

MASTER'S EXAMINATION IN MATHEMATICS
12 September 2000

INSTRUCTIONS.

MS Candidates: Answer a total of eight questions, with at most two from Basic Analysis and at most two from Basic Algebra.

MA Candidates: Answer a total of eight questions, with no restriction on category.

- I. Basic Analysis
- II. Basic Algebra
- III. Analysis
- IV. Advanced Algebra
- V. Applied Mathematics
- VI. Topology

PLEASE WRITE YOUR ANSWER TO EACH QUESTION ON A SEPARATE PAGE. USE $8\frac{1}{2} \times 11$ INCH PAPER. PLEASE WRITE YOUR NAME AT THE TOP OF EACH PAGE. THIS WILL MAKE IT EASIER TO SEPARATE AND SORT THESE PAGES FOR GRADING AND REASSEMBLING THEM AFTERWARDS.

I. BASIC ANALYSIS

1. Do one of the following:

- (i) Define what it means for a function f on a closed interval to be integrable and show that an increasing function is integrable.
- (ii) Define what it means for a subset of \mathbf{R}^n to be compact in terms of coverings and show that the continuous image of a compact set is compact, namely, show that if $A \subset \mathbf{R}^n$ and $f : A \rightarrow \mathbf{R}^m$ is a continuous function, then $f(A)$ is compact.

2. Do one of the following:

- (i) Show that a sequence of real numbers has a monotone subsequence. (A sequence x_n is monotone if either $x_n \leq x_{n+1}$ for each n or $x_n \geq x_{n+1}$ for each n .)
- (ii) State the Inverse Function Theorem for a function $f : \mathbf{R} \rightarrow \mathbf{R}$. Prove the theorem.

3. Do one of the following:

- (i) State and prove the Ratio Test.
- (ii) Find all values of p for which so that the improper integrals are finite.

$$(a) \int_0^1 \frac{1}{x^p} dx \qquad (b) \int_e^\infty \frac{1}{x \ln^p x} dx$$

4. Do one of the following:

- (i) Prove that the uniform limit of continuous functions is continuous.
- (ii) If $\sum_{k=0}^\infty a_k c^k$ is convergent and $|x| < |c|$, show that $\sum_{k=0}^\infty a_k x^k$ is absolutely convergent.

II. BASIC ALGEBRA

1. Let \mathbf{P} denote the vector space over the real numbers consisting of all polynomials in the variable x with real coefficients. \mathbf{P} is an inner product space relative to the inner product $\langle \cdot, \cdot \rangle$ defined as follows: for polynomials $p(x), q(x) \in \mathbf{P}$,

$$\langle p(x), q(x) \rangle = \int_{-1}^1 p(x)q(x) dx.$$

Apply the Gram-Schmidt process to the set $\{1, x, x^2, x^3\}$ to produce an orthonormal set of vectors in \mathbf{P} .

2. Let A be an abelian normal subgroup of the group G and let B be any subgroup of G . Prove that $A \cap B$ is a normal subgroup of the group AB where $AB = \{ab \in G \mid a \in A, b \in B\}$.
3. (a) For a positive integer N , $\mathbb{Z}/N\mathbb{Z}$ will denote the additive abelian group whose elements are residue classes modulo N . Show that if m and n are relatively prime positive integers, then

$$\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}.$$

direct sum $\mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$.

- (b) Give an explicit example to illustrate that the previous isomorphism does *not* hold true if m and n are not relatively prime.
4. Show that if G is a group and H is a subgroup of index 2, then H is normal in G .

III. ANALYSIS

1. Let A_1, A_2, \dots be Lebesgue measurable sets in \mathbb{R}^n . Let m be a positive integer. Let E_m be the set of x belonging to A_k for *at least* m values of k . Prove that E_m is measurable and

$$m\lambda(E_m) \leq \sum_{k=1}^{\infty} \lambda(A_k).$$

(λ denotes Lebesgue measure).

2. Let $f \in L^1(\mathbb{R}^n)$. Prove that for each $\epsilon > 0$ there are disjoint intervals I_1, \dots, I_m in \mathbb{R}^n such that

$$\int_{\mathbb{R}^n} \left| f - \sum_{j=1}^m a_j x_{I_j} \right| d\lambda < \epsilon$$

for some a_1, \dots, a_m in \mathbb{C} .

Hint: First let f be indicator function of a measurable set.

3. Show that

$$\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3} dy = \frac{1}{2} \text{ and } \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3} = -\frac{1}{2}.$$

Deduce that $\int_{[0,1]^2} \frac{|x-y|}{(x+y)^3} d\lambda = +\infty$.

4. Let $t \in \mathbb{R} \sim \mathbb{Z}$ be fixed. Use Parseval's identity for the periodic function equal to e^{itx} for $-\pi < x < \pi$ to obtain

$$\frac{\pi^2}{\sin^2 t\pi} = \sum_{n=-\infty}^{\infty} \frac{1}{(t-n)^2}.$$

IV. ADVANCED ALGEBRA

1. Show that every rotation of 3 space has an axis of rotation. Do this by showing that if A is a 3×3 orthogonal matrix with determinant 1 then A has a real eigenvector with eigenvalue 1.
2. Let α be any root of the irreducible polynomial $x^3 + x + 1 \in \mathbb{Q}[x]$. In the field extension $\mathbb{Q}(\alpha)$ compute the following:
 - (a) $(1 + \alpha)(3 - \alpha - \alpha^2)$
 - (b) $\frac{1}{3 - \alpha - \alpha^2}$(In this exercise, 'compute' means to find an expression of the form $a + b\alpha + c\alpha^2$ where $a, b, c \in \mathbb{Q}$.)
3. Show that every finite integral domain is a field.
4. Calculate the Galois over \mathbb{Q} of $x^3 - 2$.

V. APPLIED MATHEMATICS

1. Let X be the space of continuous real-valued functions on $[-1, 1]$. Define a bilinear form on X by

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x)(1-x^2)dx.$$

- (a) Prove that this defines an inner product on X .
- (b) Use the Gram-Schmidt procedure to produce orthonormal functions from the sequence $\{1, x, x^2, x^3\}$ in the order given.

2. Find $g(x, t)$ such that

$$u(x) = \int_0^{\pi/2} g(x, t)f(t)dt$$

solves the boundary value problem

$$\begin{cases} u'' + u = f(x), & 0 < x < \frac{\pi}{2} \\ u(0) = 0 = u\left(\frac{\pi}{2}\right) \end{cases}$$

3. (a) Give general conditions on f and u so that the boundary value problem

$$\begin{cases} u'' + 4u = f & \text{in } (0, \pi) \\ u(0) = u(\pi) = 0 \end{cases}$$

has a unique solution.

- (b) Give an expression for the solution u in terms of f .

4. Let f be a complex-valued function on $(-\pi, \pi)$ for which $\int_{-\pi}^{\pi} |f(x)|^2 dx < \infty$. Define for $h = 0, \pm 1, \pm 2, \dots$

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-ikx} dx.$$

Prove that $\sum_{k=-\infty}^{\infty} |c_k|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$ (Bessel's inequality).

VI. TOPOLOGY

1. Let X be the unit square $[0, 1] \times [0, 1]$ lexicographically ordered ($(a, b) < (c, d)$ if $a < c$, or if $a = c$ and $b < d$)

Show that X is compact in the order topology by showing that if \mathcal{G} is a covering of X by open intervals then there is a finite $\mathcal{F} \subset \mathcal{G}$ with \mathcal{F} covering X . (you may assume that closed intervals of \mathbb{R} are compact).

2. Let X be a linearly ordered space with the linear topology. Show that X is regular. That is: For each $x \in X$ and each closed $A \subset X$ with $x \notin A$, show that there are disjoint open sets U and V such that $x \in U$ and $A \subset V$.
3. Prove the following version of the tube lemma: Lemma: If \mathcal{G} is a covering of $X \times Y$ with no finite subcover, and if Y is compact, then there is an $x \in X$ such that the slice $\{x\} \times Y$ is not covered by any finite subcollection of \mathcal{G} .
4. Prove the following: If $A \subset \mathbb{R}^2$ is countable, show that $\mathbb{R}^2 - A$ is path connected. (Hint: How many lines pass through a given point in \mathbb{R}^2).