MASTER'S EXAMINATION IN MATHEMATICS
Saturday, 21 January 2006

INSTRUCTIONS.

Answer a total of eight questions, with at most two from Basic Analysis.

I. Basic Analysis

II. Groups and Linear Algebra

III. Analysis

IV. Fields, Rings and Modules

V. Applied Mathematics

VI. Topology

DO NOT USE A RESULT MORE ADVANCED THAN THE ONE YOU ARE TRYING TO PROVE.

PLEASE WRITE YOUR ANSWER TO EACH QUESTION ON A SEPARATE PAGE. USE $8\frac{1}{2} \times 11$ INCH PAPER. PLEASE WRITE YOUR ASSIGNED NUMBER AT THE TOP OF EACH PAGE. THIS WILL MAKE IT EASIER TO SEPARATE AND SORT THESE PAGES FOR GRADING AND REASSEMBLING THEM AFTERWARDS.

I. BASIC ANALYSIS

1. Prove that in $\mathbb{R}$ every Cauchy sequence converges.

2. A twice differentiable function $f : \mathbb{R} \to \mathbb{R}$ is called convex if $f''(x) > 0$ for all $x \in \mathbb{R}$. Prove that a bounded convex function $f : \mathbb{R} \to \mathbb{R}$ is constant.

3. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ continuously differentiable with the property that $\forall\ x \in \mathbb{R}^n$ rank $DF(x) = n$ (where $DF(x)$ is the derivative matrix of F at x). Prove that the image $F(\mathbb{R}^n)$ is an open subset of $\mathbb{R}^n$.

4. Let V be a normed vector space over $\mathbb{R}$ and U a connected open subset of V. Prove that U is path connected.

II. GROUPS AND LINEAR ALGEBRA

1. Prove there is a homomorphism from A_4 (the alternating group on four elements) onto the cyclic group of order three.

2. Determine how many abelian groups there are of order twenty-four, and prove your answer.

3. Let M be a two-by-two real matrix with *distinct* eigenvalues α and β. Prove there exist two-by-two matrices A and B such that $MA = \alpha A$ and $MB = \beta B$ and such that $A + B$ is the (two-by-two) identity matrix. Thus prove that $M^n = \alpha^n A + \beta^n B$.

4. Let G be the group of all non-singular, n by n real matrices, under the operation of matrix multiplication. Prove that G has a quotient group which is isomorphic to the *additive* group of all real numbers.

III. ANALYSIS

1. Suppose $f(x) = \sum_{k=1}^{\infty} u_k(x)$ where the functions, u_k are all differentiable and suppose for all $x, |u_k'(x)| \leq K_k$ where $\sum_{k=1}^{\infty} K_k < \infty$. Can it be concluded that
$$f'(x) = \sum_{k=1}^{\infty} u_k'(x).$$
Give a complete explanation consisting either of a proof or a counter example.

2. If A is a nonempty subset of $\mathbb{R}^n, d(\mathbf{x}, A) \equiv \inf\{d(\mathbf{x}, \mathbf{y}) : \mathbf{y} \in A\}$. Verify that $|d(\mathbf{x}, A) - d(\mathbf{y}, A)| \leq d(\mathbf{x}, \mathbf{y})$.

3. Suppose $\{A_k\}$ is a sequence of measurable sets and $\sum_{k=1}^{\infty} \lambda(A_k) < \infty$ where λ is a measure. Now let $S \equiv \{\mathbf{x} : \mathbf{x} \text{ is an element of infinitely many of the} A_k\}$. Show that S is a measurable set and that $\lambda(S) = 0$.

4. Show
$$\frac{\pi^2}{6} = \sum_{m=1}^{\infty} \frac{1}{m^2}.$$
Hint: You might try to use the Fourier series of $f(x)$ where $f(x) = x^2$ on $[-\pi, \pi]$ and is extended to be 2π periodic on the whole line.

5. Suppose $\{f_n\}$ is a sequence of measurable real valued functions. Define $A \equiv \{\mathbf{x} : \{f_n(\mathbf{x})\} \text{ converges.}\}$ Is A measurable? Explain why or give a counter example.

6. Suppose f is a lower semicontinuous function defined on $\mathbb{R}$ which has the property that
$$\lim_{|x| \to \infty} f(x) = \infty.$$
Prove that f achieves its minimum value on $\mathbb{R}$.

7. Suppose $f : X \to X$ where X is a complete metric space and suppose
$$d(f(x), f(y)) \leq r d(x, y)$$
where $r < 1$. Show f has a unique fixed point.

8. Construct an example of a subset of $[0, 1]$ which has Lebesgue measure equal to $\frac{5}{7}$ and which is compact and contains no intervals having positive length.

9. Let $\phi(0) = 0$, ϕ is strictly increasing, and C^1. Let $f : \Omega \to [0, \infty)$ be measurable. Then
$$\int_{\Omega} (\phi \circ f) d\mu = \int_0^{\infty} \phi'(\alpha) \mu[f > \alpha] d\alpha.$$
Here $(\Omega, \mathcal{S}, \mu)$ is a measure space.

10. Suppose f is a nonnegative Lebesgue measurable function which is in $L^2(\mathbb{R})$. Suppose also that for all $x \in \mathbb{R}$, $f(x) \geq 1/10$. Does it follow that $\int f(x) dm < \infty$? Either prove or disprove with a counter example.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be everywhere differentiable. Give an example which shows that f' does not need to be continuous. Show however that f' must be Borel measurable.

IV. FIELDS, RINGS AND MODULES

1. Let R be the ring
$$R = \mathbb{Z}[x, y]/I$$
where I is the ideal
$$I = (2, x^3 + x^2 + x + 1, y - x^3 - 1).$$
   (a) Determine whether R contains zero divisors. If so, give an explicit example of a zero divisor.
   (b) Determine *all* of the ideals of R.
   (c) Write R as a direct product of fields, if possible. If this is not possible, explain why not.

2. Let R be a commutative ring with $1 \neq 0$. Suppose I and J are ideals of R such that $I + J = R$. Prove that
$$R/(I \cap J) \cong R/I \oplus R/J.$$
After proving the previous isomorphism, show that this implies
$$\big(R/(I \cap J)\big)^\times \cong (R/I)^\times \oplus (R/J)^\times.$$
where, for a ring S, $S^\times$ denotes the group of units of S.

3. Let F be the splitting field of $x^4 - 5$ over $\mathbb{Q}$. Find $\mathrm{Gal}(F/\mathbb{Q})$ and express it as a subgroup of S_4 by identifying each element of $\mathrm{Gal}(F/\mathbb{Q})$ with a element of S_4 in cycle notation.

4. Let α be a root of the polynomial $x^3 - 2 \in \mathbb{Z}[z]$, and let β be a root of the polynomial $x^2 + x + 1 \in \mathbb{Z}[z]$. Determine the minimal polynomial of $\alpha + \beta$ over $\mathbb{Q}$.

V. APPLIED MATHEMATICS

1. (a) Use the definition of Laplace transform

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt$$

to show that

$$\mathcal{L}(u_2(t)f(t-2)) = e^{-2s}F(s)$$

where $u_c(t) = u(t-c)$ and $u(t)$ is the Heaviside function.

(b) Determine the solution of the initial value problem

$$y'' + 4y = f(t), \quad y(0) = 0, y'(0) = 1$$

where

$$f(t) = \delta(t-4)\log(1+\sin(\pi t)) + \begin{cases} 1, & 0 \le t < 1, \\ -1, & 1 \le t < 2, \\ 0, & \text{otherwise.} \end{cases}$$

2. Consider the system $\dfrac{dx}{dt} = 2y - xy, \; \dfrac{dy}{dt} = x + y.$

(a) Find all equilibrium (or critical) points of the system.

(b) Find the *linearized* systems about the equilibrium points and discuss the stability of these points.

(c) Give a phase portrait of the above first order system.

3. (a) Use Gram-Schmidt orthogonalization method, or otherwise, express x^3 as a linear combinations of polynomials P_0, P_1, P_2 and P_3 such that $P_n = x^n +$ lower order terms and

$$\int_{-1}^{1} P_i(x)P_j(x)\, dx = 0 \quad \text{if } i < j.$$

(b) Find a numerical integration scheme $\int_{-1}^{1} f(x)\, dx = w_1 f(x_1) + w_2 f(x_2)$ that is exact when the integrand is a polynomial of degree less than or equal to three.

4. (a) Determine the characteristic curves of $u_{tt} = e^t u_{xx}$.

(b) Find a solution of:

$$u_t = u_{xx} + 2x \quad \text{for } 0 < x < 1, t > 0$$

with initial and boundary conditions

$$u(x,0) = 0 \;\text{ for } 0 < x < 1; \qquad u(0;t) = u(1;t) = 0 \;\text{ for } t > 0$$

(Hint: First subtract a suitable time-independent solution. You may write the Fourier coefficients unevaluated integrals.)

5. The Green's function of the Laplace operator $\nabla^2 = \partial_{xx} + \partial_{yy}$ for the upper half plane is given by

$$G(x, y; \xi, \eta) = \frac{1}{2} \ln \frac{(x-\xi)^2 + (y-\eta)^2}{(x-\xi)^2 + (y+\eta)^2}$$

(a) State the equation and boundary condition satisfied by the function $\frac{1}{2\pi} G(x, y; 1, 2)$.

(b) Let $f(x, y)$ be a function defined on the upper half plane. Using the above Green's function to find the solution of

$$\nabla^2 u = f(x, y), \quad -\infty < x < \infty, y > 0$$

$$u(x, 0) = 0, \quad -\infty < x < \infty.$$

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}, \quad s > 0$	e^{at}	$\frac{1}{s-a}, \quad s > a$
t^n, $n > 0$ integer	$\frac{n!}{s^{n+1}}, \quad s > 0$	$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
$\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$	$\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \lvert a\rvert$	$\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \lvert a\rvert$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$	$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$t^n e^{at}$, $n > 0$ integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$	$u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
$u_c(t) f(t-c)$	$e^{-cs} F(s)$	$e^{ct} f(t)$	$F(s-c)$
$f(ct)$	$\frac{1}{c} F\left(\frac{s}{c}\right), \quad c > 0$	$(f * g)(t)$	$F(s)G(s)$
$\delta(t-c)$	e^{-cs}	18. $(-t)^n f(t)$	$F^{(n)}(s) = \frac{d^n}{ds^n} F(s)$

VI. TOPOLOGY

1. *Tychonoff's Theorem*: Prove that if X and Y are compact, then $X \times Y$ is also compact.

2. Show that the set of rational numbers is not a complete metric space.

3. Let K_n be a sequence of connected sets in a topological space so that K_n meets K_{n+1}. Prove that $\bigcup_{n=1}^{\infty} K_n$ is connected.

4. *Lebesgue Covering Lemma*: Suppose that X is a compact metric space with metric d and $\mathcal{G}$ is an open covering of X. Prove there is a positive number δ such that if $C \subset X$ and $\operatorname{diam}(C) < \delta$, then some element of $\mathcal{G}$ contains C.