Algebraic topology qualifying exam August 1999

Answer eight questions, four from part I and four from part II. Give as much detail in your answers as you can.

Part I

1. Prove the Zig-Zag lemma: let $0 \to C \to D \to E \to 0$ be a short exact sequence of chain complexes with the above maps being $f: C \to D$, $g : D \to E$. Show that there is a long exact sequence of homology groups that arises from this situation.

2. A) State and prove the Mayer-Vietoris theorem for singular or simplicial (non-reduced) homology. B) Show that if $S(X)$ is the suspension of a topological space X, then $H_p(S(X))$ and $H_{p-1}(X)$ are isomorphic in reduced homology.

3. A) Let K, L be simplicial complexes and $f,g : |K| \to |L|$ homotopic maps. Show that $f_* = g_*$, where * indicates the induced map of homology groups. B) State and prove the Brouwer fixed point theorem.

4. State and prove Poincaré duality for orientable triangulated compact homology n-manifolds.

5. State and prove the universal coefficient theorem for cohomology.

Part II

6. A) Calculate the cohomology ring of $T \times P$, where T is the torus and P the projective plane. B) Show that if M is an orientable compact manifold of odd dimension, then it has zero Euler characteristic. C) What is the Euler characteristic of $T \times T$, where T is the torus.

7. Let S^n be the n-sphere in R^{n+1}defined by $x_1^2+\ldots+ x_{n+1}^2= 1$. Let E_1 be the subset defined by $x_{n+1} \geq 0$ and E_2 by $x_{n+1} \leq 1/2$. Let F be the intersection of E_1 and E_2. Show that $H_r(S^n,E_2)$ is isomorphic to $H_r(E_1,F)$ for all $r \geq 0$. C) Let S^{n-1} be defined by $x_{n+1}=0$. Show that $H_r(S^n,E_2)$ is isomorphic to $H_r(E_1, S^{n-1})$ for all $r \geq 0$.

8. Find all the homology groups (with integer coefficients) of the 2-skeleton of the 4-simplex.

9. Let X denote the solid torus $B^2 \times S^1$. Let $f:X \to X$ be the imbedding which maps X around itself twice. Let $X_0= X$, $X_1 = f(X_0)$, $X_2 = f(X_1)$ etc. Let S (the solenoid) be the intersection of all of the X_i. Find the first Cech cohomology group of S.

10. Let A be the union of two once-linked embedded circles in the 3-sphere S^3. Let B be the union of two unlinked circles. Show that the cohomology groups of $S^3 \setminus A$ and $S^3 \setminus B$ are isomorphic, but that their cohomology rings are not.