Math 113 (Calculus II)
Final Exam KEY

Short Answer. Fill in the blank with the appropriate answer.

1. (10 points)

   a. Let $y = f(x)$ for $x \in [a, b]$. Give the formula for the length of the curve formed by the graph of this function. $\int_a^b \sqrt{1 + (f'(x))^2}\, dx$

   b. Find the vertex of the graph of $y = x^2 + 4x$. $(-2, -4)$

   c. The equation $2x^2 + 3x + y^2 - y = 7$ is called a ellipse.

   d. If $r = f(\theta), \theta \in [a, b]$ is an equation of a curve in polar coordinates, give the formula for the area enclosed by this curve. $\int_a^b \frac{1}{2} f^2(\theta)\, d\theta$

   e. What does the ratio test predict with regard to the convergence of the series $\sum_{n=1}^{\infty} \frac{n^2}{n^4 + 1}$? The ratio test fails.

   f. What is the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{4^n}{n} x^n$? $\frac{1}{4}$

   g. The integral $\int_0^\infty \frac{dx}{1+x^2}$ equals $\frac{\pi}{2}$.

   h. Here is an antiderivative: $\int \frac{\sqrt{9-x^2}}{x^2} dx$. Tell what substitution to use in order to find this antiderivative.$x = 3 \sin \theta$

   i. The integral $\int_0^1 \frac{1}{x^{2/3}} dx$ equals 3

   j. The antiderivative $\int x \sin(x)\, dx$ equals $(\sin x - x \cos x + C)$

Multiple Choice. In the grid below fill in the correct answer to each question.

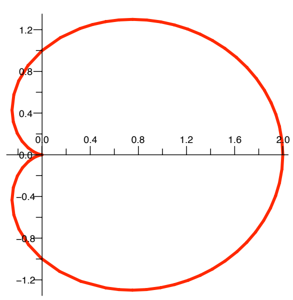
2	A	B	C	D	**E**	F	G	H	I	J
3	A	**B**	C	D	E	F	G	H	I	J
4	A	B	C	D	E	**F**	G	H	I	J
5	A	B	C	D	E	**F**	G	H	I	J
6	A	B	C	D	**E**	F	G	H	I	J
7	A	B	C	D	**E**	F	G	H	I	J
8	**A**	B	C	D	E	F	G	H	I	J
9	A	B	C	D	E	**F**	G	H	I	J

2. Find the area enclosed by the polar curve $r = 2 + \sin\theta$ for $\theta \in [0, 2\pi]$.

a) $\frac{5}{2}\pi$ b) $\frac{11}{3}\pi$ c) 2π

d) 4π e) $\frac{9}{2}\pi$ f) None of the above.

3. Identify the equation which goes with the polar graph,



a) $r = 1 + 2\cos\theta$ b) $r = 1 + \cos\theta$ c) $r = 2 + \sin\theta$

d) $r = 2\sin(2\theta)$ e) $r = 2\cos(2\theta)$ f) None of the above.

4. Which of the following series has as its interval of convergence $(-\infty, 1]$?

a) $\sum x^2/n!$ b) $\sum e^{-n}x^{2n}$ c) $\sum \frac{(x-1)^n}{2^n}$

d) $\sum x^{n-1}/n^2$ e) $\sum (x+1)^{n+1}/n^n$ f) None of these

5. Determine whether the following series is convergent. If it is, find its sum.

$$\sum_{n=1}^{\infty}\left(e^{1/n}-e^{1/(n+1)}\right)$$

a) It is divergent. b) e c) $\frac{1}{e}$

d) $e-e^{1/2}$ e) $\ln(1)$ f) None of these.

6. If we differentiate the power series

$$\frac{x^3}{3!}-\frac{x^5}{5!}+\frac{x^7}{7!}-\frac{x^9}{9!}+...$$

the resulting function is

a) e^{-x} b) $\cos 2x$ c) $\sin x$

d) only expressible as a series. e) $-(\cos x-1)$ f) $\cosh x+1$

7. Find the area between $y=2x$ and $y=x^2$ for $x\in[-1,3]$.

a) $\frac{8}{3}$ b) $\frac{4}{3}$ c) $-\frac{4}{3}$

d) 3 e) 4 f) None of the above

8. Let A denote the region between the graphs of $y = \cos(x)+1$ and the line $y=1$ for $x\in[0,\pi/2]$. A solid is obtained by revolving A about the x axis. find the volume of the solid.

a) $\frac{\pi^2}{4}+2\pi$ b) $\frac{\pi^2}{2}+2\pi$ c) $\frac{\pi^2}{2}$

d) $\frac{\pi^2}{4}$ e) $\frac{\pi^2}{4}-1$ f) None of these

9. Which of the following integrals represents the surface area of the surface generated by revolving the curve $y=e^{2x}$ for $x\in[0,1]$ about the line $y=-1$?

a) $\int_0^1 2\pi\left(e^{2x}-1\right)\sqrt{1+4e^{4x}}dx$ b) $\int_0^1\left(e^{2x}+1\right)dx$ c) $\int_0^1 2\pi\left(e^{2x}+1\right)\left(1+2e^{2x}\right)dx$

d) $\int_0^1 2\pi\left(e^{2x}+1\right)\sqrt{1+e^{4x}}dx$ e) $\int_0^1\left(e^{4x}+1\right)dx$ f) None of the above.

Free Response. For problems 10 - 17, write your answers in the space provided. Use the back of the page if needed, indicating that fact. Neatly show all work.

10. (8 points) A ball of radius 5 has a hole of radius 3 drilled completely through it such that the axis of the hole is a diameter of the ball. What is the volume of what is left?

    Solution:

    $2\pi \int_3^5 2x\sqrt{25-x^2}dx = \frac{256}{3}\pi$

    You could also do it by cross sections, (washers).

    $2\int_0^4 \pi \left(\left(\sqrt{25-y^2}\right)^2 - 9\right) dy = \frac{256}{3}\pi$

11. (8 points) A spherical tank having radius 10 feet is filled with a fluid which weighs 100 pounds per cubic foot. This tank is half full. Find the work in foot pounds needed to pump the fluid out of a hole in the top of the tank.

    Solution: $\int_{-10}^{0} (10-y)\, 100\pi\, (100-y^2)\, dy = \frac{2750\,000}{3}\pi$

12. (6 points) Determine whether the following series converges and explain your answer.

$$\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$$

    Solution: Notice that

$$\lim_{n\to\infty} \frac{\frac{1}{3n^2+2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{3n^2+2} = \frac{1}{3}.$$

    Since

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

    converges (p series, p=2¿1), the original series converges by the limit comparison test.

    The integral test or comparison test could also be used.

13. (6 points) The economic trickle-down effect is based on the idea that injection of money into the economy reaches far beyond the initial receiver of funds. Say that one person receives a dollar and spends 4/5 of it. The person who receives that portion spends 4/5 of it in turn and so on and on "forever." The total cash flow from the first dollar is the sum of all these passed-on funds including the first dollar. What is that sum?

    Solution: Add up $\sum_{n=0}^{\infty}(4/5)^n$ a geometric series.
    This equals 5.

14. (6 points) Compute the first 4 nonzero terms of the MacLaurin series for the function $f(x) = \ln(x^2+1)$.

    Solution: One can use the formula

$$\sum \frac{f^n(0)}{n!} x^n$$

to get

$$x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4}$$

Or, Since students know the series for

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n,$$

they only need to replace x by x^2 to get

$$\ln(1+x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^{2n}.$$

15. (8 points) A cube with 2 foot long sides is sitting on the bottom of an aquarium in which the water is 5 feet deep. Find the hydrostatic force on one of the sides of the cube. Use 62.5 pounds per cubic foot as the weight density of water.

    Solution:

$$\int_3^5 62.5h \cdot 2\,dh = 62.5h^2|_{3^5} = 16 \cdot 62.5 = 990.$$

16. Find the following

    (a) (6 points) $\int_0^{\pi/2} \sin^3(x)\cos^3(x)\,dx$

    Solution: $\int_0^{\pi/2} \sin^3(x)\cos^3(x)\,dx = \frac{1}{12}$

    (b) (6 points) $\int \frac{x+5}{x^2-1} dx$.

    Solution: Using partial fractions, we see that

$$\frac{x+5}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1},$$

or

$$x + 5 = A(x+1) + B(x-1).$$

By plugging in ± 1 for x, we see that $A = 3$ and $B = -2$. Thus,

$$\int \frac{x+5}{x^2-1} dx = \int (\frac{3}{x-1} - \frac{2}{x+1}\,dx = 3\ln|x-1| - 2\ln|x+1|.$$

    (c) (6 points) $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$

    Solution: Let $x = 2\tan\theta$. Then, $dx = 2\sec^2\theta$. The integral becomes

$$\int \frac{2\sec^2\theta}{\tan^2\theta \cdot 2\sec\theta}\,d\theta = \int \csc\theta\cot\theta\,d\theta = -\csc\theta + C.$$

17. (6 points) Here is a parameterized curve called a cycloid. find the equation of the tangent line when $\theta = \pi/3$.

$$x(\theta) = \theta - \sin\theta$$
$$y(\theta) = 1 - \cos(\theta)$$

Solution: You need the slope, dy/dx.

$$dy/dx = \frac{\sin(\theta)}{1-\cos(\theta)}$$

when $\theta = \pi/3$ this is

$$\frac{\sin(\pi/3)}{1-\cos(\pi/3)} = \sqrt{3}$$

and so the equation of the line is

$$y - \frac{1}{2} = \sqrt{3}\left(x - \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right)$$

END OF EXAM