

PhD QUALIFYING EXAM - ALGEBRA
Saturday, 14 September 2002

INSTRUCTIONS. Each problem is worth 20 points. Do as many as you can. Calculators may **NOT** be used.

1. Show that every group of order $992 (= 32 \times 31)$ is solvable.
2. Find the Galois group of the splitting field of the polynomial $x^3 - x - 1$ if the group field is
 - (a) \mathbf{R}
 - (b) \mathbf{Q}
 - (c) $Z/2Z$
 - (d) $Z/5Z$
 - (e) $Z/23Z$
3. List all abelian groups of order 200. Each group on your list should be displayed as a direct product of cyclic groups, and no two groups on the list should be isomorphic.

4. Are the 4×4 matrices

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$

similar? Explain your reasoning.

5. Determine all ideals of the ring $Z[x]/(2, x^3 + 1)$.
6. Fermat proved that the number $2^{37} - 1 = 137438953471$ was composite by finding a small prime factor p . Suppose you know $200 < p < 300$. What is p ?
7. Determine the minimal polynomial of the element $\alpha = \sqrt{2} + \sqrt{5}$. (In other words, find $\text{Irr}(\alpha, Q)$.)
8. (a) Prove carefully that the rings $Q[x]/(x^2 - 2)$ and $Q[x]/(x^2 - 3)$ are not isomorphic.
(b) Find an example of a commutative ring with unity R such that $R[x]/(x^2 - 2)$ and $R[x]/(x^2 - 3)$ are isomorphic. Justify your answer briefly.
9. Let ζ be a primitive 13th root of unity. For each element α listed below, find $[Q(\alpha) : Q]$.
 - (a) $\alpha = \zeta$
 - (b) $\alpha = \zeta + \zeta^{12}$
 - (c) $\alpha = \zeta + \zeta^2$
10. An R -module M is said to be irreducible if $M \neq 0$ and M has no submodules except 0 and M . Let V be a finite-dimensional vector space over a field k , and let $T : V \rightarrow V$ be a linear transformation. We know that T gives V the structure of a $k[x]$ -module. Prove that V is irreducible as a $k[x]$ -module if and only if the characteristic polynomial of T is irreducible in $k[x]$.