Ph.D. Qualifying Exam in Algebra

BYU Department of Mathematics

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Do every problem. Give appropriately complete proofs for all of your answers. In your solutions do not make assumptions or use facts which would make a problem trivial.

(1) Show that if p is a prime then every group of order p^n has a nontrivial center.
Recall that the *center* of a group is the set of group elements which commute with every group element.

(2) Show that a finite simple group cannot have order 30.

(3) Show that every subgroup of a solvable group is solvable.

(4) Show that every vector space contains a basis.

(5) Show that if R is a commutative ring, M a Noetherian R-module and $u : M \to M$ a surjective R-module homomorphism then u is an isomorphism. Recall that an R-module is *Noetherian* if every strictly ascending chain of submodules is finite. Hint: Consider $\{\ker(u^n)\}$.

(6) Show that the intersection of all proper nontrivial ideals of a commutative ring with unity is an ideal in which every nonzero element is a zero-divisor. Hint: Show that this ideal is principal and consider the square of each element.

(7) Show that if F is a subfield of the finite field K then the Galois group of K over F is cyclic.

(8) Find a nonzero polynomial with coefficients in $\mathbb{Z}$ which has $\sqrt[3]{2} - 3\sqrt[3]{4} + 1$ as a root.

(9) Give an example of an nonseparable irreducible polynomial with coefficients in a field.

(10) Suppose R is a commutative ring and $f \in R[x]$ is a polynomial which has a root in R. Show that R is Noetherian if and only if $\frac{R[x]}{(f(x))}$ is Noetherian.