Ph.D. Qualifying Exam–Algebra

Summer 2004

Answer as many questions as you can. Give an appropriate amount of detail for each proof. Your judgment in this matter is an important part of the exam.

1. Prove that a finite integral domain is a field.

2. Prove that any group of order 42 is solvable.

3. Let A and B be normal subgroups of a group G, and assume that G/A and G/B are abelian. Prove that $A \cap B \lhd G$ and that $G/(A \cap B)$ is abelian.

4. Determine the rational and Jordan canonical forms of the matrix below, over a field of characteristic not equal to 2:
$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

5. Let R be a commutative ring with identity, and let P and Q be projective R-modules. Prove that $P \otimes_R Q$ is a projective R-module.

6. Prove that if R is a commutative ring with 1 such that $R[t]$ is a PID, then R is a field.

7. State and prove Eisenstein's criterion for monic polynomials with integer coefficients.

8. Give an example (with a very brief explanation) of each of the following:
   a: A UFD that is not a PID.
   b: A projective R-module that is not a free R-module.
   c: A division ring that is not a field.
   d: A normal extension of fields that is not Galois.
   e: Galois extensions M/L and L/K such that M/K is not Galois.

9. Let M/L and L/K be algebraic extensions of fields. Prove that M/K is also algebraic.

10. Determine (with proof) the Galois group of the polynomial $x^p - x + 1$ over $\mathbb{F}_p$, the finite field with p elements.