MASTER'S EXAMINATION IN MATHEMATICS
Saturday, 18 January 2008

INSTRUCTIONS.

Do all twelve problems. You are expected to make an attempt to solve each problem. Partial credit can only be given if a problem is attempted. No notes, calculators or other aids are allowed. Appropriately justify your work. Do not use theorems or other results that would cause a problem to become trivial.

I. Algebra

II. Analysis

III. Topology

DO NOT USE A RESULT MORE ADVANCED THAN THE ONE YOU ARE TRYING TO PROVE.

PLEASE WRITE YOUR ANSWER TO EACH QUESTION ON A SEPARATE PAGE. USE $8\frac{1}{2}\times 11$ INCH PAPER. PLEASE WRITE YOUR ASSIGNED NUMBER AT THE TOP OF EACH PAGE. THIS WILL MAKE IT EASIER TO SEPARATE AND SORT THESE PAGES FOR GRADING AND REASSEMBLING THEM AFTERWARDS.

I. ALGEBRA

1. Let A be a real orthogonal matrix. Prove that if λ is a real eigenvalue of A, then λ equals either 1 or -1.

2. (a) Let A be a commutative ring with 1. An element $a \in A$ is said to be nilpotent if $a^n = 0$ for some positive integer n. Prove that the nilpotent elements of A form an ideal in A.

   (b) Does the result of part (a) still hold if the hypothesis of commutativity is dropped? Prove or disprove.

3. Prove there is no simple group of order 148. (Recall that a group G is simple if the only normal subgroups are the identity subgroup and the whole group.)

4. Let F be a field with 81 elements. Does the polynomial $x^3 - x + 1$ have a root in this field? (The polynomial should be considered as having coefficients in $Z/3$.)

II. ANALYSIS

1. Let $f : [0, \infty) \to \mathbb{R}$ be bounded and continuous. Show that there exists $c \in [0, \infty)$ such that
$$\int_0^\infty f(x)\, e^{-x}\, dx = f(c).$$

2. Let $f_n(x) = nxe^{-nx}$. Prove that $f_n \to 0$ pointwise, but not uniformly on $[0, 1]$ as $n \to \infty$.

3. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly continuous. Prove that there exists positive constants a and b such that
$$|f(x)| \leq a|x| + b, \quad \forall x \in \mathbb{R}.$$

4. State the Lebesgue dominated convergence theorem for functions on $\mathbb{R}^n$. If $f_n(x) = a(b - c)x^{na-1} + b(c - a)x^{nb-1} + c(a - b)x^{nc-1}$, where a, b and c are positive integers, show that
$$\int_0^X \sum_{n=1}^\infty f_n(x)dx \text{ and } \sum_{n=1}^\infty \left(\int_0^X f_n(x)dx \right)$$
exist when $0 < X \leq 1$ and evaluate them (i) when $0 < X < 1$, (ii) when $X = 1$.

5. State Fubini's theorem for $\mathbb{R}^m \times \mathbb{R}^n$. Integrate the function $(\sin x)e^{-xy}$ over the set $(0, a) \times (0, \infty)$. Show that
$$\int_0^a \frac{\sin x}{x}\, dx = \frac{\pi}{2} - \cos a \int_0^\infty \frac{e^{-ay}}{1 + y^2} - \sin a \int_0^\infty \frac{ye^{-ay}}{1 + y^2}\, dy.$$

III. TOPOLOGY

1. Let $[a, b]$ be a closed interval in the real numbers $\mathbb{R}$ and $\mathcal{G}$ be a collection of open intervals that covers $[a, b]$. Show some finite subcollection of $\mathcal{G}$ cover $[a, b]$.

2. Prove the Lebesgue Number Lemma. Let $\mathcal{G}$ be an open covering of the metric space X with metric d. If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than δ, there exists an element of $\mathcal{G}$ containing it.

3. Prove the Brouwer Fixed-Point Theorem for the disc B^2. If $f : B^2 \to B^2$ is a continuous function, then there is a point $x \in B^2$ so that $f(x) = x$.