Ph.D. Qualifying Exam–Algebra

Fall 2004

Answer as many questions as you can. Give an appropriate amount of detail for each proof. Your judgment in this matter is an important part of the exam.

1. Prove that if R is a commutative ring with 1, then every maximal ideal of R is prime.

2. Let p and q be distinct primes. Prove that any group of order pq is solvable.

3. Define the commutator subgroup G' of a group G, and prove that if N is a normal subgroup of G such that G/N is abelian, then G' is a subgroup of N.

4. Determine (with proof) a complete set of representatives of the conjugacy classes of the group $\mathrm{GL}_3(\mathbb{F}_2)$. Be sure that your list has no repetition.

5. Let m and n be positive integers. Compute (with justification) $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$.

6. Let R be a commutative ring with 1. Prove carefully that every proper ideal of R is contained in a maximal ideal of R.

7. Let R be a commutative ring with 1, and let $I_1, \ldots, I_n$ be ideals of R. If
$$J = I_1 \cap I_2 \cap \cdots \cap I_n$$
is a prime ideal of R, show that at least one of the ideals I_k, with $k \in \{1, \ldots, n\}$, is prime.

8. Find the Galois group of $x^5 - 4x^3 + 2$ over $\mathbb{Q}$.

9. Prove that no finite field is algebraically closed.

10. Determine (with proof) the number of irreducible degree 6 polynomials in $\mathbb{F}_2[x]$.