APPLIED MATHEMATICS
PH.D. PRELIMINARY EXAMINATION 2005

Option: Ordinary Differential Equations

Instructions: Give solutions to five of the following eight problems.

1. A solution $x(t)$, $0 \le t < \infty$, is called recurrent if $x(t_n) \to x(0)$ for some sequence $t_n \to \infty$. Prove that a gradient dynamical system has no nonconstant recurrent solution.

2. Sketch the phase portraits of

$$\dot{x} = y,$$
$$\dot{y} = x - x^2 + \epsilon y$$

when $\epsilon < 0$, $\epsilon = 0$ and $\epsilon > 0$.

3. Prove the following instability theorem: Let V be a C^1 real-valued function defined on a neighborhood U of an equilibrium point $\bar{x}$ of $x' = f(x)$. Suppose $V(\bar{x}) = 0$ and $\dot{V}(x) > 0$ in $U - \bar{x}$. If there is a convergent sequence $x_n \to \bar{x}$ such that $V(x_n) > 0$, then $\bar{x}$ is unstable.

4. Let U be an open subset of $\mathbb{R}^2$ and V be an open subset of U with $\bar{V} \subset U$. Let f is a C^1 function on U. Prove that if V contains a bounded positive orbit of $x' = f(x)$, then U contains an equilibrium point.

5. Find the normal form (up to order 3) of the following equation

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} + \begin{pmatrix} x^2 + xy^2 \\ y^2 + x^3 \end{pmatrix}$$

6. Consider the following boundary value problem

$$x'' + \lambda x = f(\lambda, x), \qquad 0 \le t \le \pi$$
$$x(0) = x(\pi) = 0.$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function, $f(\lambda, 0) = 0$, $D_x f(\lambda, 0) = 0$, $\lambda \in \mathbb{R}$ is the bifurcation parameter. Find bifurcation points.

7. Consider $x' = f(x)$ and its perturbed equation $x' = f(x) + \epsilon h(t, x)$ where $x \in \mathbb{R}^n$, f and h are C^2 functions, and $h(t, x)$ is $T-$periodic in t, $0 < \epsilon$ is a parameter. Prove that if $x' = f(x)$ has a hyperbolic equilibrium point p^*, then the perturbed equation $x' = f(x) + \epsilon h(t, x)$ has a unique periodic solution $p(t, \epsilon)$ such that

$$p(t, \epsilon) - p^* = O(\epsilon).$$

8. Consider $x' = Ax + f(x)$ where $x \in \mathbb{R}^n$, A is a $n \times n$ matrix, and f is a C^1 function with $f(0) = 0$ and $f'(0) = 0$. Assume that the set of eigenvalues $\sigma(A) = \sigma_c \cup \sigma_s$, where $\sigma_c = \{\lambda \in \sigma(A) \mid Re\ \lambda = 0\}$ and $\sigma_c = \{\lambda \in \sigma(A) \mid Re\ \lambda < 0\}$. Let $\mathbb{R}^n = E_c \oplus E_s$ be the corresponding decomposition. We further assume that there exists a global center manifold

$$W^c = \{p + h(p) | p \in E_c\}$$

where $h : E_c \to E_s$ is a C^1 function with $h(0) = 0$, $h'(0) = 0$ and $Lip(h) < 1$. Prove if $\dim(E_c) = 2$, then for each bounded solution $x(t, x_0)$ the omega limit set $\omega(x_0)$ of x_0 is a periodic orbit if the $\omega(x_0)$ contains no equilibrium point.