ALGEBRA QUALIFYING EXAM
September 9, 2000

Do 7 of the 9 problems.

1. Prove that there is no simple group of order 36.

2. Prove that every nontrivial group has a factor group which is simple.

3. Exhibit an irreducible polynomial in $\mathbb{Q}[x]$ which has $\sqrt[3]{2} + \sqrt[2]{3}$ as a root.

4. Prove that if R is a commutative ring with unity, then R is Noetherian if and only if the polynomial ring $R[x]$ is Noetherian.

5. Prove that the set of algebraic integers (roots of *monic* polynomials in $\mathbb{Q}[x]$) form a subring of the algebraic numbers.

6. Let p be a prime and let $G = GL_2(\mathbb{Z}/p)$, the group of invertible 2×2 matrices with entries in $\mathbb{Z}/p$.

   (a) find a Sylow p-subgroup of G.

   (b) Find the number of Sylow p-subgroups of G.

7. State and prove the Chinese Remainder Theorem.

8. Prove that $x^5 - 6x + 3$ has exactly 3 real roots and then use this fact to calculate its Galois group over $\mathbb{Q}$.

9. Let A be an $n \times n$ matrix with entries in $\mathbb{C}$. Show that if $A^2 = A$, then there are subspaces K and L of $\mathbb{C}^n$ so that $L \oplus K = \mathbb{C}^n$, and so that $A|_L$ is the identity and $A|_K \equiv 0$.