

MATH 113 - WINTER 1998  
Departmental Final Exam

Problems 1 through 11 are multiple choice. Select the best answer and fill in the corresponding bubble. Please make certain that your name and social security number are coded on the bubble sheet.

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- 2 pts 1. The series  $\sum_{n=1}^{\infty} a_n$  is guaranteed to converge if
- a) the sequence  $\{a_n\}$  converges      b)  $\lim_{n \rightarrow \infty} a_n = L$  for some constant  $L < 1$
- c)  $\lim_{n \rightarrow \infty} a_n = 0$       d)  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$
- e)  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 0$       f)  $\lim_{n \rightarrow \infty} |a_n|^{1/n} = 1$
- g) for all  $n, a_n > 0$  and  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = r$ , for some  $r$  such that  $0 \leq r < 1$ .
- h) none of the above is enough to guarantee that the series converges
- i) any of the conditions (a)-(g) is enough to guarantee that the series converges

- 1pt ea. 2&3. Fill in the blanks in the following statement to make it correct. The expression you put in the first blank will be your answer to (2), and the expression you put in the second blank will be your answer to (3):

"We say  $\lim_{n \rightarrow \infty} a_n = L$  if and only if for every  $\varepsilon > 0$ , there is a positive integer  $k$  such that \_\_\_\_\_ whenever \_\_\_\_\_."

- a)  $|a_n - n| < \varepsilon$       b)  $|a_k - \varepsilon| < L$       c)  $|a_n - L| < k$
- d)  $|a_n - k| < L$       e)  $|a_n - L| < \varepsilon$       f)  $|a_n - \varepsilon| < k$
- g)  $n > k$       h)  $k > n$       i)  $n > \varepsilon$
- j)  $k > \varepsilon$

- 2 pts 4. The improper integral  $\int_1^{\infty} \frac{1}{x^p} dx$  diverges if and only if

- a)  $p < 1$       b)  $p \leq 1$       c)  $p \geq 1$
- d)  $p > 1$       e)  $p \geq 2$

In problems (5)-(9), assume that  $f$  and  $g$  are continuous functions on  $[a,b]$  with  $f(x) \geq g(x) \geq 0$ , and assume that the derivative of  $f$  is also continuous on  $[a,b]$ . Choose from the following list, the correct formula for the stated task:

- a)  $\int_a^b \pi[f(x)]^2 dx$
- b)  $\int_a^b 2\pi x[f(x)]dx$
- c)  $\int_a^b f(x)dx$
- d)  $\int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2} dx$
- e)  $\int_a^b \sqrt{1+[f'(x)]^2} dx$
- f)  $\int_a^b x[f(x) - g(x)]dx$
- g)  $\int_a^b 2\pi x[f(x) - g(x)]dx$
- h)  $\int_a^b \frac{1}{2}([f(x)]^2 - [g(x)]^2)dx$
- i)  $\int_a^b \pi(f(x) - g(x))^2 dx$
- j) none of these

- 2 pts 5. To find the volume of the solid generated by revolving about the  $x$ -axis the region between the graph of  $y = f(x)$  and the  $x$ -axis, for  $x$  in  $[a,b]$ .
- 2 pts 6. To find the area of the surface generated by revolving about the  $x$ -axis the graph of  $y = f(x)$ , for  $x$  in  $[a,b]$ .
- 2 pts 7. To find the length of the curve formed by the graph of  $y = f(x)$ , for  $x$  in  $[a,b]$ .
- 2 pts 8. To find the volume of the solid generated by revolving about the  $y$ -axis the region between the graph of  $y = f(x)$  and the  $x$ -axis, for  $x$  in  $[a,b]$ .
- 2 pts 9. To find the volume of the solid generated by revolving about the  $x$ -axis the region between the graphs of  $f(x)$  and  $g(x)$ , for  $x$  in  $[a,b]$ .

Problems (10) and (11) have the same set of choices.

- a)  $\cos y$                       b)  $x \cos y + \sin y$                       c)  $\sin y$   
d)  $x \cos y$                       e)  $x + \sin y$                       f)  $1$   
g)  $1 + \cos y$                       h)  $-x \cos y$

2 pts    10.    If  $f(x,y) = x \sin y$ , what is  $f_x = \frac{\partial f}{\partial x}$ ?

2 pts    11.    If  $f(x,y) = x \sin y$ , what is  $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$ ?

*Problems 12-21 are write-in. Use the paper provided and write neatly.*

4 pts ea 12.    Compute the following four integrals:

i)     $\int_1^6 x^2 \ln x \, dx$

ii)     $\int \frac{x^3 + 4}{x^2 + x} \, dx$

iii)     $\int_0^4 x^3 \sqrt{16 - x^2} \, dx$

iv)  $\int \sin(3x)\cos(2x)dx$

3 pts ea 13. Determine whether the following three series converge or diverge. Justify your answers (e.g., by citing an appropriate convergence test).

i)  $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

ii)  $\sum_{n=1}^{\infty} \frac{3}{n+2}$

iii)  $\sum_{n=1}^{\infty} \frac{2}{e^n}$

6 pts 14. For which real numbers  $x$  does the power series  $\sum_{n=1}^{\infty} \frac{x^n}{4^n n}$  converge?

- 6 pts 15. Find the fifth degree Taylor polynomial  $p_5(x)$  of  $e^x$  about 0 and use it to estimate  $e^2$ .
- 6 pts 16. Graph  $r = \cos(3\theta)$  in polar coordinates. Set up, but do not evaluate, an integral (or sum of integrals) representing the area of the region enclosed by this curve.
- 6 pts 17. Find the arclength of the graph of  $y = \ln(\sec x)$ ,  $0 \leq x \leq \pi/4$ .

8 pts 18. Let

$$f(x,y) = \begin{cases} \frac{x^2 \sin y}{x^2 + y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

a) Find  $\frac{\partial f}{\partial x}(0,0)$  and  $\frac{\partial f}{\partial y}(0,0)$ .

b) Find the directional derivative of  $f$  at the origin in the direction of the vector  $u = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ .

6 pts 19. Compute  $\int_0^1 \int_y^1 2ye^{x^3} dx dy$ . You may consider interchanging the order of integration.

- 8 pts 20. The function  $f$  is positive, increasing, and concave downward on the interval  $[a,b]$ . The integral

$$I = \int_a^b f(x)dx$$

is estimated using various methods, all with the same number of subintervals: the left-hand sum is  $L$ , the right-hand sum is  $R$ , the trapezoidal rule estimate is  $T$ , and the midpoint sum is  $M$ . Arrange the numbers  $L, R, T, M$ , and  $I$  in order from smallest to largest.

- 9 pts 21. Find the area  $A$ , and the centroid or center of mass  $(\bar{x}, \bar{y})$  of the plane region bounded above by the line  $y = x + 2$  and bounded below by the parabola  $y = x^2$