PhD. Analysis Exam

Do 10 of the following 15 problems.

1. Suppose $\{f_n\}$ is a sequence of measurable real valued functions. Define $A \equiv \{\mathbf{x} : \{f_n(\mathbf{x})\} \text{ converges.}\}$ Is A measurable? Explain why or give a counter example.

2. A sequence $\{x_n\}_{n=1}^{\infty}$ of points in a Banach space, X is weakly bounded if for every $x^* \in X'$, the set of complex numbers, $\{x^*(x_n)\}_{n=1}^{\infty}$ is bounded. Show that a weakly bounded sequence is in fact bounded.

3. State the open mapping theorem and using this theorem, give a proof of the closed graph theorem.

4. Suppose $(\Omega, \mathcal{F}, \mu)$ is a measure space and let $f : \Omega \to \mathbb{R}$ be a measurable function. Suppose $g : \mathbb{R} \to \mathbb{R}$ is Borel measurable. Does it follow that $g \circ f$ is measurable? Give either a proof or a counter example.

5. Give an example in which the Vitali convergence theorem applies but the Dominated convergence theorem does not apply.

6. The maximal function of $f \in L^1(\mathbb{R}^n)$ is given by

$$Mf(\mathbf{x}) \equiv \sup \left\{ \frac{1}{m_n(B(\mathbf{0},r))} \int_{B(\mathbf{x},r)} |f(\mathbf{y})| \, dm_n(y) : r > 0 \right\}.$$

   Using some version of the Vitali covering theorem or other method, establish the weak $(1,1)$ estimate,

$$\overline{m_n}(\{\mathbf{x} : |Mf(\mathbf{x})| > \delta\}) < \frac{C^n}{\delta} ||f||_{L^1(\mathbb{R}^n)}$$

   where C is some constant which is independent of n. Here $\overline{m_n}$ is the outer measure determined by n dimensional Lebesgue measure.

7. Let $\{f_n\}_{n=1}^{\infty}$ be a set of functions which are bounded in $L^5(\Omega)$ where $(\Omega, \mathcal{F}, \mu)$ is a finite measure space. Suppose also that

$$\lim_{n\to\infty} f_n(\mathbf{x}) = f(\mathbf{x}).$$

   Can you conclude that

$$\lim_{n\to\infty} \int f_n(\mathbf{x}) \, d\mu = \int f(\mathbf{x}) \, d\mu?$$

   Explain why or why not.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be everywhere differentiable. Give an example which shows that f' does not need to be continuous. Show however that f' must be Borel measurable.

9. Using the Cauchy integral formula, give a short proof of the fundamental theorem of algebra which states that every non constant polynomial has a zero in the complex plane.

10. An entire function, $f(z+a) = f(z)$ and $f(z+ib) = f(z)$ for a, b two positive real numbers. Suppose also that $f(a) = 1$. Find a formula for $f(z)$ valid for all $z \in \mathbb{C}$. Explain why your formula is correct. No guessing.

11. Suppose f is an entire function which satisfies $|f(z)| \leq C\left(1 + |z|^{1/2}\right)$ and $f(0) = 0$. Find a formula for $f(z)$ which is valid for all z and justify your answer.

12. Suppose $\{f_n\}$ is a sequence of functions which are analytic on Ω, a bounded region such that each f_n is also continuous on $\overline{\Omega}$. Suppose that $\{f_n\}$ converges uniformly on $\partial\Omega$. Show that then $\{f_n\}$ converges uniformly on $\overline{\Omega}$ and that the function to which the sequence converges is analytic on Ω and continuous on $\overline{\Omega}$.

13. You want an entire function, $f(z)$ which has the property that $f(x) = e^x$ for $x \in \mathbb{R}$. Show one such function is

$$f(z) = e^x (\cos(y) + i \sin(y))$$

Next explain why this is the only function which can satisfy these conditions.

14. Find $\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2} dx$.

15. Show $f(z) = \frac{z-i}{z+i}$ maps the upper half plane onto the unit $\{z \in \mathbb{C} : |z| < 1\}$. Could there exist an entire function which maps $\mathbb{C}$ onto the upper half plane?