MASTER'S EXAMINATION IN MATHEMATICS
Saturday, 11 May 2002

INSTRUCTIONS.

Answer a total of eight questions, with at most two from Basic Analysis and at most two from Basic Algebra.

I. Basic Analysis

II. Groups and Linear Algebra

III. Advanced Analysis

IV. Rings, Modules, Fields and Galois Theory

V. Applied Mathematics

VI. Topology

PLEASE WRITE YOUR ANSWER TO EACH QUESTION ON A SEPARATE PAGE. USE $8\frac{1}{2} \times 11$ INCH PAPER. PLEASE WRITE YOUR NAME AT THE TOP OF EACH PAGE. THIS WILL MAKE IT EASIER TO SEPARATE AND SORT THESE PAGES FOR GRADING AND REASSEMBLING THEM AFTERWARDS.

I. BASIC ANALYSIS

1. Let $f : (a,b) \to \mathbb{R}$ be uniformly continuous. Prove that $\exists\ F : [a,b] \to \mathbb{R}$ continuous such that $f(x) = F(x)\ \forall\ x \in (a,b)$.

2. Let $(c_k) \subseteq \mathbb{R}$ be a sequence an suppose $a \in \mathbb{R}$ so that $\Sigma c_k a^k$ converges. Prove that for any $0 \leq r < |a|$ the power series $\Sigma c_k x^k$ converges uniformly on $[-r, r]$.

3. Given $f : \mathbb{R}^n \to \mathbb{R}^n$ 1 to 1 and smooth (Df is never the zero matrix), and $K \subseteq \mathbb{R}^n$ compact. Prove that for some $c > 0$, $\|f(u) - f(v)\| \geq c\|u - v\| \quad \forall\ u, v \in K$.

4. Prove the Fundamental Theorem of Calculus.

II. GROUPS AND LINEAR ALGEBRA

1. Let G be a group and $H < G$ with the index, $[G : H] = n$. Prove that there is a homomorphism $\phi_H : G \to S_n$ with kern $\phi < H$.

2. Let G be a group of order 245. Show that G has a normal subgroup of order 49.

3. Let A be an $n \times n$ matrix. The minor diagonal is the line from the lower left corner to the upper right corner $= \{a_{ij} | i + j = n + 1\}$. The untranspose of A, $A^u = A$ reflected through the minor diagonal. If $B = A^u$, then $a_{ij} = b_{(n+1)-j,(n+1)-i}$. Prove that $\det A = \det A^u$.

4. (a) Classify up to isomorphism all abelian groups of order 72.
   (b) Which of these is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_{18}$?

III. ADVANCED ANALYSIS

1. 1. Let $f : [0, 1] \to \mathbb{R}$ be a function of bounded variation.

   (a) Show that for every $c \in (0, 1)$, the left- and right-hand limits of f at c exist.

   (b) Show that f has at most countably many discontinuities.

2. State and prove the Riemann-Lebesgue Lemma for Fourier Transforms of functions $f \in L^1(\mathbb{R}^n)$. (You may assume the continuity of translation in $L^1(\mathbb{R}^n)$.)

3. Let $(X, \mathcal{M}, \mu)$ be a measure space satisfying $\mu(X) = 1$, and let $f : X \to (0, \infty)$ and $g : X \to (0, \infty)$ be $\mathcal{M}$-measurable functions satisfying $f(x)g(x) \geq 1$ for every $x \in X$. Show that

$$\left(\int_X f \, d\mu\right) \left(\int_X g \, d\mu\right) \geq 1.$$

4. Which of the following six statements must be true if $1 \leq p \leq q \leq \infty$? Give explicit counterexamples for the statements you claim are not necessarily true. (You do not need to prove or justify the statements you claim *are* necessarily true.)

   (a) $L^p(\mathbb{R}) \subseteq L^q(\mathbb{R})$

   (b) $L^q(\mathbb{R}) \subseteq L^p(\mathbb{R})$

   (c) $L^p([0, 1]) \subseteq L^q([0, 1])$

   (d) $L^q([0, 1]) \subseteq L^p([0, 1])$

   (e) $l^p(\mathbb{Z}) \subseteq l^q(\mathbb{Z})$

   (f) $l^q(\mathbb{Z}) \subseteq l^p(\mathbb{Z})$

IV. RINGS, MODULES, FIELDS AND GALOIS THEORY

1. If F is a finite field, show that $|F| = p^n$ for some prime p.

2. What is the Galois group of $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ over $\mathbb{Q}$?

3. In a principal ideal domain, show that every nonzero prime ideal is maximal.

4. Show that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

V. APPLIED MATHEMATICS

1. Consider the variational principle: Find u such that

$$\int_0^{\pi} F(x,u,u')\,dx = \min \text{ subject to } u(0)=0 \text{ and } u(\pi)=0.$$

   Derive the Euler equation of this variational principle, i.e. find a differential equation which is an equivalent problem to the minimization problem.

2. The singular value decomposition (SVD) of a (real) $m \times n$ matrix A is defined by

$$A = U\Sigma V^T$$

   where U and V are orthogonal and Σ is a diagonal matrix. Show how to use the SVD to derive a diagonal linear system that is equivalent to

$$Ax = b \tag{1}$$

   (where $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$). Describe how to answer the following questions using the diagonal system: Are there solutions of (1)? If so, what is the dimension of the (affine) solution space, and what is a general form of the solution?

3. Consider a lake in which people regularly go fishing. We wish to model the fish-fisherman interaction. Assumptions:

   (a) Fish grow logistically in the absence of fishing. Assume that for small fish populations the growth is directly proportional to the population and that as the fish population approaches 1 the growth rate goes to zero.

   (b) The presence of fishermen depresses fish growth at a rate jointly proportional to the fish and the fishermen populations. Let the constant of proportionality be 1.

   (c) Fishermen are attracted to the lake at a rate directly proportional to the amount of fish in the lake.

   (d) Fishermen are discouraged from the lake at a rate directly proportional to the number of fishermen already there.

   Write down a mathematical model for this situation. Calculate the steady states, comment on their stability and draw a phase portrait showing the null-clines.

4. Find traveling wave solutions of

$$\frac{\partial u}{\partial t} = -c^2 \frac{\partial^2 u}{\partial x^2} - 2u.$$

VI. TOPOLOGY

1. Show that if S is a locally compact regular space which is the union of a countable collection G of closed sets, then some element of G has an open nonempty subset.

2. If every uncountable subset of a metric space S has a limit point, then S is separable.

3. If H and K are disjoint closed subsets of a normal space S, then there exists a continuous real valued function f defined over S such that

   (a) $0 \leq f(x) \leq 1$ for $x \in S$

   (b) $f(x) = 0$ for $x \in H$, and

   (c) $f(x) = 1$ for $x \in K$.

4. Show that in a connected linear ordered space with the linear topology, every bounded infinite point set has a limit point.