MASTER'S EXAMINATION IN MATHEMATICS
Saturday, 2 June 2007

**INSTRUCTIONS.**

**Do all twelve problems.** You are expected to make an attempt to solve each problem. Partial credit can only be given if a problem is attempted. No notes, calculators or other aids are allowed. Appropriately justify your work. Do not use theorems or other results that would cause a problem to become trivial.

I. Algebra

II. Analysis

III. Topology

**DO NOT USE A RESULT MORE ADVANCED THAN THE ONE YOU ARE TRYING TO PROVE.**

PLEASE WRITE YOUR ANSWER TO EACH QUESTION ON A SEPARATE PAGE. USE $8\frac{1}{2}\times 11$ INCH PAPER. PLEASE WRITE YOUR ASSIGNED NUMBER AT THE TOP OF EACH PAGE. THIS WILL MAKE IT EASIER TO SEPARATE AND SORT THESE PAGES FOR GRADING AND REASSEMBLING THEM AFTERWARDS.

# I. ALGEBRA

1. Let $H$ be a subgroup of finite index $n$ in a group $G$. Let $C = \{H, g_2H, g_3H, \ldots, g_nH\}$ be the set of cosets of $H$ in $G$. Then $G$ acts on $C$ by $g(g_iH) = g_jH$ where $g_jH = (gg_i)H$, thus giving a homomorphism $\pi : G \to S_n; g(i) = j$. Let

$$Core(H) = \bigcap_{g \in G} gHg^{-1}$$

denote the intersection of all of the conjugates of $H$.

   (a) Show that
$$Core(H) = \ker(\pi).$$

   (b) Show that
$$[G : Core(H)] < \infty.$$

2. (a) Let $F \subset K, K \subset L$ be finite extensions of fields. Show that
$$[L : F] = [L : K][K : F].$$

   (b) Determine (with proof) the Galois group of $(x^3 - 2)(x^2 - 3)$.

3. Let $V$ be a finite-dimensional vector space over the field $F$. Let $V^* =$Hom$(V, F)$ denote the set of linear transformations $f : V \to F$. Prove that $V$ and $V^*$ have the same dimension.

4. Let $f : [a, b] \to \mathbb{R}$ be continuous and $f : (a, b) \to \mathbb{R}$ be differentiable. Prove that if $f(a) = f(b)$, then $\exists\ c \in (a, b)$ with $f'(c) = 0$.

## II. ANALYSIS

1. Give complete statements of the following theorems:

   (a) Fatou's Lemma

   (b) Lebesgue's Dominated Convergence Theorem

2. Give specific examples of the following:

   (a) A differentiable function $f : [0, 1] \to \mathbb{R}$ that is not absolutely continuous.

   (b) An absolutely continuous function $f : [0, 1] \to \mathbb{R}$ that is not Lipschitz continuous.

3. (a) Let $f : [a, b] \to \mathbb{R}$ be a real valued function on the closed interval $[a, b] \subset \mathbb{R}$. Accurately state what it means for $f$ to be integrable on the interval $[a, b]$.

   (b) Assume $f(x)$ is an integrable function on $[a, b]$, and let $A$ be a fixed real number. Let $g(x)$ be defined by

$$g(x) = \begin{cases} A & \text{if } x = a, \\ f(x) & \text{if } a < x \leq b. \end{cases}$$

   Use an argument with upper and lower sums to prove that $g$ is integrable on $[a, b]$.

   *Caution*: You should prove that $g$ is integrable without quoting a theorem that trivializes the question. For example, you may *not* quote a standard theorem which says: If $f : [a, b] \to \mathbb{R}$ is bounded, and $f$ is integrable on $[c, b]$ for all $c \in (a, b)$, then $f$ is integrable on $[a, b]$.

4. Let $S$ be the subset of $\mathbb{R}^3$ defined by

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \text{ and } z > 0\}.$$

   Evaluate the surface integral $\displaystyle\int_S f \, d\sigma$ where $f(x, y, z) = z^2$.

5. Use complex analysis to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x \, dx}{x^4 + 1}.$$

## III. TOPOLOGY

1. Recall that a topological space in which no point is isolated (e.g. no point is open) is called *perfect*. Show that a nonempty perfect subset of the unit interval must be uncountable.

2. Let $Y$ be the union of three line segments joined at one end to form a "Y", also known as a "triod", or the cone over three points. Show that there does not exist a continuous injective map from $Y$ into the line, $\mathbb{R}$.

3. State the Seifert-van Kampen Theorem and use it to calculate (with proof) the fundamental group of the projective plane, $\mathbb{P}^2$.