Name______________________________

Student Number______________________________

Section Number______________________________

Instructor______________________________

Math 113 – Fall 2004

Departmental Final Exam

Instructions:

- The time limit is 3 hours.
- Problem 1 consists of 22 short answer questions.
- Problems 2 through 9 are multiple choice questions.
- For problems 10 through 18 give the best answer and *justify* it with suitable reasons and/or relevant work.
- Work on scratch paper will not be graded.
- Please write neatly.
- Notes, books, and calculators are not allowed.
- Expressions such as ln(1), e^0, $\sin(\pi/2)$, etc. must be simplified for full credit.

For administrative use only:

1	/22
M.C.	/24
10	/6
11	/6
12	/6
13	/6
14	/6
15	/6
16	/6
17	/6
18	/6
Total	/100

Math 113 – Fall 2004

Departmental Final Exam

Part I: Short Answer and Multiple Choice Questions

1. Fill in the blanks with the correct answer.

   [*arbitrary constant of integration assumed in indefinite integrals below*]

   (a) The integral $\int_0^{\pi/2} \sin x\, dx$ equals $\underline{-\cos x|_0^{\pi/2} = 1}$

   (b) The integral $\int \cos x \sin^2 x\, dx$ equals $\underline{\frac{1}{3}\sin^3 x}$

   (c) The integral $\int_0^{\pi/2} \sin^2 x\, dx$ equals $\underline{\frac{1}{2}(x - \cos x \sin x)\Big|_0^{\pi/2} = \frac{\pi}{4}}$

   (d) The integral $\int_0^{\pi/4} \sec^2 x\, dx$ equals $\underline{\tan x|_0^{\pi/4} = 1}$

   (e) The integral $\int \tan x\, dx$ equals $\underline{-\ln(\cos x) = \ln(\sec x)}$

   (f) The integral $\int \frac{x}{\sqrt{1-x^2}}\, dx$ equals $\underline{[u = 1 - x^2] \Rightarrow -\sqrt{1-x^2}}$

   (g) The integral $\int \frac{e^{1/x}}{x^2}\, dx$ equals $\underline{[u = \frac{1}{x}] \Rightarrow -e^{1/x}}$

   (h) The integral $\int_0^{\infty} \frac{dx}{1+x^2}$ equals $\underline{\tan^{-1} x|_0^{\infty} = \frac{\pi}{2}}$

   (i) The integral $\int_0^{1} \frac{dx}{\sqrt{1-x^2}}$ equals $\underline{\sin^{-1} x|_0^{1} = \frac{\pi}{2}}$

   (j) The integral $\int_1^{2/\sqrt{3}} \frac{dx}{x\sqrt{x^2-1}}$ equals $\underline{-\tan^{-1} \frac{1}{\sqrt{x^2-1}}\Big|_1^{2/\sqrt{3}} = \frac{\pi}{6}}$

   (k) The integral $\int_0^{1} \frac{dx}{x^2}$ equals $\underline{\infty\text{: divergent integral}}$

   (l) The integral $\int_1^{\infty} \frac{dx}{x^2}$ equals $\underline{-x^{-1}|_0^{1} = 1}$

   (m) The radius of convergence of $\sum_{n=0}^{\infty} x^n$ is $\underline{\left[\text{sum} = \frac{1}{1-x}\right] \Rightarrow 1}$

   (n) The area between the x-axis and the parametric curve $x = t + t^2,\ y = t - t^2,\ 0 \le t \le 1$

   is given by $\int y\, dx$ and has value $\underline{\int_0^1 y \frac{dx}{dt}\, dt = \frac{1}{3}}$

(o) The centroid of the region $0 \le y \le x^2,\ -1 \le x \le 1$ is located at $[M = \frac{2}{3},\ M_x = \frac{1}{5}] \to (0, \frac{3}{10})$

(p) Indicate which convergence test one would use in determining the convergence/ divergence of

i. $\sum \frac{1}{n^2}$ p-series test

ii. $\sum \frac{1}{\sqrt{n}}$ p-series test

iii. $\sum \frac{1}{n^2 + n}$ comparison test

iv. $\sum \frac{1}{n^2 - n}$ limit comparison test

v. $\sum \frac{(-1)^n}{\sqrt{n}}$ alternating series test

vi. $\sum \left(\frac{1}{n}\right)^n$ ratio test/ root test

(q) The MacLaurin series for $\cos x$ is $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$.

(Write at least 5 nonzero terms.)

Problems 2 through 9 are multiple choice. Each multiple choice problem is worth 3 points. In the grid below fill in the square corresponding to each correct answer.

[note: there is a typo in 3c, so there are two "correct" answers]

2	A	**B**	C	D	E	F	G	H	I	J
3	**A**	B	**C**	D	E	F	G	H	I	J
4	A	B	**C**	D	E	F	G	H	I	J
5	A	B	C	**D**	E	F	G	H	I	J
6	A	B	C	**D**	E	F	G	H	I	J
7	**A**	B	C	D	E	F	G	H	I	J
8	A	B	**C**	D	E	F	G	H	I	J
9	A	B	C	**D**	E	F	G	H	I	J

2. Find $\int_1^2 x^3 \ln(5x)\, dx$

   (a) $4\ln 2 + \frac{15}{4}\ln 5 - \frac{15}{17}$

   (b) $4\ln 2 + \frac{15}{4}\ln 5 - \frac{15}{16}$

   (c) $4\ln 2 + \frac{3}{4}\ln 5 - \frac{15}{16}$

   (d) $4\ln 2 + \frac{11}{4}\ln 5 - \frac{15}{16}$

   (e) $4\ln 3 + \frac{15}{4}\ln 5 - \frac{15}{16}$

   (f) $2\ln 2 + \frac{15}{4}\ln 5 - \frac{15}{16}$

   (g) None of the above

   Answer:

   (b): $4\ln 2 + \frac{15}{4}\ln 5 - \frac{15}{16}$

3. The sum $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if

   (I) $\displaystyle\int_1^{\infty} x^{-p}\, dx$ converges

   (II) $p > 1$

   (III) $p \leq 1$

   (IV) $p \geq 1$

   (V) $p < 1$

   Which of the above are true?

(a) (I),(II)

(b) (II) only

(c) (I) and (II)

(d) (III) only

(e) (I) and (IV)

(f) (IV) only

(g) (I) and (V)

(h) (V) only

(i) None of the above

Ans: (a) [note: *typo in (c)*]

4. Find $\int_0^1 x^2 \sin(2\pi x)\, dx$.

(a) 2π

(b) π^2

(c) $-\dfrac{1}{2\pi}$

(d) 0

(e) $\dfrac{-1}{2\pi} + \dfrac{1}{2\pi^2}$

(f) $\dfrac{1}{2\pi}$

(g) $\dfrac{1}{2\pi^2}$

(h) $\pi^2 - \dfrac{1}{2\pi}$

(i) $\dfrac{1}{2}$

(j) None of the above

Ans: (c) $-\dfrac{1}{2\pi}$

5. A ball of radius 5 has a hole of radius 2 drilled completely through it such that the axis of the hole is a diameter of the ball. What is the volume of what is left?

(a) $7\sqrt{21}\pi$

(b) $21\sqrt{21}\pi$

(c) $\frac{116}{3}\sqrt{29}\pi$

(d) $28\sqrt{21}\pi$

(e) $14\sqrt{21}\pi$

(f) $28\sqrt{21}\pi^2$

(g) None of the above

Answer: (d) $28\sqrt{21}\pi$

6. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} n\,(-1)^n \left(\frac{1}{2}\right)^n$.

(a) $-\frac{13}{18}$

(b) $-\frac{20}{9}$

(c) $\frac{7}{9}$

(d) $-\frac{2}{9}$

(e) $\frac{1}{9}$

(f) $-\frac{5}{9}$

(g) $\frac{5}{18}$

(h) None of the above

Answer: (d) $-\frac{2}{9}$

7. Write the equation $r = 2\cos\theta - \sin\theta$ in rectangular coordinates and simplify your answer.

(a) $x^2 + y^2 = 2x - y$

(b) $x^2 + y^2 = 2xy^2 - y$

(c) $x^2 + 2y^2 = x - 2y$

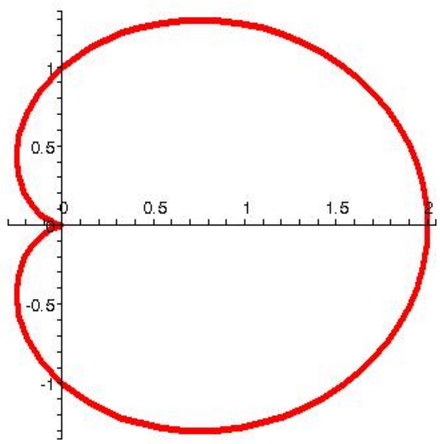
(d) $2x^2 + y^2 = 2x + y$

(e) $(x + y)^2 = 2x - y$ (i)

(f) None of the above (j)

Answer: (a) $x^2 + y^2 = 2x - y$

8. Identify the equation which goes with the polar graph,



(a) $r = 2\cos\theta,\ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(b) $r = 3 + \sin(5\theta)\,,\ \theta \in [0, 2\pi]$

(c) $r = 1 + \cos\theta, \theta \in [0, 2\pi]$

(d) $r = \frac{4}{2+\cos\theta},\ \theta \in [0, 2\pi]$

(e) $r = 1 + \sin\theta,\ \theta \in [0, 2\pi]$

(f) $r = \sin(3\theta)\,,\ \theta \in \left[0, \frac{\pi}{3}\right]$

(g) $r = 1 + \cos(2\theta)\,,\ \theta \in [0, 2\pi]$

(h) None of the above

Answer: (c) 8

9. Which of the following integrals represents the surface area of the surface generated by revolving the arc $y = \sin x,\ 0 \le x \le \pi$ about the line $y = 2$.

(a) $\int_0^\pi 2\pi(1 - \cos x)\sqrt{1 + \sin^2 x}\,dx$

(b) $\int_0^\pi 2\pi(1 - \sin x)\sqrt{1 + \cos^2 x}\,dx$

(c) $\int_0^\pi 2\pi(2 - \cos x)\sqrt{1 + \sin^2 x}\,dx$

(d) $\int_0^\pi 2\pi(2 - \sin x)\sqrt{1 + \cos^2 x}\,dx$

(e) $\int_0^\pi 2\pi(1 - \cos x)\sqrt{1 + \cos^2 x}\,dx$

(f) $\int_0^\pi 2\pi(1 - \sin x)\sqrt{1 + \cos^2 x}\,dx$

(g) $\int_0^\pi 2\pi(2 - \cos x)\sqrt{1 + \cos^2 x}\,dx$

(h) $\int_0^\pi 2\pi(2 - \sin x)\sqrt{1 + \sin^2 x}\,dx$

Ans: (d)

10. Evaluate the definite integral $\int_0^1 x^3\sqrt{1-x^2}\,dx$.

Ans:

$$u = 1 - x^2 \Rightarrow \int_0^1 x^3\sqrt{1-x^2}\,dx = \int_1^0 (1-u)u^{1/2}\frac{-du}{2} = \frac{2}{5}$$

11. (a) Evaluate the indefinite integral $\int \left(\frac{3x}{2+3x^2} + \frac{x}{3-x}\right) dx$

(b) Derive the partial fraction expansion for $\frac{3x^2-x-2}{(x+1)(x^2+1)}$.

Ans:

With $u = 2+3x^2,\ 3x\,dx = du/2$,

$$\int \frac{3x}{2+3x^2}\,dx = \int \frac{1}{u}\frac{du}{2} = \frac{1}{2}\ln(2+3x^2)$$

$$\int \frac{x}{3-x}\,dx = \int \frac{x-3+3}{3-x}\,dx = \frac{3}{3-x} - 1\,dx = -x - 3\ln|3-x|$$

$$\frac{3x^2-x-2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$A(x^2+1) + (Bx+C)(x+1) = 3x^2 - x - 2$$

Setting $x = -1$:

$$A(2) = 2 \Rightarrow A = 1$$

Setting $x = 0$:

$$A + C = -2 \Rightarrow C = -3$$

Setting $x = 1$:

$$2A + (B+C)(2) = 0 \Rightarrow B = -C - A = 2$$

So

$$\frac{3x^2-x-2}{(x+1)(x^2+1)} = \frac{1}{x+1} + \frac{2x-3}{x^2+1}$$

12. Find the length of the curve $f(x) = \ln(\cos x)$ for $0 \le x \le \pi/4$.

Ans: arc length is given by

$$\int_0^{\pi/4} \sqrt{1+(f'(x))^2}\,dx = \int_0^{\pi/4} \sqrt{1+\left(\frac{-\sin x}{\cos x}\right)^2}\,dx$$

$$= \int_0^{\pi/4} \sec x\,dx = [\ln|\sec x + \tan x|\,]_0^{\pi/4} = \ln(1+\sqrt{2})$$

13. Use the specified method to find the volume of the of the solid of revolution formed when the region bounded by $y = x^2$ and $y = x$ is revolved

(a) about the x- axis (discs)

(b) about the y-axis (shells)

Ans:

Curves intersect at $x = 0, 1$ with $y = x$ lying above $y = x^2$.

Disc about x-axis:

$$\int_0^1 \pi x^2 - \pi(x^2)^2 \, dx = \pi \left. \frac{x^3}{3} - \frac{x^5}{5} \right|_0^1 = \pi(\frac{1}{3} - \frac{1}{5} = \frac{2\pi}{15}$$

Shell about y-axis:

$$\int_0^1 2\pi x(x - x^2) \, dx = 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{\pi}{6}$$

14. A solid of uniform density has as its base the unit circle in the plane, and cross sections of the solid perpendicular to the x-axis are equilateral triangles. Find the mass of the solid.

Ans:

For fixed $x \in [-1, 1]$, area of corresponding equilateral triangle has base $2y = 2\sqrt{1 - x^2}$ and height $\sqrt{3}y$.

Assume density is k then mass is

$$\int_{-1}^1 \frac{1}{2}(2y)(\sqrt{3}y) \, k \, dx = 2\sqrt{3}k \int_0^1 (1 - x^2) \, dx = \frac{4k}{\sqrt{3}}.$$

15. Suppose Achilles can run 40 times as fast as the tortoise and that the tortoise has a lead of 100 paces at the beginning of a race. How far will Achilles run before he overtakes the tortoise?

Ans: Notice that when Achilles has travelled 100 paces, the tortoise will have travelled $\frac{1}{40}$ of that amount, or $\frac{100}{40}$ paces. When Achilles has travelled $\frac{100}{40}$ paces, the tortoise has travelled $\frac{100}{40^2}$ paces. Thus, the total number of paces that Achilles has travelled to catch up with the toroise is

$$100 + \frac{100}{40} + \frac{100}{40^2} + \cdots = \frac{100}{1 - \frac{1}{40}} = \frac{4000}{39}$$

paces.

Note: This answer can also be calculated without any knowledge of geometric series. Do you see how?

16. Write down the Taylor series for $f(x) = \dfrac{1 + x}{1 - x}$ about $x = 0$. You should include the nth term in the Taylor series. Hence find the fifth derivative of $f(x)$ at $x = 0$:

$$\frac{d^5}{dx^5} \left(\frac{1 + x}{1 - x} \right) \bigg|_{x=0}$$

Ans: Repeated differentiation of the function yields

$$f^n(x) = \frac{2 \cdot n!}{(1-x)^n + 1}.$$

Thus, $f^n(0) = 2 \cdot n!$ for $n > 0$. Since $f(0) = 1$, the series is given by

$$1 + 2\sum_{n=1}^{\infty} x^n.$$

Note: I can think of 3 different ways to find this series without finding the Taylor series directly. Can you find them?

17. Consider the ellipse $x^2 + 4y^2 = 1$.

(a) Give the foci of the ellipse.
Ans: Notice that we can write the above as

$$x^2 + \frac{y^2}{\frac{1}{4}} = 1.$$

Thus, $a = 1$ and $b = 1/2$, so $c = \sqrt{1 - 1/4} = \frac{\sqrt{3}}{2}$. Hence the foci are at $(-\frac{\sqrt{3}}{2}, 0)$ and $(\frac{\sqrt{3}}{2}, 0)$

(b) Find the area of the ellipse.
Ans. Since

$$y = \frac{1}{2}\sqrt{1-x^2},$$

the area is given by

$$\int_{-1}^{1} \frac{1}{2}\sqrt{1-x^2}\,dx = \int_{0}^{1} \sqrt{1-x^2}\,dx$$

This integral is easily solved using trigonometric integration with $x = \sin(\theta)$. With this substitution, the integral becomes

$$\int_0^{\pi/2} \cos^2(\theta)\,d\theta = \frac{\pi}{4}.$$

(c) Write the equation of an ellipse in polar form .
Ans.

$$x = \cos(t)$$
$$y = \frac{1}{2}\sin(t)$$

(d) Write down (but do not evaluate) the integral that represents the perimeter of the above ellipse in the parametric form $x = t, y = y(t),\ -1 \le t \le 1$.

$$2\int_{-1}^{1} \sqrt{1 + \frac{x^2}{4(1-x^2)}}\,dx = 2\int_{-1}^{1} \sqrt{\frac{1-3x^2}{4(1-x^2)}}\,dx.$$

18. Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 2(1 + \cos\theta)$.

$$\int_{2\pi/3}^{4\pi/3} \frac{1}{2} \cdot 1^2 \, d\theta - \int_{2\pi/3}^{4\pi/3} \frac{1}{2} 4(1 + \cos\theta)^2 =, d\theta$$

$$= \frac{7\sqrt{3}}{2} - \frac{5\pi}{3}.$$