Ph.D. QUALIFIER EXAMINATION: ANALYSIS

Fall 2004

Instructions: Answer *exactly* 6 of the 10 questions given. If you do more than 6 questions, your grade will be determined by the first 6 questions that you answered.

Some Notation.

1. $\mathbb{R}^k$ – Euclidean k-dimensional space
2. $\mathbb{C}$ – the complex numbers
3. $(X, \mathcal{M}, \mu)$ – a measure space where X is a set, $\mathcal{M}$ is a σ-algebra of subsets of X, and μ is a measure on $\mathcal{M}$
4. a.e.$[\mu]$ – almost every with respect to the measure μ
5. m – Lebesgue measure on $\mathbb{R}^k$
6. $\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}$ – the L^p-norm of a μ-measurable function $f : X \to \mathbb{C}$
7. $\|f\|_\infty$ – the essential supremum of f
8. p, q – conjugate exponents where $\dfrac{1}{p} + \dfrac{1}{q} = 1$
9. $L^p(\mu)$ – the space of μ-measurable functions $f : X \to \mathbb{C}$ with $\|f\|_p < \infty$
10. $L^p(\mathbb{R}^k)$ – the space of Lebesgue measurable functions $f : \mathbb{R}^k \to \mathbb{C}$ with $\|f\|_p < \infty$
11. $|\lambda|$ – the total variation of a measure λ.
12. $\lambda \ll \mu$ – the measure λ is absolutely continuous with respect to the measure μ
13. $\lambda \perp \mu$ – the measures λ and μ are mutually singular
14. $\dfrac{d\lambda}{d\mu}$ – the Radon-Nikodym derivative of λ with respect to μ where $\lambda \ll \mu$
15. Lip α – the space of complex functions f on $[a, b]$ for which $\sup_{x \neq y} \dfrac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty$; here $0 < \alpha \leq 1$
16. $f * g$ – the convolution of f and g: $(f * g)(x) = \int_{-\infty}^{\infty} f(x - y) g(y) \, dy$
17. $C_0(\mathbb{R})$ – the continuous complex functions on $\mathbb{R}$ which vanish at infinity
18. $\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-ixt} \, dm(x)$ – the Fourier transform

1. State and prove Lebesgue's Dominated Convergence Theorem. [You may assume Fatou's Lemma in your proof.]

2. Construct a sequence of continuous functions f_n on $[0,1]$ such that $0 \le f_n \le 1$ and

$$\lim_{n\to\infty} \int_0^1 f_n(x)\ dx = 0,$$

but the sequence $\{f_n(x)\}$ does not converge for any $x \in [0,1]$.

3. Suppose $1 \le p < q < r \le \infty$. Prove that if $f \in L^p(\mu) \cap L^r(\mu)$, then $f \in L^q(\mu)$.

4. Suppose that X and Y are Banach spaces. Suppose that $\Lambda : X \to Y$ is a linear mapping with the property that for every sequence $\{x_n\}$ in X such that $x = \lim x_n$ and $y = \lim \Lambda x_n$ exist, it follows that $y = \Lambda x$. Prove that Λ is continuous. [You may assume that a continuous, one-to-one linear mapping from one Banach space *onto* another Banach space has an inverse that is a continuous linear mapping.]

5. Let $\{f_n\}$ be a sequence of continuous complex functions on a nonempty complete metric space X such that $f(x) = \lim f_n(x)$ exists for every $x \in X$ (i.e. $f_n \to f$ pointwise). Prove for every $\epsilon > 0$ there is a nonempty open set V and a positive integer N such that $|f(x) - f_n(x)| \le \epsilon$ whenever $x \in V$ and $n \ge N$.

6. Suppose that μ and λ are measures on a σ-algebra $\mathcal{M}$ with μ positive and λ complex. Prove that $\lambda \ll \mu$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|\lambda(E)| < \epsilon$ for all $E \in \mathcal{M}$ with $\mu(E) < \delta$.

7. Let μ be a complex Borel measure on $\mathbb{R}^k$. Define the symmetric derivative, $D\mu$, of μ with respect to m. Define a Lebesgue point of an $L^1(\mathbb{R}^k)$ function. Prove that if $\mu \ll m$ and f is the Radon-Nikodym derivative of μ with respect to m, then

$$D\mu = f \text{ a.e.}[m], \quad \text{and} \quad \mu(E) = \int_E (D\mu)\ dm.$$

[You may assume that almost every $x \in \mathbb{R}^k$ is a Lebesgue point of an $L^1(\mathbb{R}^k)$ function.]

8. Suppose p and q are conjugate exponents with $1 < p < \infty$, and set $\alpha = 1/q$. Prove that if f is absolutely continuous on $[a,b]$ and $f' \in L^p$, then $f \in \text{Lip}\ \alpha$.

9. Prove that if $f, g \in L^1(\mathbb{R})$, then $f * g$ is $L^1(\mathbb{R})$ with $\|f * g\|_1 \le \|f\|_1 \|g\|_1$.

10. Prove that if $f \in L^1(\mathbb{R})$, then $\hat{f} \in C_0(\mathbb{R})$ and $\|\hat{f}\|_\infty \le \|f\|_1$. [You may assume for each $x \in \mathbb{R}$ that the map $y \to f(x-y)$ from $\mathbb{R}$ to $L^1(\mathbb{R})$ is uniformly continuous.]