Math 113 – Fall 2003 – Solutions

Departmental Final Exam

Part I: Multiple Choice

Problems 1 through 7 are multiple choice. Each multiple choice problem is worth 4 points. In the grid below fill in the square corresponding to each correct answer.

1	A	B	C	D	E	■	G	H	I	J
2	A	B	C	D	■	F	G	H	I	J
3	A	B	C	■	E	F	G	H	I	J
4	A	■	C	D	E	F	G	H	I	J
5	A	B	C	D	E	F	G	H	■	J
6	A	B	C	D	E	■	G	H	I	J
7	A	B	C	D	E	■	G	H	I	J

1. In the decomposition of $\dfrac{4x^2+x+1}{(x-1)(x+1)(x-3)}$ as $\dfrac{A}{x-1}+\dfrac{B}{x+1}+\dfrac{C}{x-3}$, the value of C is

   (a) $3/2$
   (b) $-1/2$
   (c) $1/2$
   (d) $-2/3$
   (e) -5
   $\boxed{\text{(f)}}$ 5
   (g) $-3/2$
   (h) $1/5$
   (i) 2
   (j) None of the above

2. Find parametric equations for the circle

   $$(x-2)^2+(y+3)^2=25.$$

   In each of the choices assume $0 \le t \le 2\pi$.

   (a) $x = 2\cos t,\ y = 3\sin t$
   (b) $x = 2\cos t,\ y = -3\sin t$
   (c) $x = 10\cos t,\ y = -15\sin t$
   (d) $x = 2/5+\cos t,\ y = -3/5+\sin t$
   $\boxed{\text{(e)}}$ $x = 2+5\cos t,\ y = -3+5\sin t$
   (f) $x = 5+2\cos t,\ y = 5-3\sin t$
   (g) $x = 5+2\cos^2 t,\ y = 5-3\sin^2 t$
   (h) $x = 2+5\cos^2 t,\ y = -3+5\sin^2 t$

3. Which of the following integrals represents the arc length of the graph $y = \ln x$ from the point $(1,0)$ to the point $(2, \ln 2)$?

   (a) $\int_1^2 \sqrt{1+\ln x}\, dx$
   (b) $\int_0^{\ln 2} \sqrt{1+x^2}\, dx$
   (c) $\int_1^2 \sqrt{1+(\ln x)^2}\, dx$
   $\boxed{\text{(d)}}$ $\int_1^2 \sqrt{1+\frac{1}{x^2}}\, dx$
   (e) $\int_1^2 \sqrt{1-\ln x}\, dx$
   (f) $\int_0^{\ln 2} \sqrt{1-x^2}\, dx$
   (g) $\int_1^2 \sqrt{1-(\ln x)^2}\, dx$
   (h) $\int_1^2 \sqrt{1-\frac{1}{x^2}}\, dx$
   (i) $\int_1^2 x\sqrt{1-\frac{1}{x}}\, dx$
   (j) $\int_1^2 x\sqrt{1+\frac{1}{x}}\, dx$

4. Find the volume generated when the region above the x axis and below the graph $y = 4 - x^2$ is rotated around the y axis.

(a) 12π	(e) 10π	(i) 6π
[(b)] 8π	(f) 16π	(j) None of the above.
(c) 4π	(g) $512\pi/15$	
(d) $1024\pi/15$	(h) $256\pi/15$	

5. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$ equals

(a) $\dfrac{3^{n+1}}{n+1}$	(e) $\cos(3)$	[(i)] $e^3 - 1$
(b) $\ln 3$	(f) $\ln(3) - 1$	(j) None of the above
(c) $\ln 2$	(g) e^3	
(d) ∞	(h) 3^e	

6. The interval of convergence for the power series $\displaystyle\sum_{n=0}^{\infty} \frac{3^n(x-2)^n}{n}$ is

(a) $[-3, 3)$	(e) $(-1/3, 1/3]$	(i) $(-1, 1)$
(b) $(-\infty, \infty)$	[(f)] $[5/3, 7/3)$	(j) None of the above
(c) $(-1, 5]$	(g) $[-5, 1)$	
(d) $[-7/3, -5/3)$	(h) $(-2, 2)$	

7. Find the focus of the parabola $y^2 - 4y - 8x - 12 = 0$.

(a) $(-2, 2)$	(e) $(-2, 0)$	(i) $(-4, 2)$
(b) $(-2, 4)$	[(f)] $(0, 2)$	(j) None of the above.
(c) $(4, -2)$	(g) $(4, 2)$	
(d) $(2, 4)$	(h) $(0, -2)$	

The answers to the multiple choice MUST be entered on the grid on the previous page. Otherwise, you will not receive credit.

Part II: Written Solutions

For problems 8 - 19, write your answers in the space provided. Neatly show your work for full credit.

8. Compute $\displaystyle\int x^2 \cos x\, dx$

Solution. Use integration by parts:

$$u = x^2 \qquad dv = \cos x\, dx$$
$$du = 2x\, dx \qquad v = \sin x$$

$$\int x^2 \cos x\, dx = x^2 \sin x - \int 2x \sin x\, dx$$

Use integration by parts again:

$$u = 2x \qquad dv = \sin x\, dx$$
$$du = 2\, dx \qquad v = -\cos x$$

$$\begin{aligned}\int x^2 \cos x\, dx &= x^2 \sin x - \left[-2x\cos x - \int (-2)\cos x\, dx\right] \\ &= x^2 \sin x + 2x\cos x - 2\int \cos x\, dx \\ &= \boxed{x^2 \sin x + 2x \cos x - 2\sin x + C}\end{aligned}$$

□

9. Compute $\displaystyle\int_0^1 \frac{dx}{(1+x^2)^2}$

Solution. Use the substitution

$$x = \tan\theta \qquad x = 0 \quad \leftrightarrow \quad \theta = 0$$
$$dx = \sec^2\theta\, d\theta \qquad x = 1 \quad \leftrightarrow \quad \theta = \pi/4$$

Then

$$\begin{aligned}\int_0^1 \frac{dx}{(1+x^2)^2} &= \int_0^{\pi/4} \frac{\sec^2\theta}{(1+\tan^2\theta)^2 d\theta} = \int_0^{\pi/4} \frac{\sec^2\theta}{\sec^4\theta} d\theta \\ &= \int_0^{\pi/4} \cos^2\theta\, d\theta = \int_0^{\pi/4} \frac{1+\cos(2\theta)}{2}\, d\theta \\ &= \frac{1}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Bigg|_0^{\pi/4} \\ &= \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2}\cdot 1\right) = \boxed{\frac{\pi}{8} + \frac{1}{4}}\end{aligned}$$

□

10. Compute $\displaystyle\int_1^4 \frac{dx}{1+\sqrt{x}}$

Solution. Use the substitution:

$$\begin{aligned} x &= u^2 & \qquad x = 1 \quad &\leftrightarrow \quad u = 1 \\ dx &= 2u\,du & \qquad x = 4 \quad &\leftrightarrow \quad u = 2 \end{aligned}$$

Then

$$\begin{aligned} \int_1^4 \frac{dx}{1+\sqrt{x}} &= \int_1^2 \frac{2u\,du}{1+u} \\ &= 2\int_1^2 \left(1 - \frac{1}{u+1}\right) du \\ &= 2[u - \ln(u+1)]\Big|_1^2 \\ &= 2[2 - \ln 3 - (1 - \ln 2)] \\ &= \boxed{2(1 + \ln 2 - \ln 3)} \quad \text{or} \quad \boxed{2(1 + \ln(2/3))} \end{aligned}$$

□

11. Compute $\displaystyle\int_0^{\pi/4} \sec^6(x)\,dx$

Solution.

$$\begin{aligned} \int_0^{\pi/4} \sec^6 x\,dx &= \int_0^{\pi/4} \sec^4 x \sec^2 x\,dx \\ &= \int_0^{\pi/4} (1 + \tan^2 x)^2 \sec^2 x\,dx \\ &= \int_0^{\pi/4} (1 + 2\tan^2 x + \tan^4 x) \sec^2 x\,dx \\ &= \left(\tan x + \frac{2}{3}\tan^3 x + \frac{1}{5}\tan^5 x\right)\Bigg|_0^{\pi/4} \\ &= 1 + \frac{2}{3} + \frac{1}{5} - 0 \\ &= \frac{15 + 10 + 3}{15} \\ &= \boxed{\frac{28}{15}} \end{aligned}$$

□

12. Find the surface area generated by rotating the graph $y = x^3$, from $x = 1$ to $x = 2$, around the x axis.

Solution. The surface area is

$$A = 2\pi \int_1^2 y\sqrt{1 + (y')^2} dx$$

where $y = x^3$ and $y' = 3x^2$. Then

$$A = 2\pi \int_1^2 x^3 \sqrt{1 + (3x^2)^2}\, dx = 2\pi \int_1^2 x^3 \sqrt{1 + 9x^4}\, dx$$

Use the substitution

$$\begin{aligned} u &= 1 + 9x^4 & \qquad x = 1 \quad &\leftrightarrow \quad u = 10 \\ du &= 36x^3\, dx & \qquad x = 2 \quad &\leftrightarrow \quad u = 145 \\ x^3\, dx &= \frac{1}{36} du \end{aligned}$$

Then

$$\begin{aligned} A &= 2\pi \int_{10}^{145} \sqrt{u}\, \frac{1}{36}\, du \\ &= \frac{\pi}{18} \frac{2}{3} u^{3/2} \Big|_{10}^{145} \\ &= \boxed{\frac{\pi}{27}\left(145^{3/2} - 10^{3/2}\right)} \quad \text{or} \quad \boxed{\frac{\pi}{27}\left(145\sqrt{145} - 10\sqrt{10}\right)} \end{aligned}$$

□

13. A plate covering the region below $y = 1 + x^2$ over $[1, 3]$ has density $\delta(x) = 4x$. Find the mass of the plate.

Solution.

$$\begin{aligned} \text{Mass} &= \int_1^3 \delta(x)\, dA = \int_1^3 4x(1 + x^2)\, dx \\ &= \int_1^3 (4x + 4x^3)\, dx = \left(2x^2 + x^4\right) \Big|_1^3 \\ &= (18 + 81) - (2 + 1) = \boxed{96} \end{aligned}$$

□

14. Determine whether the series $\sum \frac{(-1)^n}{1+\ln n}$ is absolutely convergent, conditionally convergent, or divergent. Cite appropriate convergence test(s) and explain why it/they apply.

Solution. The series is *conditionally convergent.*

First, we'll check that the sum of absolute values diverges.

$$\sum \left| \frac{(-1)^n}{1+\ln n} \right| = \sum \frac{1}{1+\ln n}$$

$1 + \ln n < 1 + n$ so that $\frac{1}{1+\ln} > \frac{1}{1+n}$. The harmonic series $\sum \frac{1}{n+1}$ diverges. So, $\sum \frac{1}{1+\ln n}$ also diverges by the *basic comparison test.*

Second, we'll check that the alternating sum converges.

$$\frac{1}{1+\ln n} > \frac{1}{1+\ln(n+1)} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{1+\ln n} = 0.$$

So, $\sum \frac{(-1)^n}{1+\ln n}$ converges by the *alternating series test.* □

15. What is the coefficient of x^{20} in the Maclaurin series for $f(x) = x^2 e^{x^3}$?

Solution.

$$\begin{aligned} x^2 e^{x^3} &= x^2 \left(1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \frac{x^{12}}{4!} + \frac{x^{15}}{5!} + \frac{x^{18}}{6!} + \cdots\right) \\ &= x^2 + \cdots + \frac{x^{20}}{6!} + \cdots \end{aligned}$$

The coefficient of x^{20} is $\boxed{\frac{1}{6!}}$ or $\boxed{\frac{1}{720}}$.

□

16. Find the Taylor polynomial of degree 3 centered at $c = \dfrac{\pi}{6}$ for the function $f(x) = \cos x$.

Solution.

$$\begin{aligned}
f(x) &= \cos x & f(\pi/6) &= \frac{\sqrt{3}}{2} \\
f'(x) &= -\sin x & f'(\pi/6) &= -\frac{1}{2} \\
f''(x) &= -\cos x & f''(\pi/6) &= -\frac{\sqrt{3}}{2} \\
f'''(x) &= \sin x & f''(\pi/6) &= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
P_3(x) &= \frac{\sqrt{3}}{2} + \frac{(-1/2)}{1!}\left(x - \frac{\pi}{6}\right) + \frac{(-\sqrt{3}/2)}{2!}\left(x - \frac{\pi}{6}\right)^2 + \frac{(1/2)}{3!}\left(x - \frac{\pi}{6}\right)^3 \\
&= \boxed{\frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{6}\right)^2 + \frac{1}{12}\left(x - \frac{\pi}{6}\right)^3}
\end{aligned}$$

□

17. Find the z coordinate of the centroid of the (solid) half-ball

$$0 \le z \le \sqrt{4 - (x^2 + y^2)}.$$

Solution. The centroid $\bar{z}$ is

$$\begin{aligned}
\bar{z} = \frac{\int z\,dV}{\int dV} &= \frac{\int_0^2 z\pi(\sqrt{4 - z^2})^2\,dz}{\frac{2}{3}\pi(2)^3} \\
&= \frac{\int_0^2 z(4 - z^2)\,dz}{16/3} = \frac{3}{16}\int_0^2 (4z - z^3)\,dz \\
&= \frac{3}{16}\left(2z^2 - \frac{z^4}{4}\right)\bigg|_0^2 = \frac{3}{16}(8 - 4) = \boxed{\frac{3}{4}}
\end{aligned}$$

□

18. Find the area between $y = x$ and $y = x^2$ for $0 \leq x \leq 3$.

Solution.

$$\begin{aligned}
A &= \int_0^3 |x - x^2| \, dx \\
&= \int_0^1 (x - x^2) \, dx + \int_1^3 (x^2 - x) \, dx \\
&= \left(\frac{x^2}{2} - \frac{x^3}{3} \right) \Bigg|_0^1 + \left(\frac{x^3}{3} - \frac{x^2}{2} \right) \Bigg|_1^3 \\
&= \left(\frac{1}{2} - \frac{1}{3} \right) - 0 + \left(9 - \frac{9}{2} \right) - \left(\frac{1}{3} - \frac{1}{2} \right) \\
&= \frac{1}{2} - \frac{1}{3} + 9 - \frac{9}{2} - \frac{1}{3} + \frac{1}{2} \\
&= 1 - \frac{2}{3} + \frac{9}{2} = \frac{1}{3} - \frac{9}{2} = \frac{2 + 27}{6} \\
&= \boxed{\frac{29}{6}}
\end{aligned}$$

□

19. Tell whether the integral $\displaystyle\int_0^\infty \frac{dx}{1 + x^2}$ converges, and determine its value if it does converge.

Solution.

$$\begin{aligned}
\int_0^\infty \frac{dx}{1 + x^2} &= \lim_{b \to \infty} \int_0^b \frac{dx}{1 + x^2} \\
&= \lim_{b \to \infty} \arctan(x) \Big|_0^b \\
&= \lim_{b \to \infty} \big(\arctan(b) - \arctan(0) \big) \\
&= \frac{\pi}{2} - 0 \\
&= \boxed{\frac{\pi}{2}}
\end{aligned}$$

The integral *converges.* □