Mathematics 112

Departmental Competency Examination

April 10, 1997

NO CALCULATORS ARE ALLOWED

1. (12 points) Determine whether each of the following limits exist. If it exists, compute it; if it doesn't, explain why.

(i) $\displaystyle\lim_{x\to 1^-} \frac{|x-1|}{4(x-1)}$

(ii) $\displaystyle\lim_{x\to 2} \frac{x^4-16}{x-2}$

(iii) $\displaystyle\lim_{x\to 0} \frac{\sin(2x)}{5x}$

(iv) $\displaystyle\lim_{x\to 3^-} \frac{x^2+4}{x^2-9}$

(v) $\displaystyle\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$

(vi) $\displaystyle\lim_{x\to \infty} \frac{\sin^2 x}{x}$

2. (15 points) Compute the derivative, $g'(x)$ for each of the following functions $g(x)$:

(i) $g(x) = (x^2+1)\tan x$

(ii) $g(x) = (x^4-2)^{100}$

(iii) $g(x) = \dfrac{(x^3-4)}{(x^2+1)}$

(iv) $g(x) = e^{\cos(2x^5+3)}$

(v) $g(x) = 10^{(x^2+1)}$

3. (9 points) Compute each of the following:

(a) $\int_0^{\pi/6} \sin(3x)dx$

(b) $\int_0^1 x^2 e^{-x^3} dx$

(c) $\frac{d}{dx} \left\{ \int_0^{x^3} (1 + \sin t)^{2/3} dx \right\}$

4. (8 points) Compute the area bounded by the graphs of $y = \sin x,\ y = \cos x,\ x = 0$ and $x = 2\pi$. Set up the integrals expressing the area and compute.

5. Do each of the following:

   (a) (2 points) State Rolle's Theorem

   (b) (2 points) State the Mean Value Theorem for derivatives

   (c) (3 points) Assuming Rolle's Theorem, prove the Mean Value Theorem

   (d) (3 points) Verify the Mean Value Theorem for the function $f(x) = x^3 - 6x$ on the interval $[a, b]$, where $a = -2$ and $b = 2$.

6. (6 points) Determine a function f such that

$$f''(x) = x, \ f'(2) = 2 \ \text{ and } \ f(1) = -3.$$

7. (7 points) Suppose that $f : [0, 2] \to \mathbf{R}$ and $g : [0, 2] \to \mathbf{R}$ are continuous functions. State whether the following statements are True or False.

   (a) ______ $\int_0^2 f(x)g(x)dx = \left(\int_0^2 f(x)dx\right) \left(\int_0^2 g(x)dx\right)$

   (b) ______ $\int_1^2 f(x)dx = \int_0^2 f(x)dx - \int_0^1 f(x)dx$

   (c) ______ $\int_0^2 f(x)g(x)dx = f(x) \int_0^2 g(x)dx$

   (d) ______ $\int_0^1 f(x)dx = -\int_1^0 f(x)dx$

   (e) ______ If $f(x) \geq g(x)$ for every $x \in [0, 2]$, then

$$\int_0^2 f(x)dx \geq \int_0^2 g(x)dx$$

(f) ______ There exists $x_0 \in [0, 2]$ such that

$$f(x_0) = \frac{1}{2} \int_0^2 f(x)dx$$

(g) ______ $\left| \int_0^2 f(x)dx \right| = \int_0^2 |f(x)|dx$

8. Consider the function $f(x) = x^4 - 8x^2$ on the interval $[-4, 4]$.

(a) (2 points) Compute $f'(x)$ and $f''(x)$.

(b) (3 points) Determine the local extrema, end point extrema and absolute extrema of f on $[-4, 4]$.

(c) (2 points) Determine the points of inflection of the graph of f.

(d) (2 points) Sketch the graph of f on $[-4, 4]$.

9. (8 points) Sketch the graph of a function that satisfies ALL of the conditions given below.

   $f(-4) = 4$; $f(-1) = -1$; $f(2) = 2$; $f'(-1) = 0$

   $f'(x) = 0$ if $x < -3$; $f'(x) < 0$ on $(-3, -1)$;

   $f'(x) > 0$ on $(-1, \infty)$; $f''(x) > 0$ on $(-3, 2)$;

   $f''(x) < 0$ on $(2, \infty)$ and $f''(2) = 0$

10. (8 points) In a desert there is a cabin on the side of a straight road. There is a hiker in the desert 4 miles from the cabin, and the cabin is at the point on the road that is closest to the hiker. The hiker is out of water and needs to get to a spring that is 5 miles down the road from the cabin. Suppose that the hiker can travel 5 miles per hour on the road, but only 3 miles per hour when he travels off-road. If the hiker is free to choose his own path (and, in particular, does not need to go by the cabin), what path should he take to get to the spring as quickly as possible? How long will it take him to get there on that path?

11. A particle moving along a straight path has a velocity of $v(t)$ measured in miles per hour. The illustration below shows the graph of $v(t)$. Answer the following questions relative to the illustration.

(a) (2 points) At which of the times A, B, C, or D is the particle moving the fastest?

(b) (2 points) At which of the times A, B, C, or D is the particle farthest from its original location (at the time 0 hours)?

(c) (2 points) At which of the times A, B, C, or D does the particle have the greatest positive acceleration?

(d) (2 points) At which of the times A, B, C, or D is the acceleration closest to 0?