Ph.D. Qualifying Exam

Work at least 7 problems from Real analysis and 3 from complex analysis. Clear explanation is required at all stages of each answer. You may quote standard theorems other than the one you may be trying to prove.

Real Analysis Questions

1. Let f_n, f be measurable functions with values in $\mathbb{C}$. We say that f_n converges in measure if

$$\lim_{n\to\infty} \mu(x \in \Omega : |f(x) - f_n(x)| \geq \varepsilon) = 0$$

for each fixed $\varepsilon > 0$. Prove the theorem of F. Riesz. If f_n converges to f in measure, then there exists a subsequence $\{f_{n_k}\}$ which converges to f a.e. **Hint:** Choose n_1 such that

$$\mu(x : |f(x) - f_{n_1}(x)| \geq 1) < 1/2.$$

Choose $n_2 > n_1$ such that

$$\mu(x : |f(x) - f_{n_2}(x)| \geq 1/2) < 1/2^2,$$

$n_3 > n_2$ such that

$$\mu(x : |f(x) - f_{n_3}(x)| \geq 1/3) < 1/2^3,$$

etc. Now consider what it means for $f_{n_k}(x)$ to fail to converge to $f(x)$. Use appropriate theorems to prove this.

2. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and suppose $f, g : \Omega \to [-\infty, \infty]$ are measurable. Prove the sets

$$\{\omega : f(\omega) < g(\omega)\} \text{ and } \{\omega : f(\omega) = g(\omega)\}$$

are measurable.

3. Let $m(W) > 0$, W is measurable, $W \subseteq [a, b]$. Show, using the axiom of choice, there exists a nonmeasurable subset of W. **Hint**: Let $x \sim y$ if $x - y \in \mathbb{Q}$. Observe that $\sim$ is an equivalence relation on $\mathbb{R}$. Let $\mathcal{C}$ be the set of equivalence classes. Let $\mathcal{D} = \{C \cap W : C \in \mathcal{C}$ and $C \cap W \neq \emptyset\}$. If $C \cap W \in \mathcal{D}$, pick exactly one element of $C \cap W$. Denote by A the collection of these points.

4. If S is an uncountable set of irrational numbers, is it necessary that S has a rational number as a limit point? **Hint:** Consider the proof that a countable set of numbers has measure zero when applied to the rational numbers.

5. Show that (X, τ) is locally compact if and only if it has the property that whenever $K \subseteq V$ for V open and K compact, it follows there exists a continuous map, f which equals one on K, has compact support in V and maps every point of X into the interval, $[0, 1]$. You may use Urysohn's lemma and the one point compactification if you want.

6. Suppose $X \subseteq \mathbb{R}^n$ is a measurable set such that $m(X) > 0$ where here m is Lebesgue measure. Also let D be a countable dense set in $\mathbb{R}^n$ and let $Q \equiv \prod_{i=1}^n [a_i, b_i)$. Establish the result of Smítal, which says that under these conditions, $m((X + D) \cap Q) = m(Q)$. **Hint:** Let

$$\mathcal{M} \equiv \{E \text{ Borel such that if } m(E) > 0$$
$$\text{then } m((E + D) \cap Q) = m(Q)\}$$

and show $\mathcal{M}$ is a monotone class containing the sets of $\mathcal{E}$ where $\mathcal{E}$ consists of finite disjoint unions of half open boxes.

7. Let E be a Lebesgue measurable set in $\mathbb{R}$. Suppose $m(E) > 0$. Consider the set

$$E - E = \{x - y : x \in E, y \in E\}.$$

Show that $E - E$ contains an interval. **Hint:** Let

$$f(x) = \int \mathcal{X}_E(t)\mathcal{X}_E(x + t)dt.$$

Note f is continuous at 0 and $f(0) > 0$. Remember continuity of translation in L^p.

8. Suppose that $\mathbf{f} : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and bounded. Show there exists a solution on $[0, T]$ to the initial value problem

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}), \mathbf{y}(0) = \mathbf{y}_0.$$

This is a version of the Peano existence theorem. **Hint:** Let $h > 0$ and consider the functions $\mathbf{y}_h$ defined by $\mathbf{y}_h(t) = \mathbf{y}_0$ if $t \in [-h, 0]$ and $\mathbf{y}_h(t) = \mathbf{y}_0 + \int_0^t \mathbf{f}(s, \mathbf{y}_h(s - h))\,ds$ for $t > 0$. Argue the functions $\mathbf{y}_h$ are equicontinuous and uniformly bounded. Then apply the Ascoli Arzela theorem.

9. Suppose $\mathbf{f} : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies the Lipschitz condition,

$$|\mathbf{f}(t, \mathbf{y}) - \mathbf{f}(t, \mathbf{z})| \leq K |\mathbf{z} - \mathbf{y}|.$$

Show there exists a unique solution on $[0, T]$ to the initial value problem,

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}), \mathbf{y}(0) = \mathbf{y}_0.$$

Hint: Define a norm on $C([0, T])$ by $||\mathbf{f}||_\lambda \equiv \max\left\{e^{\lambda t} |\mathbf{f}(t)| : t \in [0, T]\right\}$, where λ is chosen in such a way that the mapping, $T : C([0, T]) \to C([0, T])$ given by $T\mathbf{y}(t) \equiv \mathbf{y}_0 + \int_0^t \mathbf{f}(s, \mathbf{y}(s))\,ds$ is a contraction map with respect to this norm. Then use the contraction mapping fixed point theorem.

10. Suppose $T : X \to X$ where X is a complete metric space. Suppose that for all n large enough, T^n is a strict contraction,

$$d(T^n x, T^n y) \leq r d(x, y)$$

where $r \in (0, 1)$. Show that T has a unique fixed point. **Hint:** Consider the sequence, $\left\{(T^n)^k x\right\}_{k=1}^\infty$.

11. A set of functions, $\Phi \subseteq L^1$, is uniformly integrable if for all $\varepsilon > 0$ there exists a $\sigma > 0$ such that $\left|\int_E f\, d\mu\right| < \varepsilon$ whenever $\mu(E) < \sigma$. Prove Vitali's Convergence theorem: Let $\{f_n\}$ be uniformly integrable, $\mu(\Omega) < \infty$, $f_n(x) \to f(x)$ a.e., and $|f(x)| < \infty$ a.e. Then $f \in L^1$ and $\lim_{n\to\infty} \int_\Omega |f_n - f| d\mu = 0$. **Hint:** Use Eggorov's theorem.

12. Suppose $f \in L^\infty \cap L^1$. Show $\lim_{p\to\infty} ||f||_{L^p} = ||f||_\infty$.

13. Suppose $\lambda(E) = \int_E f d\mu$ where λ and μ are two measures and $f \in L^1(\mu)$. Show $\lambda << \mu$.

14. Suppose f is continuous on $(0,\infty)$, and for some $\gamma > 0$,
$$\left|f(x)\, e^{-\gamma x}\right| \leq C$$
for all $x \in (0,\infty)$, and suppose also that for all t large enough,
$$\int_0^\infty f(x)\, e^{-tx} dx = 0.$$
Show $f(x) = 0$ for all $x \in (0,\infty)$. Can it be concluded that $f(x) = 0$ for all x? **Hint:** Using the Stone Weierstrass theorem argue that the functions, $x \to \sum_{k=0}^n a_k e^{-xkt}$ for $t > 0$ are dense in $C(X)$ where X is the space $[0,\infty]$ with a subbasis for the topology given by sets of the form $[0,a)$ and $(b,\infty]$. Then note that $C(X)$ is dense in $L^2\left(e^{-\delta x}dx\right)$.

15. Let $(\Omega, \mathcal{S}, \mu)$ be any σ finite measure space, $f \geq 0$, f real-valued, and measurable. Let ϕ be an increasing C^1 function with $\phi(0) = 0$. Show
$$\int_\Omega \phi \circ f d\mu = \int_0^\infty \phi'(t)\mu([f(\dot{x}) > t])dt.$$
Hint:
$$\int_\Omega \phi(f(x))d\mu = \int_\Omega \int_0^{f(x)} \phi'(t)dtd\mu$$
$$= \int_\Omega \int_0^\infty \mathcal{X}_{[0,f(x))}(t)\phi'(t)dtd\mu.$$
Argue $\phi'(t)\mathcal{X}_{[0,f(x))}(t)$ is product measurable and use Fubini's theorem. The function $t \to \mu([f(x) > t])$ is called the distribution function.

16. The result of this Problem is sometimes called the Vitali Covering Theorem. Let $E \subseteq \mathbb{R}^n$ be Lebesgue measurable, $m(E) < \infty$, and let $\mathcal{F}$ be a collection of balls that cover E in the sense of Vitali. This means that if $\mathbf{x} \in E$ and $\varepsilon > 0$, then there exists $B \in \mathcal{F}$, diameter of $B < \varepsilon$ and $\mathbf{x} \in B$. Show there exists a countable sequence of disjoint balls of $\mathcal{F}$, $\{B_j\}$, such that $m(E \setminus \cup_{j=1}^\infty B_j) = 0$. **Hint:** Let $E \subseteq U$, U open and
$$m(E) > (1 - 10^{-n})m(U).$$
Let $\{B_j\}$ be disjoint,
$$E \subseteq \cup_{j=1}^\infty \hat{B}_j,\ B_j \subseteq U.$$
Thus
$$m(E) \leq 5^n m(\cup_{j=1}^\infty B_j).$$
Then
$$m(E) > (1 - 10^{-n})m(U)$$
$$\geq (1 - 10^{-n})[m(E \setminus \cup_{j=1}^\infty B_j) + m(\cup_{j=1}^\infty B_j)]$$
$$\geq (1 - 10^{-n})[m(E \setminus \cup_{j=1}^\infty B_j) + 5^{-n}m(E)].$$
Hence
$$m(E \setminus \cup_{j=1}^\infty B_j) \leq$$
$$(1 - 10^{-n})^{-1}(1 - (1 - 10^{-n})5^{-n})m(E).$$
Let
$$(1 - 10^{-n})^{-1}(1 - (1 - 10^{-n})5^{-n}) < \theta < 1$$
and pick N_1 large enough that
$$\theta m(E) \geq m(E \setminus \cup_{j=1}^{N_1} \bar{B}_j).$$
Let $\mathcal{F}_1 = \{B \in \mathcal{F} : B_j \cap B = \emptyset,\ j = 1, \cdots, N_1\}$. If $E \setminus \cup_{j=1}^{N_1} \overline{B}_j \neq \emptyset$, then $\mathcal{F}_1 \neq \emptyset$ and covers $E \setminus \cup_{j=1}^{N_1} \overline{B}_j$ in the sense of Vitali. Repeat the same argument, letting $E \setminus \cup_{j=1}^{N_1} \bar{B}_j$ play the role of E.

17. For Lebesgue measure on $\mathbb{R}^n$ show translation is continuous on $L^p(\mathbb{R}^n)$. That is, show that if $T_r f(t) = T(t - r)$ then
$$\lim_{r\to 0} ||T_r f - f||_p = 0.$$

18. Suppose λ is a Radon measure on $\mathbb{R}^n$, and $\lambda(S) < \infty$ where $m_n(S) = 0$ and $\lambda(E) = \lambda(E \cap S)$. (If $\lambda(E) = \lambda(E \cap S)$ where $m_n(S) = 0$ we say $\lambda \perp m_n$.) Show that for m_n a.e. $\mathbf{x}$ the following holds. If $B_i \downarrow \{\mathbf{x}\}$, then $\lim_{i\to\infty} \frac{\lambda(B_i)}{m_n(B_i)} = 0$. **Hint**: You might try this. Set $\varepsilon, r > 0$, and let $E_\varepsilon = \{\mathbf{x} \in S^C :$ there exists $\{B_i^{\mathbf{x}}\}, B_i^{\mathbf{x}} \downarrow \{\mathbf{x}\}$ with $\frac{\lambda(B_i^{\mathbf{x}})}{m_n(B_i^{\mathbf{x}})} \geq \varepsilon\}$. Let K be compact, $\lambda(S \setminus K) < r\varepsilon$. Let $\mathcal{F}$ consist of those balls just described that do not intersect K and which have radius < 1. This is a Vitali cover of E_ε. Let $B_1, \cdots, B_k$ be disjoint balls from $\mathcal{F}$ and
$$\overline{m}_n(E_\varepsilon \setminus \cup_{i=1}^k B_i) < r.$$
Then
$$\overline{m}_n(E_\varepsilon) < r + \sum_{i=1}^k m_n(B_i) < r + \varepsilon^{-1} \sum_{i=1}^k \lambda(B_i) =$$
$$r + \varepsilon^{-1} \sum_{i=1}^k \lambda(B_i \cap S) \leq r + \varepsilon^{-1}\lambda(S \setminus K) < 2r.$$
Since r is arbitrary, $m_n(E_\varepsilon) = 0$. Consider $E = \cup_{k=1}^\infty E_{k^{-1}}$ and let $\mathbf{x} \notin S \cup E$.

19. Suppose $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous, $|f(x) - f(y)| \leq K|x-y|$ show there exists $h \in L^{\infty}(\mathbb{R})$ such that

$$f(y) - f(x) = \int_x^y h(t)\,dt.$$

Hint: Let $F(x) = Kx + f(x)$ and let λ be the measure representing $\int f dF$. Show $\lambda << m$. What does this imply about the differentiability of a Lipschitz continuous function?

20. Consider the construction employed to obtain the Cantor set, but instead of removing the middle third interval, remove only enough that the sum of the lengths of all the open intervals which are removed is less than one. That which remains is called a fat Cantor set. Show it is a compact set which has measure greater than zero which contains no interval and has the property that every point is a limit point of the set. Let P be such a fat Cantor set and consider

$$f(x) = \int_0^x \mathcal{X}_{P^C}(t)\,dt.$$

Show that f is a strictly increasing function which has the property that its derivative equals zero on a set of positive measure.

21. Let $\phi \in C_c^{\infty}(\mathbb{R}^n)$ and let $\phi_m(x) = \phi(mx)m^n$. Suppose $\phi \geq 0$ and $\int_{\mathbb{R}^n} \phi(y)dy = 1$. Show, $\int_{\mathbb{R}^n} \phi_m(y)dy = 1$, If $f \in L^p(\mathbb{R}^n)$, then $\lim_{m\to\infty} f * \phi_m = f$ in $L^p(\mathbb{R}^n)$, and $f * \phi_m$ is a function in $C^{\infty}(\mathbb{R}^n)$.

22. A set of functions, $\mathfrak{S} \subseteq L^1(\Omega)$, defined on a measure space, $(\Omega, \mathcal{S}, \mu)$ is said to be uniformly integrable if for all $\epsilon > 0$ there exists $\alpha > 0$ such that if $\mu(E) < \alpha$, then for all $f \in \mathfrak{S}$

$$\left|\int_E f d\mu\right| < \varepsilon. \tag{1}$$

Show that if $(\Omega, \mathcal{S}, \mu)$ is a finite measure space then $\mathfrak{S}$ is uniformly integrable and there exists a constant,$C < \infty$ such that

$$\int |f|\,d\mu \leq C \tag{2}$$

for all $f \in \mathfrak{S}$ if and only if for all $\epsilon > 0$ there exists $\alpha > 0$ such that for all $f \in \mathfrak{S}$,

$$\int_{[|f|\geq\alpha]} |f|\,d\mu \leq \epsilon \tag{3}$$

Show that if 3. holds and if $f_n(\omega) \to f(\omega)$ a.e., then it is automatically the case that $|f(\omega)| < \infty$ a.e.

23. If $f : \mathbb{R}^n \to [0,\infty]$ is Lebesgue measurable, show there exists $g : \mathbb{R}^n \to [0,\infty]$ such that $g = f$ a.e. and g is Borel measurable.

24. Suppose that $\theta \in [0,1]$ and $r, s, q > 0$ with

$$\frac{1}{q} = \frac{\theta}{r} + \frac{1-\theta}{s}.$$

show that

$$(\int |f|^q d\mu)^{1/q} \leq ((\int |f|^r d\mu)^{1/r})^{\theta}((\int |f|^s d\mu)^{1/s})^{1-\theta}.$$

If $q, r, s \geq 1$ this says that

$$||f||_q \leq ||f||_r^{\theta}||f||_s^{1-\theta}.$$

Hint:

$$\int |f|^q d\mu = \int |f|^{q\theta}|f|^{q(1-\theta)}d\mu.$$

Now note that $1 = \frac{\theta q}{r} + \frac{q(1-\theta)}{s}$ and use Holder's inequality.

25. Let B be a Borel set in $\mathbb{R}^n$ and let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^n$. Suppose B has the following property. For each $\mathbf{x} \in \mathbb{R}^n$, $m(\{t : \mathbf{x}+t\mathbf{v} \in B\}) = 0$. Then show $m_n(B) = 0$. Note the condition on B says roughly that B is thin in one direction.

26. Let f:$\mathbb{R}^n \to \mathbb{R}$ be defined by $f(\mathbf{x}) \equiv \left(1 + |\mathbf{x}|^2\right)^k$. Find the values of k for which f is in $L^1(\mathbb{R}^n)$. **Hint:** Use polar coordinates.

27. Suppose A is covered by a finite collection of Balls, $\mathcal{F}$. Show that then there exists a disjoint collection of these balls, $\{B_i\}_{i=1}^p$, such that $A \subseteq \cup_{i=1}^p \widehat{B}_i$ where 5 can be replaced with 3 in the definition of $\widehat{B}$. **Hint:** Since the collection of balls is finite, we can arrange them in order of decreasing radius.

28. Let X, Y be normed linear spaces. Show that if Y is a Banach space, then $\mathcal{L}(X,Y)$ is also a Banach space with respect to the operator norm.

29. The theorem of de la Vallee Poussin states that if $\mathfrak{S}$ is a family of measurable functions defined on a finite measure space, (Ω, μ) and there exists a function, g, which is positive and increasing on $(0,\infty)$ with $\lim_{t\to\infty} g(t) = \infty$ and

$$\sup\left\{\int_{\Omega} |f|\, g(|f|)\,d\mu : f \in \mathfrak{S}\right\} < \infty,$$

then $\mathfrak{S}$ is a uniformly integrable subset of $L^1(\Omega)$ having

$$\sup\left\{\int |f(\omega)|\,d\mu : f \in \mathfrak{S}\right\} < \infty. \tag{0.1}$$

Prove this theorem.

30. Let E be a Lebesgue measurable set. We say $\mathbf{x} \in E$ is a point of density if

$$\lim_{r\to 0} \frac{m(E \cap B(\mathbf{x},r))}{m(B(\mathbf{x},r))} = 1.$$

Show that a.e. point of E is a point of density.

31. Show there exist functions defined on $[0,1]$ which are continuous but nowhere differentiable. **Hint:** If $f'(x)$ exists show there exists a constant, K, such that $|f(x)-f(y)| \leq K|x-y|$ for all $y \in [0,1]$. Now let U_n consist of the functions in $C([0,1])$ with the property that for all $x \in [0,1]$ there exists y with $|f(x)-f(y)| > n|x-y|$. Argue that U_n is open and dense in $C([0,1])$.

32. Let $\phi : \mathbb{R} \to \mathbb{R}$ be convex. This means

$$\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1-\lambda)\phi(y)$$

whenever $\lambda \in [0,1]$. Show that if ϕ is convex, then ϕ is continuous.

33. Let $f \in C_c(0,\infty)$ and define $F(x) = \frac{1}{x}\int_0^x f(t)dt$. Show

$$||F||_{L^p(0,\infty)} \leq \frac{p}{p-1}||f||_{L^p(0,\infty)} \text{ whenever } p > 1.$$

Hint: Use $xF' = f - F$ and integrate $\int_0^\infty |F(x)|^p dx$ by parts. Now suppose $f \in L^p(0,\infty)$, $p > 1$, and f not necessarily in $C_c(0,\infty)$. Note that $F(x) = \frac{1}{x}\int_0^x f(t)dt$ still makes sense for each $x > 0$. Is the inequality still valid? Why? This inequality is called Hardy's inequality.

34. Show the Vitali Convergence theorem implies the Dominated Convergence theorem for finite measure spaces. Give an example of a situation in which the Vitali Convergence theorem applies, but the Dominated Convergence theorem does not.

35. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $\mathfrak{S} \subseteq L^1(\Omega)$. We say that $\mathfrak{S}$ is uniformly integrable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $f \in \mathfrak{S}$

$$|\int_E f d\mu| < \varepsilon \text{ whenever } \mu(E) < \delta.$$

Show that $|\mathfrak{S}| \equiv \{|f| : f \in \mathfrak{S}\}$ is uniformly integrable if $\mathfrak{S}$ is. Also show that $\mathfrak{S}$ is uniformly integrable if $\mathfrak{S}$ is finite.

36. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and suppose $f \in L^1(\Omega)$ has the property that whenever $\mu(E) > 0$,

$$\frac{1}{\mu(E)}|\int_E f d\mu| \leq C.$$

Show $|f(\omega)| \leq C$ a.e.

37. Let $(\Omega, \mathcal{S}, \mu)$ be an arbitrary measure space and define $\bar{\mu} : \mathcal{P}(\Omega) \to [0,\infty]$ by

$$\bar{\mu}(S) = \inf\{\mu(E) : E \supseteq S \text{ and } E \in \mathcal{S}\}.$$

Show $\bar{\mu}$ is an outer measure. If $\overline{\mathcal{S}}$ is the set of $\bar{\mu}$ measurable sets in the sense of Caratheodory, show $\overline{\mathcal{S}} \supseteq \mathcal{S}$ and $\bar{\mu} = \mu$ on $\mathcal{S}$. This shows how any measure space can be completed and it also shows that the completion $\bar{\mu}$ comes from an outer measure.

38. Let $\{E_i\}$ be a sequence of measurable sets with the property that

$$\sum_{i=1}^{\infty} \mu(E_i) < \infty.$$

Let $S = \{\omega \in \Omega$ such that $\omega \in E_i$ for infinitely many values of $i\}$. Show $\mu(S) = 0$ and S is measurable.

39. If $f \in L^p, 1 < p < \infty$, show $Mf \in L^p$ and

$$||Mf||_p \leq A(p,n)||f||_p.$$

Hint: Let

$$f_1(\mathbf{x}) \equiv \begin{cases} f(\mathbf{x}) \text{ if } |f(\mathbf{x})| > \alpha/2, \\ 0 \text{ if } |f(\mathbf{x})| \leq \alpha/2. \end{cases}$$

Argue $[Mf(\mathbf{x}) > \alpha] \subseteq [Mf_1(\mathbf{x}) > \alpha/2]$. Then

$$\int (Mf)^p dx = \int_0^\infty p\alpha^{p-1} m([Mf > \alpha])d\alpha$$

$$\leq \int_0^\infty p\alpha^{p-1} m([Mf_1 > \alpha/2])d\alpha.$$

40. Let Ω be a Lebesgue measurable subset of $\mathbb{R}^n$, let $p > 1$, and let

$$p' \equiv \frac{p}{p-1}, \text{ if } p < \infty \text{ and } p' \equiv 1 \text{ if } p = \infty.$$

Let $\mathcal{D}$ be a dense countable set in $L^{p'}(\Omega)$. Now suppose $\{f_n\}$ is a bounded sequence in $L^p(\Omega)$. Show, using the Cantor diagonal process that there exists a subsequence, $\{f_{n_l}\}$ such that the sequence of numbers,

$$\left\{\int_\Omega f_{n_l} g dm\right\},$$

converges for each $g \in \mathcal{D}$ and that therefore, the above sequence of numbers converges for every $g \in L^{p'}(\Omega)$, not just for $g \in \mathcal{D}$. Define for each $g \in L^{p'}(\Omega)$,

$$\Lambda g \equiv \lim_{l\to\infty} \int_\Omega f_{n_l} g dm.$$

Now verify that $\Lambda \in \left(L^{p'}(\Omega)\right)'$ and conclude there exists $f \in L^p(\Omega)$ such that

$$\int fg dm = \lim_{l\to\infty} \int_\Omega f_{n_l} g dm.$$

Note this gives a simple proof that each bounded sequence in $L^p(\Omega)$ for $\infty > p > 1$ has a weakly convergent subsequence and that therefore, bounded sets are weakly sequentially precompact. Also this shows that bounded sequences in $L^\infty(\Omega)$ have weak$*$ convergent subsequences implying that bounded sets in this space are weak$*$ sequentially precompact.

41. State and prove the Riesz representation theorem for Hilbert space.

42. Show $|f(\mathbf{x})| \leq Mf(\mathbf{x})$ at every Lebesgue point of f whenever $f \in L^1(\mathbb{R}^n)$.

43. Let $\frac{1}{p} + \frac{1}{p'} = 1,\ p > 1,$ let $f \in L^p(\mathbb{R}),\ g \in L^{p'}(\mathbb{R})$. Show $f * g$ is uniformly continuous on $\mathbb{R}$ and $|(f * g)(x)| \leq ||f||_{L^p} ||g||_{L^{p'}}$. Be sure to consider all measurability questions relating to the convolution of the two functions.

44. Show that if g is Borel measurable and f is measurable, then $g \circ f$ is also measurable. Give an example of a Lebesgue measurable function, f and a continuous function g such that $f \circ g$ is not Lebesgue measurable. **Hint:** The second part of this can be accomplished through the use of that strange function based on the Cantor set which is continuous, climbs from 0 to 1 and yet has its derivative equal to zero on the complement of the Cantor set. Call this function, h. Then consider $g_1(x) = h(x) + x$. Show g_1 maps the Cantor set into a set having positive measure and then use the theorem which says there is a nonmesurable subset of this set. Now consider $g \equiv g_1^{-1}$, a continuous function along with $f \equiv \mathcal{X}_S \circ g_1$.

45. Give a proof of Gronwall's inequality which states that if for all $t \in [0, T]$, $u(t) \leq u_0 + \int_0^t f(s) u(s) ds$ where f is an integrable function, then
$$u(t) \leq u_0 e^{F(t)}$$
where $F(t) = \int_0^t f(s) ds$.

46. Suppose λ, μ are two finite measures defined on a σ algebra $\mathcal{S}$. Show $\lambda = \lambda_S + \lambda_A$ where λ_S and $\lambda_A$are finite measures satisfying
$$\lambda_S(E) = \lambda_S(E \cap S)\ ,\ \mu(S) = 0 \text{ for some } S \subseteq \mathcal{S},$$
$$\lambda_A << \mu.$$
This is called the Lebesgue decomposition. **Hint**: This is just a generalization of the Radon Nikodym theorem. Let
$$S = \{x : h(x) = 1\},\ \lambda_S(E) = \lambda(E \cap S),$$
$$\lambda_A(E) = \lambda(E \cap S^c).$$
We write $\mu \perp \lambda_S$ and $\lambda_A << \mu$ in this situation.

47. $B(p, q) = \int_0^1 x^{p-1}(1 - x)^{q-1} dx, \Gamma(p) = \int_0^\infty e^{-t} t^{p-1} dt$ for $p, q > 0$. The first of these is called the beta function, while the second is the gamma function. Show a.) $\Gamma(p + 1) = p\Gamma(p)$; b.) $\Gamma(p)\Gamma(q) = B(p, q)\Gamma(p + q)$.

48. Let f be in $L^1_{loc}(\mathbb{R}^n)$. Show Mf is Borel measurable.

49. Let $\{f_n\}$ be a sequence of real or complex valued measurable functions. Let
$$S = \{\omega : \{f_n(\omega)\} \text{ converges}\}.$$
Show S is measurable.

Complex Analysis Questions

1. If $\gamma(t) = x(t) + iy(t)$ is a C^1 curve having values in U, an open set of $\mathbb{C}$, and if $f : U \to \mathbb{C}$ is analytic, we can consider $f \circ \gamma$, another C^1 curve having values in $\mathbb{C}$. Also, $\gamma'(t)$ and $(f \circ \gamma)'(t)$ are complex numbers so these can be considered as vectors in $\mathbb{R}^2$ as follows. The complex number, $x + iy$ corresponds to the vector, $\langle x, y \rangle$. Suppose that γ and η are two such C^1 curves having values in U and that $\gamma(t_0) = \eta(s_0) = z$ and suppose that $f : U \to \mathbb{C}$ is analytic. Show that the angle between $(f \circ \gamma)'(t_0)$ and $(f \circ \eta)'(s_0)$ is the same as the angle between $\gamma'(t_0)$ and $\eta'(s_0)$ assuming that $f'(z) \neq 0$. Thus analytic mappings preserve angles at points where the derivative is nonzero. Such mappings are called isogonal. . **Hint:** To make this easy to show, first observe that $\langle x, y \rangle \cdot \langle a, b \rangle = \frac{1}{2}(z\overline{w} + \overline{z}w)$ where $z = x + iy$ and $w = a + ib$.

2. Suppose that for some constants $a, b \neq 0,\ a, b \in \mathbb{R}$, $f(z + ib) = f(z)$ for all $z \in \mathbb{C}$ and $f(z + a) = f(z)$ for all $z \in \mathbb{C}$. If f is analytic, show that f must be constant. Can you generalize this? **Hint:** This uses Liouville's theorem.

3. Suppose f is an entire function and that f has the property that whenever we write $f(z)$ as a power series expanded about a point w, it follows that at least one of the coefficients in the power series must equal zero. Show that f must be a polynomial. **Hint:** Define a set, A_n to be the points, w such that if $f(z) = \sum_{k=0}^\infty a_k (z - w)^k$, it follows $a_n = 0$. Thus A_n consists of the points where the power series of f centered at these points has the nth coefficient equal to zero. Argue that some A_n is uncountable and therefore has a limit point.

4. Consider the two polynomials $z^5 + 3z^2 - 1$ and $z^5 + 3z^2$. Show that on $|z| = 1$, we have the conditions for Rouche's theorem holding. Now use Rouche's theorem to verify that $z^5 + 3z^2 - 1$ must have two zeros in $|z| < 1$.

5. We say a real valued function, u is subharmonic if $u_{xx} + u_{yy} \geq 0$. Show that if u is subharmonic on a bounded region, (open connected set) U, and continuous on $\overline{U}$ and $u \leq m$ on ∂U, then $u \leq 0$ on U. State and prove a theorem about the uniqueness of the solutions to the equation, $u_{xx} + u_{yy} = 0$ in U and $u = f$ on ∂U. **Hint for the first part:** If not, u achieves its maximum at $(x_0, y_0) \in U$. Let $u(x_0, y_0) > m + \delta$ where $\delta > 0$. Now consider $u_\varepsilon(x, y) = \varepsilon x^2 + u(x, y)$ where ε is small enough that $0 < \varepsilon x^2 < \delta$ for all $(x, y) \in U$. Show that u_ε also achieves its maximum at some point of U and that therefore, $u_{\varepsilon xx} + u_{\varepsilon yy} \leq 0$ at that point implying that $u_{xx} + u_{yy} \leq -\varepsilon$, a contradiction.

6. Use Rouche's theorem to prove the fundamental theorem of algebra which says that if $p(z) = z^n + a_{n-1}z^{n-1} \cdots + a_1 z + a_0$, then p has n zeros in $\mathbb{C}$. **Hint:** Let $q(z) = -z^n$ and let γ be a large circle, $\gamma(t) = re^{it}$ for r sufficiently large.

7. It is desired to find an analytic function, $L(z)$ defined for all $z \in \mathbb{C} \setminus \{0\}$ such that $e^{L(z)} = z$. Is this possible? Explain why or why not.

8. If f is analytic, show that $z \rightarrow \overline{f(\overline{z})}$ is also analytic.

9. Let $f : U \rightarrow \mathbb{C}$ be analytic and $f(z) = u(x, y) + iv(x, y)$. Show u, v and uv are all harmonic although it can happen that u^2 is not. Recall that a function, w is harmonic if $w_{xx} + w_{yy} = 0$.

10. Prove Liouville's theorem from the Cauchy integral formula.

11. Consider the polynomial, $z^{11} + 7z^5 + 3z^2 - 17$. Use Rouche's theorem to find a bound on the zeros of this polynomial. In other words, find r such that if z is a zero of the polynomial, $|z| < r$. Try to make r fairly small if possible.

12. Verify that $\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$. **Hint:** Use polar coordinates.