MASTER'S EXAMINATION IN MATHEMATICS
Saturday, 20 January 2007

INSTRUCTIONS.

Do all twelve problems. You are expected to make an attempt to solve each problem. Partial credit can only be given if a problem is attempted. No notes, calculators or other aids are allowed. Appropriately justify your work. Do not use theorems or other results that would cause a problem to become trivial.

I. Algebra

II. Analysis

III. Topology

DO NOT USE A RESULT MORE ADVANCED THAN THE ONE YOU ARE TRYING TO PROVE.

PLEASE WRITE YOUR ANSWER TO EACH QUESTION ON A SEPARATE PAGE. USE $8\frac{1}{2} \times 11$ INCH PAPER. PLEASE WRITE YOUR ASSIGNED NUMBER AT THE TOP OF EACH PAGE. THIS WILL MAKE IT EASIER TO SEPARATE AND SORT THESE PAGES FOR GRADING AND REASSEMBLING THEM AFTERWARDS.

I. ALGEBRA

1. Recall that $\mathbb{R}[x]$ denotes the ring of polynomials in x with real coefficients. Let I be the principal ideal generated by the polynomial $x^2 - 1$. Let $\psi : \mathbb{R}[x] \to \mathbb{R} \times \mathbb{R}$ be defined by $\psi(p(x)) = (p(1), p(-1))$ and prove that $\mathbb{R}[x]/I \cong \mathbb{R} \times \mathbb{R}$.

2. Prove that the Galois group of the splitting field of $x^4 - 2$ over Q has order 8 and contains an element of order four.

3. Let A and B be $n \times n$ matrices. Let λ be an eigenvalue of A, and let $\vec{x}$ be an eigenvector of A belonging to λ. Prove that if $BA = AB$ and the dimension of the eigenspace $N(A - \lambda I)$ is one, then there is an eigenvalue η of B for which $\vec{x}$ is an eigenvector of B belonging to η.

4. (i) Let $\mathbb{Z}$ denote the additive group of integers, and let $G = \mathbb{Z} \times \mathbb{Z}$ be the direct product of $\mathbb{Z}$ with itself. Let H be the subgroup of G defined by

$$H = \{(2m + n, 5n) \,|\, m, n \in \mathbb{Z}\}.$$

Prove that $G/H \equiv \mathbb{Z}/10\mathbb{Z}$.

(ii) How many isomorphism classes of abelian groups of order 72 are there?

II. ANALYSIS

1. Let $f_1, f_2, \ldots$ be a sequence of Riemann-integrable functions that converges uniformly to f on $[0,1]$. Prove that f is Riemann-integrable and that

$$\lim_{k\to\infty} \int_0^1 f_k(x)\, dx = \int_0^1 f(x)\, dx.$$

2. Give specific examples of each of the following:

   (i) A function $f : \mathbb{R}^2 \to \mathbb{R}$ for which the two first-order partial derivatives $f_1(0,0)$ and $f_2(0,0)$ exist and the directional derivative of f at $(0,0)$ in the direction $(1/\sqrt{2}, 1/\sqrt{2})$ exists, but that directional derivative is not equal to $f_1(0,0)/\sqrt{2} + f_2(0,0)/\sqrt{2}$.

   (ii) A function $f : \mathbb{R}^2 \to \mathbb{R}$ for which the total derivative of f at $(0,0)$ exists, but such that there is no neighborhood of $(0,0)$ on which the first-order partial derivatives $f_1(x,y)$ and $f_2(x,y)$ exist and are continuous.

3. Suppose $S \subseteq \mathbb{R}^n$. An *open cover* of S is a collection of open sets whose union has S as a subset. A *finite subcover* of an open cover of S is a finite subcollection of the open cover whose union also has S as a subset. We say S is *compact* if every open cover of S has a finite subcover.

   Using these definitions, but no powerful theorems like the Heine-Borel Theorem, prove that if S is compact then S is closed and bounded.

4. Assume $f \in L^1(\mathbb{R})$. Prove that

$$\lim_{h\to 0} \int |f(x+h) - f(x)| d\lambda(x) = 0$$

5. A sequence $\{f_n\}$ in L^p is said to converge weakly to a function f in L^p if $\int f_n g \to \int fg$ for all $g \in L^{p'}$, here $\frac{1}{p} + \frac{1}{p'} = 1$. Prove that if $f_n \to f$ in L^p norm, $1 \le p \le \infty$, then $\{f_n\}$ converge weakly to f in L^p. Is the converse also true?

III. TOPOLOGY

1. Every limit point compact metric space is separable.

2. Show that the complement of a countable set in $\mathbb{R}^2$ is connected.

3. Suppose that M is a complete metric space and each of $U_1, U_2, U_3, \ldots$ is a dense open set in M. Then $\bigcap_{n=1}^{\infty} U_n$ is a dense subset of M.