Math 112 (Calculus I)
Final Exam Form A KEY

Part I: Multiple Choice. Enter your answer on the scantron. Work will not be collected or reviewed.

Part II: Free Response. Show all work in the space provided. Use the back if you need more space, and indicate you have work there.. Work on scratch paper will not be reviewed.

21. (5 points) Find the derivative of $f(x) = x^2 \cos\sqrt{e^{2x}+1}$.

Solution:

$$\begin{aligned} f'(x) &= 2x\cos\sqrt{e^{2x}+1} - x^2\frac{2e^{2x}}{2\sqrt{e^{2x}+1}}\sin\sqrt{e^{2x}+1} \\ &= 2x\cos\sqrt{e^{2x}+1} - \frac{x^2e^{2x}}{\sqrt{e^{2x}+1}}\sin\sqrt{e^{2x}+1} \end{aligned}$$

22. (5 points) A new species is introduced into an environment in which it has no natural predators begins growing heartily. After two months, the 24 individuals introduced at the start have increased to 30.

(a) Assuming the growth is linear, find an expression for the number of individuals after t months.

Solution:
With linear growth, the slope, the rise over the run, is $m = (30-24)/2 = 3$. Thus,

$$y(t) = 3t + 24.$$

(b) With linear growth, how many individuals will there be after one year?

Solution:

$$y(12) = 3 \cdot 12 + 24 = 60$$

(c) Assuming the growth is exponential, find an expression for the number of individuals after t months.

Solution:
Here,

$$y(t) = 24e^{rt}.$$

Since $y(2) = 30$, we have

$$30 = 24e^{2r},$$

or

$$r = \frac{1}{2}\ln(5/4).$$

Thus,

$$y(t) = 24e^{\ln(5/4)t/2}.$$

(d) With exponential growth, how many individuals will there be after one year?

Solution:

$$y(12) = 24e^{\ln(5/4)6} = 24 \cdot \left(\frac{5}{4}\right)^6.$$

23. (5 points) Give an ϵ, δ proof of $\lim_{x\to 2}(5x-4) = 6$

Solution: Let $\epsilon > 0$ be given. Choose $\delta = \epsilon/5$. Then, if

$$0 < |x-2| < \delta,$$

$$|x-2| < \epsilon/5,$$

or

$$5|x-2| < \epsilon.$$

However,

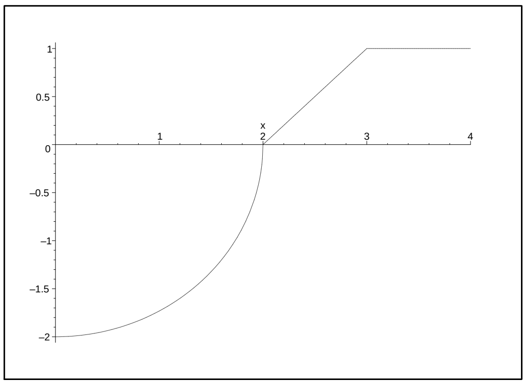
$$5|x-2| = |5x-10| = |5x-4-6|,$$

so

$$|5x-4-6| < \epsilon,$$

and we are done.

24. (5 points) Evaluate the following definite integrals from the graph of $f(x)$. The curved portion of the graph is a circular arc.



(a) $\int_0^2 f(x)\,dx$ $\quad -\pi$

(b) $\int_2^4 f(x)\,dx$ $\quad \frac{3}{2}$

(c) $\int_0^4 f(x)\,dx$ $\quad \frac{3}{2} - \pi$

(d) $\int_3^2 f(x)\,dx$ $\quad -\frac{1}{2}$

(e) $\int_2^3 2f(x)\,dx - \int_3^4 f(x)\,dx$ $\quad 0$

25. (5 points) Find the equation for the tangent line to the curve given by $x^3y + 8y^3 = 0$ through the point $(-2, 1)$.

Solution:
To find the derivative we differentiate implicitly:

$$3x^2y + x^3y' + 24y^2y' = 0$$

We plug in $x = -2$ and $y = 1$ to get

$$12 + 8y' + 24y' = 0, \text{ or } y' = -\frac{12}{32} = -\frac{3}{8}$$

Hence, the equation of the tangent line is

$$y - 1 = -\frac{3}{8}(x+2),$$

or

$$y = \frac{3}{8}x + \frac{1}{4}.$$

26. (5 points) An object moves along a coordinate line, its position at time t given by the function $x(t) = (4t-1)(t-1)^2, t \geq 0$.

   (a) When is the object moving to the right? When to the left? When does it change direction?

   Solution:
   Note that

$$x'(t) = 4(t-1)^2 + (4t-1)2(t-1) = (t-1)(4t-4+8t-2) = (t-1)(12t-6)$$

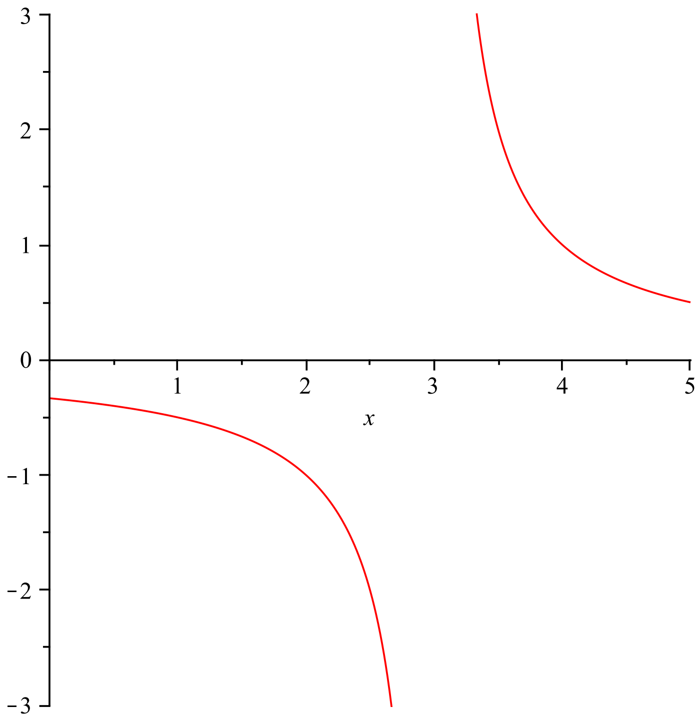
   The object is moving to the right when t is in $(-\infty, 1/2)$ and $(1, \infty)$.
   The object is moving to the left when t is in $(1/2, 1)$.
   The object changes direction at $t = 1/2$ and $t = 1$.

   (b) What is the maximum speed of the object when moving left?

   Solution:

$$|x'(3/4)| = |6(-1/4)(1/2)| = 3/4.$$

27. (5 points) Sketch the graph of $f(x) = \dfrac{x-2}{x^2-5x+6}$. Show all asymptotes.



28. (5 points) Each graph in the figure below is a position function for a particle moving along the x-axis during time $0 \leq t \leq 5$. The vertical scales are the same. Which particle has

   (a) constant velocity? d)

   (b) the greatest initial velocity? c)

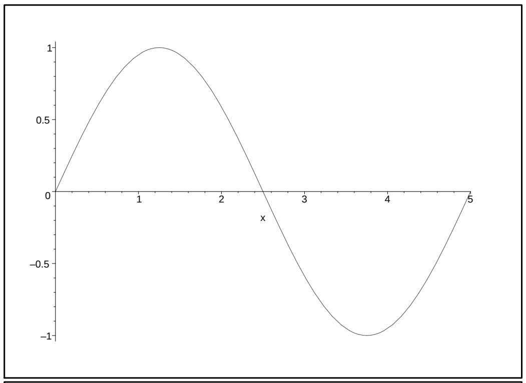
   (c) the greatest average velocity? c)

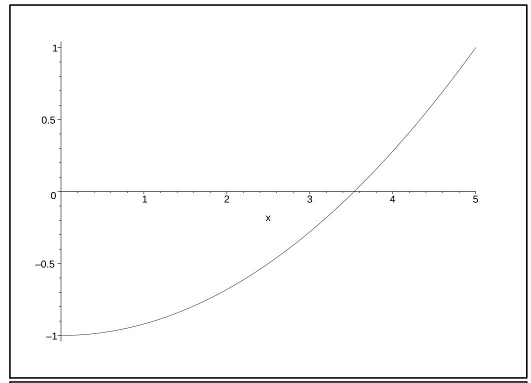
   (d) zero average velocity? a)

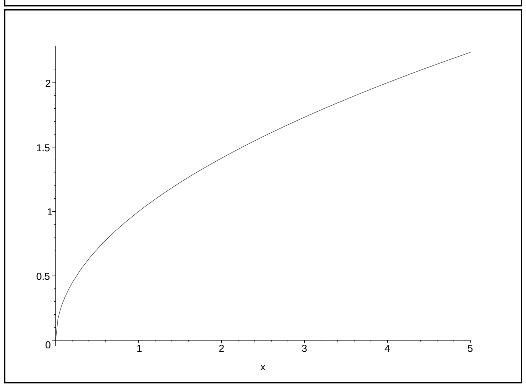
   (e) zero acceleration? d)

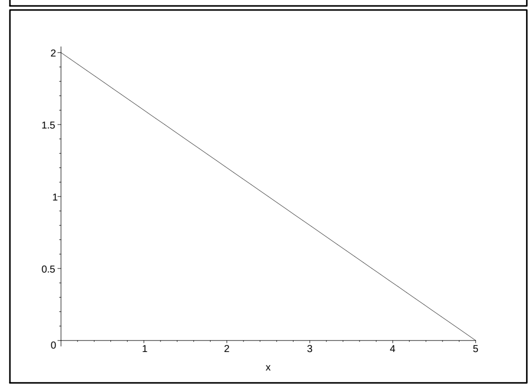
(f) positive acceleration throughout? b)

(g) negative acceleration throughout? c)

a) 

b) 

c) 

d) 

29. (5 points) Use logarithmic differentiation to evaluate the derivative of $y = \sqrt[3]{x}e^{x^2}(x^2+2)^8$.

Solution: Notice that

$$ln(y) = \ln\left(\sqrt[3]{x}e^{x^2}(x^2+2)^8\right) = \frac{1}{3}\ln(x) + x^2 + 8\ln(x^2+2).$$

Differentiating, we have

$$\frac{1}{y}y' = \frac{1}{3x} + 2x + \frac{16x}{x^2+2},$$

or

$$y' = \sqrt[3]{x}e^{x^2}(x^2+2)^8\left(\frac{1}{3x} + 2x + \frac{16x}{x^2+2}\right).$$

30. (5 points) Use l'Hospital's Rule to evaluate $\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$.

31. (5 points) Use l'Hospital's Rule to evaluate $\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$.

Solution:

$$\frac{x}{x-1} - \frac{1}{\ln x} = \frac{x\ln x - x + 1}{\ln x(x-1)},$$

so

$$\begin{aligned}\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right) &= \lim_{x\to 1}\frac{x\ln x - x + 1}{\ln x(x-1)} \\ &= \lim_{x\to 1}\frac{\ln x + 1 - 1}{\ln x + 1 - \frac{1}{x}} \\ &= \lim_{x\to 1}\frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}} = \frac{1}{2}\end{aligned}$$

END OF EXAM