MASTER'S EXAMINATION IN MATHEMATICS
29 April 2000

INSTRUCTIONS.

MS Candidates: Answer a total of eight questions, with at most two from Basic Analysis and at most two from Basic Algebra.

MA Candidates: Answer a total of eight questions, with no restriction on category.

I. Basic Analysis

II. Basic Algebra

III. Analysis

IV. Advanced Algebra

V. Applied Mathematics

VI. Topology

PLEASE WRITE YOUR ANSWER TO EACH QUESTION ON A SEPARATE PAGE. USE $8\frac{1}{2} \times 11$ INCH PAPER. PLEASE WRITE YOUR NAME AT THE TOP OF EACH PAGE. THIS WILL MAKE IT EASIER TO SEPARATE AND SORT THESE PAGES FOR GRADING AND REASSEMBLING THEM AFTERWARDS.

I. BASIC ANALYSIS

1. Do one of the following:

   (a) Prove that if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous, then $f \circ g$ is continuous. item[(b)] Prove that if f is differentiable at c, then f is continuous at c.

2. Do one of the following:

   (a) Prove that the function $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$ is differentiable at 0.

   (b) Suppose f is differentiable on (a, b) and continuous on $[a, b]$ and $f(a) = f(b)$. Prove that if $f(c) > f(a)$ for some $c \in (a, b)$, then $\exists x_1, x_2 \in (a, b)$ such that $f'(x_1) > f'(x_2)$.

3. Do one of the following:

   (a) Prove the Fundamental Theorem of Calculus.

   (b) The top of a pop can costs twice as much per square in ch as the sides or the bottom. For a fixed volume V, find the ratio of height to base radius which minimizes the cost of the can.

4. Do one of the following:

   (a) Prove that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

   (b) Prove that if $a_i \geq a_{i+1} \geq 0$, $\forall i$ and $(a_i) \to 0$, then $\sum_{i=1}^{\infty} (-1)^i a_i$ converges.

II. BASIC ALGEBRA

1. Prove that if H is a subgroup of the finite group G, then $|H|$ divides $|G|$.

2. List all of the groups (up to isomorphism) of order 6 and give a proof that there are not others.

3. Prove that if L is a linear map from R^n to R^m that $n = \text{Rank}(L) + \text{Nullity }(L)$.

4. Give an example of a matrix M so that $M^3 = 0$, but $M^2 \neq 0$.

III. ANALYSIS

1. Suppose E_k is a measurable set and $\sum_{k=1}^{\infty} \mu(E_k) < \infty$. Show that the set,

$$A \equiv \{\omega \in \Omega : \omega \in E_k \text{ infinitely often}\}$$

has measure zero and is a measurable set. **Hint:** Write the set of interest in terms of countable intersections and unions of the sets, E_k.

2. Chebyshev's inequality says that if $f \in L^1(\Omega)$ and if $A_\delta \equiv \{x \in \Omega : |f(x)| \geq \delta\}$, then

$$\mu(A_\delta) \leq \frac{1}{\delta} \int_\Omega |f(x)| d\mu.$$

Prove this inequality.

3. Suppose that $p, q, r > 0$ and that for $\theta \in [0, 1]$,

$$\frac{1}{r} = \frac{\theta}{p} + \frac{(1-\theta)}{q}.$$

Establish the following inequality:

$$\left(\int |f|^r d\mu \right)^{\frac{1}{r}} \leq \left(\int |f|^p d\mu \right)^{\frac{\theta}{p}} \left(\int |f|^q d\mu \right)^{\frac{1-\theta}{q}}.$$

4. Show that if $u(x, y) + iv(x, y) = f(z)$ is analytic, then $\nabla u \cdot \nabla v = 0$. Recall the Cauchy Riemann equations and

$$\nabla u(x, y) = \langle u_x(x, y), u_y(x, y) \rangle.$$

IV. ADVANCED ALGEBRA

1. Let R be a ring. Prove that the following three conditions are equivalent:

   (a) Given any chain of ideals in R

   $$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

   there exists a positive integer m such that $I_k = I_m$ for all $m \geq k$.

   (b) Every nonempty set of ideals of R contains a maximal element under inclusion.

   (c) Every ideal of R is finitely generated.

2. Explicitly construct a finite field with 49 elements.

3. Let A be a Hermitian $n \times n$ matrix such that $A^2 = A$. Prove that A is positive semidefinite.

4. A group G is called residually finite if $\forall g \in G \quad g \neq 1 \quad \exists f : G \to F$ homomorphism with F finite and $f(g) \neq 1$ in F. Prove that a residually finite group is Hopfian, that is, every onto self homomorphism is an isomorphism.

V. APPLIED MATHEMATICS

1. A viscous fluid in 3 dimensions obeys the equations of conservation of mass and conservation of momentum (Navier-Stokes)

$$\frac{D\rho}{Dt} + \rho \bigtriangledown \cdot \mathbf{v} = 0$$
$$\frac{D\mathbf{v}}{Dt} = f - \frac{1}{\rho} \bigtriangledown p + \frac{\mu}{\rho} \bigtriangleup \mathbf{v}.$$

   (a) What do these become if the fluid is incompressible, has constant density and there is no external force.

   (b) Assume, in addition, that the flow is bounded by parallel planes $z = h,\ z = -h$, is in the x-direction, and depends only on z. Find the pressure p and the velocity in the x-direction.

2. Determine all eigenvalues and eigenfunctions of the Sturm-Lionville system

$$(SL) \begin{cases} \frac{d}{dx}\left(e^{-2x}\frac{df}{dx}\right) + \lambda e^{-2x} f = 0, \\ f(0) = 0, \quad f(0) = 1. \end{cases}$$

(i.e., determine all scalars λ and all nonzero functions f satisfying (SL).)

3. (a) Reduce $u'' + 6u' + 9u = 0$ to a first order system.

   (b) Find the general solution to this sytem by finding an eigenvalue and eigenfunction and a generalized eigenvalue and eigenfunction.

   (c) Use your solution in part (b) to write the solution to the DE in (a). (Note: you can check your solution in (b) by solving the DE by standard methods.)

4. Let V be the vector space generated by the collection $(2, -1, 2, 3)^T$, $(3, -2, -2, 2)^T$, $(4, 3, -1, 1)^T$.

   a. Find the vector in V that is closest to the vector $B = (1, -2, 2, 1)^T$.

   b. Resolve the vector B into orthogonal components, one component in V and one perpendicular to V.

   c. The result of the above solves some least squares fit problem for a linear system. Write the system and tell whether it is consistent or inconsistent.

VI. TOPOLOGY

1. Show that the closed interval $[0, 1]$, of the real numbers is compact; i.e., show that for each collection $\{U_\alpha\}$ of open sets covering $[0, 1]$, a finite subcollecton covers.

2. Let $f : [0, 1] \to R$ be a function from the unit interval into the real numbers.

   (a) Define what it means for f to be *continuous*.

   (b) Define what ie means for f to be *uniformly continuous*.

   (c) Show that if f is continuous, then f must be uniformly continuous.

3. Let (p_n) be a sequence of distinct real numbers. Show that (p_n) either has a subsequence that is increasing or a subsequence that is decreasing.

4. Let X be a locally compact Hausdorff space which is not compact.

   (a) Define the *one-point compactification* of X.

   (b) Show that X is a subspace of its one-point compactification.

   (c) Show that the one-point compactification is a compact Hausdorff space.