathbf{0}$ .

Show that if there is such a solution, then  $\mathbf{b} \in$  column space of  $A$ .

4. Let  $A$  be an  $n \times n$  matrix with eigenvalues  $\lambda_1, \dots, \lambda_n$ . If these are all distinct, prove that  $A$  is similar to a diagonal matrix. State clearly any result that you use.

### III. ANALYSIS

1. Show that if  $(X, \mathcal{M}, \mu)$  is a measure space and  $1 \leq p \leq q \leq r < \infty$ , then

$$L^p(X, \mathcal{M}, \mu) \cap L^r(X, \mathcal{M}, \mu) \subseteq L^q(X, \mathcal{M}, \mu).$$

2. Let  $\mathcal{L}$  be the collection of Lebesgue measurable subsets of  $\mathbb{R}$ , let  $\lambda$  be Lebesgue measure, and let  $f \in L^1(\mathbb{R}, \mathcal{L}, \lambda)$  be given. Show that the function  $\tau : \mathbb{R} \rightarrow L^1(\mathbb{R}, \mathcal{L}, \lambda)$  defined by  $\tau(s) = f(\cdot + s)$  (i.e.,  $\tau(s)(x) = f(x + s)$ ) is continuous.
3. State Lebesgue's Dominated Convergence Theorem, and prove it using Fatou's Lemma.
4. Let  $X$  be a set and  $\mathcal{M}$  a  $\sigma$ -algebra of subsets of  $X$ .
  - (a) Give a standard definition of what it means for a function  $f : X \rightarrow \mathbb{R}$  to be  $\mathcal{M}$ -measurable.
  - (b) Use the definition from part a to prove that if  $f : X \rightarrow \mathbb{R}$  and  $g : X \rightarrow \mathbb{R}$  are  $\mathcal{M}$ -measurable then  $f + g$  is  $\mathcal{M}$ -measurable

#### IV. FIELDS, RINGS AND MODULES

1. Does the polynomial  $x^3 - x + 1$  have a root in the field with 81 elements? Justify your answer.
2. Let  $R$  be a commutative ring with 1, and let  $S$  be a subset of  $R$  with the following properties:
  - (a)  $1 \in S$ .
  - (b) 0 is not an element of  $S$ .
  - (c)  $S$  is closed under multiplication.

Prove that there exists a prime ideal of  $R$  which does not meet  $S$ . (Hint: Use Zorn's lemma.)

3. Let  $R$  be a commutative ring with 1,  $M$  and  $P$  finitely generated  $R$ -modules. Suppose we have a surjective  $R$ -module homomorphism  $f : M \rightarrow P$ .
  - (a) If  $P$  is projective, show that the kernel of  $f$  is finitely generated.
  - (b) Give an example to show that  $\ker f$  need not be finitely generated if  $P$  is not projective.
4. Let  $R$  be a commutative ring with 1. Let  $M$  be a Noetherian  $R$ -module, and let  $f : M \rightarrow M$  be a surjective  $R$ -module homomorphism. Prove that  $f$  is an isomorphism.

## V. APPLIED MATHEMATICS

1. The system

$$\begin{aligned}\frac{dx}{dt} &= ax - bxy \\ \frac{dy}{dt} &= -cy + dxy\end{aligned}$$

describes the evolution of certain predator-prey populations. Assume  $a, b, c, d$  are the positive constants (respectively)  $\frac{1}{2}, 1, 2, 1$ . Find the steady-states of the system (equilibria) and discuss them—which are stable? Are solutions periodic nearby any of the equilibria? Observe appropriate theorems as needed.

2. The following equations describe heat transfer in a large slab of homogeneous material of thickness  $a$  where the faces of the slab are held at temperature  $T$ . Find the temperature function  $u(x, t)$ . Assume that  $f(a) = f(0)$ ,  $k$  and  $T$  are constants. (State appropriate convergence theorems as needed.)

$$\begin{aligned}\frac{\partial u}{\partial t} &= k \frac{\partial^2 u}{\partial x^2} \\ u(0, t) &= T \\ u(a, t) &= T \\ u(x, 0) &= f(x)\end{aligned}$$

3. The following matrix  $A$  arises as the coefficient matrix of a system of ordinary differential equations,  $\frac{dx}{dt} = Ax$ . Find  $e^{At}$  by any method.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

4. (a) State Gauss's theorem (the divergence theorem) in standard vector notation.  
(b) State Stokes' theorem.  
(c) Verify Stokes' theorem for the velocity field  $\mathbf{v} = xz\mathbf{j}$  for the surface  $z = 4 - y^2$  cut off by the planes  $y = x, x = 0, y = 0$ .

## VI. TOPOLOGY

1. Show that if  $X$  is a topological space with a countable basis, then every uncountable set  $A \subset X$  has a limit point  $P$  which is an element of  $A$ .
2. Let  $X$  be a linearly ordered space with the linear topology. Show that  $X$  is regular. That is: for each  $x \in X$  and each closed  $A \subset X$  with  $x \notin A$ , show that there are disjoint open sets  $U$  and  $V$  such that  $x \in U$  and  $A \subset V$ .
3. Show that the product of regular spaces is regular.
4. If  $f$  is a continuous transformation of a compact metric space,  $A$ , into a metric space  $B$ , then  $f$  is uniformly continuous.

## VIII. MATHEMATICS EDUCATION

Write responses to any two of the following.

1. Describe and discuss qualitative research in mathematics education. Purposes, methodologies, theoretical frameworks, use, and importance might be among aspects addressed.
2. Describe “constructivism” and “symbolic interactionism” and discuss how each views mathematical knowledge and learning. Give examples to support your assertions.
3. Characterize some central issues in mathematics education relating to classrooms, students and teachers. Provide examples where appropriate and support your assertions.
4. Characterize the role of symbolizing (including representation, modeling, and metaphor) in mathematical thinking, learning and communication.
5. Analyze the following episode<sup>1</sup>.

Students in a fifth grade class were given the task:

*At Pizza Hut each large pizza is cut into twelve slices. Mrs. Elson ordered two large pizzas. Seven students from Mrs. Elson’s class are to eat one piece from each of the pizzas. What fraction of the two pizzas was eaten?*

Students in the classroom worked together in groups of two on this problem while the teacher circulated from group to group. Brian and Scott, together in a group, had, in fact, arrived at two different solutions based on very different approaches. Brian built a representation of the problem situation using Pattern Blocks to model the pizzas (two yellow hexagons were one pizza with six green equilateral triangles making up one hexagon.) He built a solution of  $14/24$ . Scott, meanwhile, searched for a paper-and-pencil procedure or a rule that he could apply to the situation. He came up with  $14/12$  as the solution.

When the teacher turns her attention to this group, Brian explains his solution. [In the transcript that follows, B is Brian, S is Scott and T is the teacher.]

B: So there’s twenty-four slices in both pizzas, so Mrs. E. wants seven students. . . she took seven students to Pizza Hut, so . . . she’s gonna give ’em one slice from each pizza so we would have, uh, fourteen out of the twenty-four, right, slices.

T: All right, now let me ask you this. How do you get twenty-four slices in the one pizza, and twelve slices in the other?

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<sup>1</sup>See Maher, C. A., Davis, R.B. (1990). Teacher’s learning: Building representations of children’s mathematics. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics* (pp. 79-92). Reston, VA: NCTM

B: In all.

T: Brian.

B: There's twelve slices in the one pizza, and twelve slices in the other.

T: All right, we should think of them as two separate pizzas, though, right? [Scott nods affirmatively.]

B: Yeah. . .

T: OK, you, you're putting your two pizzas together and making one . . . gi-huge-ic pizza.

S: [laughing] Gi-huge-ic pizza

T: OK, we can't have one gi-huge-ic pizza because there isn't a box that could carry it in to take it home.

B: No, just stick it in [mumbles] . . . slices.

T: We have to keep it separate. They have to go in two separate boxes.

[The teacher uses Brian's pizza model and, moving some of the pieces, asks where were the twelve slices.]

S: But my theory was wrong.

B: Yeah. . .

T: Hmm. Now it says we're gonna get one piece from each of the two pizzas

S: [yawns] We're gonna need seven pieces.

B: We're gonna hafta get more than that.

T: Well, one student gets one piece from each pizza, [Scott nods], and there's seven students. . .

S: Yeah, and they're each gonna get seven.

T: . . . and each getting that, right?

B: [as he is still working] But, they could have more slices than that, cause there's gonna be more slices left over.

S: Yeah, but then it would, it would be equal—but then there would be a slice left over for Mrs. Elson.

B: [to teacher] So do we have to count Mrs. Elson?

T: No, we just have to count what it says in the, in the problem.

B: [arranging the blocks] Scott, I think I need a little more than this.

S: Oh, OK, I think we can handle that. [Together they reach for the blocks.]

B: I think I'm really gonna need a little less than this.

T: OK, now show me what... what's... who's getting what here.

B: OK, we counted the kids from this class, like... [Brian repeats the assignment of names for pizza slices for the seven students.]

T: Uh-huh.

B: So. [Brian counts seven pieces with Scott's help and removes them.]

T: Uh-huh.

B: So, these would be gone.

T: So that's their, their pieces from pizza number one?

B: Yeah. So then...

S: Then their pieces from number two...

B: Some left over but then she wants to take from this pizza [points]. So here's the... six kids and [inaudible while removing the blocks.]

T: OK.

B: So this would be left.

T: [points to the model] So these are all the slices that are being given to the, the kids?

B: Yeah.

T: The seven kids. OK, so it's that much. All right, now let's see if we can count up how many slices?

B: Fourteen.

S: [simultaneously] Fourteen.

T: Fourteen, and how many slices made one whole pizza?

S: Seven.

T: No, how many slices. [pointing to a yellow hexagon]

B: There're twelve.

S: There're twelve, yeah, there're twelve.

T: ... made a pizza. So we're using up fourteen twelfths, right?

B: [simultaneously with Scott] Twelfths.

T: Lets put down fourteen twelfths.

B: So it would be one...

T: Alright.

B: ... and two twelfths.

S: Equals one.

T: It says [reading the problem] "what fraction." So is what we figured out [pointing], fourteen twelfths? So let's leave that answer.

B: OK.

In the teacher's later analysis of this lesson, she wrote,

After noticing a need to further develop the concept of mixed numbers, this sheet [task] was created. Scott saw right away that the  $14/12$  was eaten, but Brian decided this was  $14/24$ .

It is interesting to note that in separate interviews conducted one year later, both Brian and Scott remembered the task, but neither could recall the solution they had found or the controversy about the solution. Both, however, constructed an answer of  $14/24$ —Scott by building with pattern blocks and Brian by drawing pizzas on a sheet of paper.