MASTER'S EXAMINATION IN MATHEMATICS
Saturday, 14 September 2002

INSTRUCTIONS.

Answer a total of six questions, with at most two from Basic Analysis.

I. Basic Analysis

II. Groups and Linear Algebra

III. Analysis

IV. Fields, Rings and Modules

V. Applied Mathematics

VI. Topology

PLEASE WRITE YOUR ANSWER TO EACH QUESTION ON A SEPARATE PAGE. USE $8\frac{1}{2} \times 11$ INCH PAPER. PLEASE WRITE YOUR NAME AT THE TOP OF EACH PAGE. THIS WILL MAKE IT EASIER TO SEPARATE AND SORT THESE PAGES FOR GRADING AND REASSEMBLING THEM AFTERWARDS.

I. BASIC ANALYSIS

1. First prove Rolle's Theorem. If $f : [a, b] \to \mathbb{R}$ is continuous and differentiable on (a, b) and $f(a) = f(b)$ then $\exists\ C \in (a, b)$ with $f'(c) = 0$. Using this, prove the mean value theorem.

2. Prove that every sequence of real numbers has a monotone subsequence.

3. State and prove the alternating series test.

4. Prove Green's Theorem that is

$$\iint\limits_{\Omega} \left[\frac{\partial M}{\partial X} - \frac{\partial N}{\partial Y} \right] dA = \oint\limits_{\partial\Omega} M dy + N dx$$

where $\overleftarrow{F}(x, y) = (N(x, y), M(x, y))$ is continuously differentiable.

II. GROUPS AND LINEAR ALGEBRA

1. Prove Cauchy's Theorem. If p is prime and p divides the order of the group G, then G contains an element of order p.

2. Let A be a diagonalizable $n \times n$ matrix. Suppose that λ is an eigenvalue of A with $|\lambda|$ strictly maximal. Let V be the eigenspace of λ. Let $\overleftharpoon{u}$ be a random vector (i.e. $\overleftharpoon{u}$ is not contained in the span of the other eigenspaces of A). Show that the angle between $A^n \overleftharpoon{u}$ and V goes to 0 as $n \to \infty$.

3. Prove that a group of order p^2 is abelian (where p is prime).

4. In S_n show that every k-cycle is conjugate to every other k-cycle.

III. ANALYSIS

1. Let

$$\mathcal{A} = \{ f : [0,1] \to \mathbb{R} \mid f \text{ is absolutely continuous } \},$$
$$\mathcal{B} = \{ f : [0,1] \to \mathbb{R} \mid f \text{ has bounded variation } \},$$
$$\mathcal{C} = \{ f : [0,1] \to \mathbb{R} \mid f \text{ is continuous } \}.$$

   Give explicit counterexamples for each of the following inclusions that is false:

   (i) $\mathcal{A} \subseteq \mathcal{B}$

   (ii) $\mathcal{B} \subseteq \mathcal{A}$

   (iii) $\mathcal{A} \subseteq \mathcal{C}$

   (iv) $\mathcal{C} \subseteq \mathcal{A}$

   (v) $\mathcal{B} \subseteq \mathcal{C}$

   (vi) $\mathcal{C} \subseteq \mathcal{B}$

2. Give explicit examples of:

   (i) A sequence of Lebesgue-measurable sets $\mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \mathcal{A}_3 \supseteq \cdots$ such that

$$\lim_{k\to\infty} \lambda(\mathcal{A}_k) > \lambda\left(\bigcap_{k=1}^{\infty} \mathcal{A}_k\right).$$

   (ii) A sequence of nonnegative Lebesgue-integrable functions $f_1 \geq f_2 \geq f_3 \geq \cdots$ such that

$$\lim_{k\to\infty} \int_{\mathbb{R}} f_k \, d\lambda > \int_{\mathbb{R}} \left(\lim_{k\to\infty} f_k\right) \, d\lambda.$$

3. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f : X \to [0, \infty]$ be an $\mathcal{M}$-measurable function. Using definitions but no theorems, prove that $\int_X f \, d\mu = 0$ if and only if $\mu\left(\{x \in X \mid f(x) > 0\}\right) = 0$.

4. Let $\mathcal{F} = \bigcap_{p\in[1,\infty]} L^p(\mathbb{R})$.

   (i) For each $q \in (1, \infty)$, give an example of a sequence $\{f_k\} \subset \mathcal{F}$ such that $\lim_{k\to\infty} \|f_k\|_p = 0$ for all $p \in [1, q)$ and such that $\lim_{k\to\infty} \|f_k\|_p = \infty$ for all $p \in [q, \infty]$, or show that there is a $q \in (1, \infty)$ for which no such sequence exists.

   (ii) For each $q \in (1, \infty)$, give an example of a sequence $\{f_k\} \subset \mathcal{F}$ such that $\lim_{k\to\infty} \|f_k\|_p = \infty$ for all $p \in [1, q)$ and such that $\lim_{k\to\infty} \|f_k\|_p = 0$ for all $p \in [q, \infty]$, or show that there is a $q \in (1, \infty)$ for which no such sequence exists.

IV. FIELDS, RINGS AND MODULES

1. Determine the minimal polynomial over Q of the element $\alpha = \sqrt{2} + \sqrt{5}$. (In other words, find $\mathrm{Irr}(\alpha, Q)$.

2. Let ζ be a primitive 5th root of unity.

   (a) Determine $\mathrm{Gal}(Q(\zeta)/Q)$.

   (b) Find all intermediate fields K such that $Q \subseteq K \subseteq Q(\zeta)$. Give a primitive element for each field found.

   (c) Indicate which fields found in part (b) are Galois over Q.

3. (a) Prove carefully that the rings $Q[x]/(x^2 - 2)$ and $Q[x]/(x^2 - 3)$ are not isomorphic.

   (b) Find an example of a commutative ring with unity R such that $R[x]/(x^2-2)$ and $R[x]/(x^2 - 3)$ are isomorphic. Justify your answer briefly.

4. Let R be a commutative ring with 1. An R-module M is said to be irreducible if $M \neq 0$ and M has no submodules except 0 and M. Prove that every irreducible R-module is isomorphic to one of the form R/I, where I is a maximal ideal of R.

V. APPLIED MATHEMATICS

1. Suppose $f(t)$ is periodic with period T, show that its Laplace transform

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$$

satisfies the relation

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}\,dt$$

Consequently, find the Laplace transform of the convolution $f * g$ where $f(t)$ is the "pulse" function

$$f(t) = \begin{cases} 0, & 0 \le x < 1, \\ 1, & 1 \le x < 2 \end{cases}, \qquad f(t+2) = f(t)$$

and $g(t) \equiv 1$.

2. Use Gram-Schmidt orthogonalization process, or otherwise, find an orthonormal basis for the column space of the matrix

$$A = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Prove that the basis vectors are linearly independent and determine the rank of the matrix A.

3. Consider the wave equation for an infinite string

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

Show that the general solution has the D'Alembert form

$$u(x,t) = F(x-ct) + G(x+ct)$$

and provide a physical description of each term in the solution.

Determine F and G using the initial conditions

$$u(x,0) = e^{-x}, \qquad \frac{\partial u}{\partial t}(x,0) = \sin x$$

4. Determine the nature of the equilibrium point at the origin for the system

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = A\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{pmatrix} 2 & -1 \\ a & 0 \end{pmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

for the cases of constant a with (i) $0 < a < 1$ and (ii) $a > 1$. Provide phase portraits in both cases.

If $a = 3x(t) - 2$, what is the linearized system at the equilibrium point away from the origin?

VI. TOPOLOGY

1. Prove that if a Hausdorff space has a countable basis, then it has a countable dense subset.

2. Let X be a linearly ordered space with the linear topology. Show that X is regular. That is: For each $x \in X$ and each closed $A \subset X$ with $x \notin A$, show that there are disjoint open sets U and V such that $x \in U$ and $A \subset V$.

3. Give an example of a linearly ordered space with no first point and no last point so that every sequence is bounded, and prove your example has this property.

4. Let $f_n : X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If (f_n) converges uniformly to the function f, show f is continuous.