MASTER'S EXAMINATION IN MATHEMATICS
Saturday, 17 January 2004

INSTRUCTIONS.

Answer a total of eight questions, with at most two from Basic Analysis.

I. Basic Analysis

II. Groups and Linear Algebra

III. Analysis

IV. Fields, Rings and Modules

V. Applied Mathematics

VI. Topology

PLEASE WRITE YOUR ANSWER TO EACH QUESTION ON A SEPARATE PAGE. USE $8\frac{1}{2}\times 11$ INCH PAPER. PLEASE WRITE YOUR ASSIGNED NUMBER AT THE TOP OF EACH PAGE. THIS WILL MAKE IT EASIER TO SEPARATE AND SORT THESE PAGES FOR GRADING AND REASSEMBLING THEM AFTERWARDS.

I. BASIC ANALYSIS

1. Let $f : [a, b] \to \mathbb{R}$ continuous. Show that f is integrable.

2. Let X and Y be metric spaces and $f : X \to Y$ a continuous bijective (1 to 1 and onto) function. Show that if X is compact, then $f^{-1} : Y \to X$ is continuous.

3. Let $f : \mathbb{R}^n \to \mathbb{R}$ differentiable. Show that for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ there exist $\mathbf{c}$ on a line segment from $\mathbf{a}$ to $\mathbf{b}$ such that $f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(\mathbf{c}) \cdot (\mathbf{b} - \mathbf{a})$.

4. Show that every sequence of real numbers has a monotone subsequence.

II. GROUPS AND LINEAR ALGEBRA

1. Are the matrices $\begin{bmatrix} 1 & 3 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ and $\begin{bmatrix} 6 & 7 & 8 \\ 0 & 4 & 9 \\ 0 & 0 & 1 \end{bmatrix}$ similar? Explain.

2. Find a 3×3 orthogonal matrix with all entries nonzero. (No partial credit.)

3. Let Q be the quaternion group of order 8. Prove that every subgroup of Q is normal.

4. Let G be the alternating group A_4. Find a Sylow p-subgroup of G for each prime p dividing the order of G.

III. ANALYSIS

1. Let $X = [0,1]$ with Lebesgue measure. Define what it means for a real-valued function f with domain X to be in the space L^p where $p \geq 1$. Define the norm on L^p. Give an example of a sequence of functions f_n in L^p so that f_n converges to the zero function, but for each x in X, $f_n(x)$ does not converge.

2. Prove that the function $f(x) = \dfrac{\sin x}{x}$ is not Lebesgue integrable. Is it Riemann integrable?

3. Prove there is a bounded subset of the real numbers that is not Lebesgue measurable.

4. Stirling's formula gives an approximation of the gamma function $\Gamma(a+1) = \sqrt{2\pi} a^{a+1/2} e^{-a}$. Explain in what sense this is an approximation and use it to derive Wallis' formula: $\dfrac{\pi}{2} = \dfrac{2}{1} \cdot \dfrac{2}{3} \cdot \dfrac{4}{3} \cdot \dfrac{4}{5} \cdot \dfrac{6}{5} \cdot \dfrac{6}{7} \cdots$.

IV. FIELDS, RINGS AND MODULES

1. Let F be a field of order p^n (p a prime), and let E be a finite extension of F. Let $\alpha(x) = x^{p^n}$ ($\forall x \in E$). Prove that α generates the group $Gal(E/F)$.

2. Let F be a field and E an algebraic extension of F. If D is a domain such that $F \subseteq D \subseteq E$, prove that D is a field. Show that the assumption that E be algebraic over F is necessary. [Hint: Consider $\{f(\pi) \big| f(x) \in Q[x]\}$.]

3. Prove that every finite extension over a field is algebraic.

4. Let F be the splitting field of $x^4 - 5$ over Q. Find $Gal(F/Q)$ and express it as a subgroup of S_4.

V. APPLIED MATHEMATICS

1. (a) Find the Fourier sine series of x, on $0 \leq x < L$.
   (b) Where does the series converge to the function?
   (c) Can you differentiate the series term by term to get the Fourier cosine series for 1?
   (d) When can you differentiate a Fourier series of a function f term by term to obtain the Fourier series of f'?

2. Prove that $\mathbf{\Phi}(t)\mathbf{\Phi}(s) = \mathbf{\Phi}(t+s)$, where $\mathbf{\Phi}(t)$ is a matrix whose columns are linearly independent solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and $\mathbf{\Phi}(0) = I$.

3. Consider the following variational principle. Find u subject to $u(0) = u(1) = 0$ which minimizes
$$I(u) = \int_0^1 (u'(x))^2 + 2u(x).$$
   (a) Derive an equivalent second order boundary value problem.
   (b) Solve the boundary value problem.

4. Find the general solution for
$$\mathbf{y}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 0 \\ \sin t \end{pmatrix}$$

VI. TOPOLOGY

1. (a) State and prove the tube lemma.
   (b) Prove that the product of finitely many compact spaces is compact.

2. (a) Give an example, with proof, of a space which is connected, but not path-connected.
   (b) Prove that the unit sphere S^{n-1} in $\mathbb{R}^n$ is connected if $n > 1$.

3. Construct, with proof, a space filling curve; show that it is space filling. You may use results from question 4 if necessary.

4. Let (Y, d) be a metric space and X a space. Define the uniform metric corresponding to d. Show that the set $\mathcal{C}(X, Y)$ of continuous functions from X to Y is a complete space under the uniform metric.