

Math 112 – Winter 2006 — Key

Departmental Final Exam

PART I: FILL IN THE BLANK AND TRUE OR FALSE QUESTIONS

1. (6 points) Fill in the blanks with the correct answer. Include the constant of integration where appropriate.

(a) The derivative of x^π equals $\underline{\pi x^{\pi-1}}$

(b) The derivative of $\frac{x}{1+x}$ equals $\underline{\frac{1}{(1+x)^2}}$

(c) The derivative of $\ln(1+x^2)$ equals $\underline{\frac{2x}{1+x^2}}$

(d) The integral $\int \sin 2x \, dx$ equals $\underline{-\frac{1}{2} \cos(2x) + C}$

(e) The integral $\int_0^1 2xe^{x^2} \, dx$ equals $\underline{e - 1}$

(f) The derivative of $\tan^2 x \, dx$ equals $\underline{2 \tan(x) (1 + (\tan(x))^2)}$

2. (6 points) Mark *T* if true under all circumstances; mark *F* otherwise.

(a) T $\lim_{x \rightarrow 0} \frac{\sin 4x}{3x} = \frac{4}{3}$

(b) F If $y = f^{-1}(x)$ denotes the inverse of $f(x)$, then by chain rule $\frac{dy}{dx} = -\frac{f'(x)}{f^2(x)}$.

(c) F If $y = 10^x$ then $\frac{dy}{dx} = x 10^{x-1}$.

(d) T $\frac{d}{dx} \int_0^{2x} f(t) \, dt = 2f(2x)$.

(e) F If $f(x)$ has a removable discontinuity at $x = 0$, then $f(x)$ can be defined at $x = 0$ so that f is differentiable at $x = 0$.

(f) T If $f''(c)$ exists and $(c, f(c))$ is an inflection point for $y = f(x)$, then $f''(c) = 0$.

PART II: MULTIPLE CHOICE

Problems 3 through 9 are multiple choice. Each multiple choice problem is worth 4 points. In the grid below fill in the square corresponding to each correct answer.

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9	<input type="checkbox"/> A	<input type="checkbox"/> B	<input type="checkbox"/> C	<input type="checkbox"/> D	<input checked="" type="checkbox"/>	<input type="checkbox"/> F	<input type="checkbox"/> G	<input type="checkbox"/> H	<input type="checkbox"/> I	<input type="checkbox"/> J

3. Assume that f is a continuous function on $[a, b]$. The partition of $[a, b]$ is given as $P = \{a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_n = b\}$. Let $\Delta x_i = x_i - x_{i-1}$, $x_{i-1} \leq t_i \leq x_i$, and $\|P\| = \max\{\Delta x_1, \Delta x_2, \dots, \Delta x_n\}$. Which of the following is not equal to $\int_a^b f(x)dx$?

- (a) $\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(t_i)\Delta x_i$ (b) $\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(x_i)\Delta x_i$
 (c) $\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(x_{i-1})\Delta x_i$ (d) $\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f\left(\frac{x_i+x_{i-1}}{2}\right)\Delta x_i$
 (e) $\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f\left(\frac{x_i+x_{i-1}}{2}\right)\Delta x_i$.

4. Consider $\lim_{x \rightarrow 12} \left(\frac{x}{4} - 2\right) = 1$. If $\epsilon > 0$ is given, find the largest possible δ such that $0 < |x - 12| < \delta$ implies that $\left|\left(\frac{x}{4} - 2\right) - 1\right| < \epsilon$.

- (a) $\delta = 12\epsilon$ (b) $\delta = 4\epsilon$ (c) $\delta = 2\epsilon$ (d) $\delta = \epsilon$ (e) $\delta = 6\epsilon$
 (f) $\delta = 3\epsilon$ (g) $\delta = 5\epsilon$

5. Find the average value of $f(x) = 4x - x^2$ on $[0, 4]$

- (a) $2/3$ (b) $4/3$ (c) $\boxed{8/3}$ (d) $16/3$
(e) $32/3$ (f) 0 (g) 32 (h) 16
(i) 8 (j) None of these.

6. Find $f^{(13)}(\pi/4)$, the 13th derivative of f at $x = \pi/4$, for $f(x) = \sqrt{2} \cos x$.

- (a) -4 (b) 4 (c) -2 (d) 2 (e) $\boxed{-1}$
(f) 1 (g) 0

7. If $F'(x) = f(x)$, then $\int_a^b f(x)dx =$

- ((a) $f(a) - f(b)$ (b) $f(b) - f(a)$ (c) $f'(b) - f'(a)$
(d) $F'(b) - F'(a)$ (e) $F(a) - F(b)$ (f) $\boxed{F(b) - F(a)}$
(g) $f(b)$ (h) None of these.

8. Find $\frac{dy}{dx}$ at $x = \pi/6$ if $y = 2 \tan^{-1}(2 \sin x)$.

- (a) $2\sqrt{3}$ (b) 2 (c) $\pi/2$ (d) $\boxed{\sqrt{3}}$ (e) $4\sqrt{3}$
(f) $\pi/3$ (g) $1/\sqrt{3}$

9. The function $f(x) = x^3 - x^6$
- (a) increases on $(-\infty, 0)$, and decreases on $(0, \infty)$.
 - (b) $f^{(6)} = 0$.
 - (c) $x = 1$ is a critical point.
 - (d) is concave down on $(-\infty, 0)$, and concave up on $(0, \infty)$.
 - (e) is concave up on $(0, 5^{-1/3})$ and concave down on $(5^{-1/3}, \infty)$, and $x = 5^{-1/3}$ is an inflection point.
 - (f) has local maximum at $x = 0$, and $x = 0$ is an inflection point.
 - (g) has local minimum at $x = 2^{-1/3}$.

The answers to the multiple choice MUST be entered on the grid on page 2. Otherwise, you will not receive credit.

PART III: WRITTEN SOLUTIONS

For problems 10 - 19, write your answers in the space provided. Neatly show your work for full credit.

10. The equation $x \ln y = y^3$ implicitly defines y as a function of x . Find $\frac{dy}{dx}$.

Differentiating $x \ln y = y^3$

$$\ln y + \frac{x}{y}y' = 3y^2y'$$

$$y' = \frac{\ln y}{3y^3 - \frac{x}{y}} = \frac{y \ln y}{3y^3 - x}$$

Alternate answer: writing $x = y^3 / \ln y$, differentiating w.r.t. y and flipping the derivative,

$$\frac{dy}{dx} = \left(\frac{\ln y}{y}\right)^2 \frac{1}{3 \ln y - 1}$$

11. (a) Evaluate the definite integral $\int_{-2}^2 \sqrt{4-x^2} dx$ as an area.

Integral represents the area of an upper half circle of radius 2, so area is $\frac{1}{2}(\pi 2^2) = 2\pi$

(b) Use linearity properties and the properties of even and odd functions to evaluate the definite integral $\int_{-1}^1 (3x^4 \tan x - x^2) dx$.

$$\int_{-1}^1 (3x^4 \tan x - x^2) dx = 3 \int_{-1}^1 x^4 \tan x dx - \int_{-1}^1 x^2 dx$$

Now x^4 is even, $\tan x$ is odd, so their product is an odd function. Thus

$$\int_{-1}^1 x^4 \tan x dx = 0$$

$$\int_{-1}^1 x^2 dx = \frac{2}{3}$$

so

$$\int_{-1}^1 (3x^4 \tan x - x^2) dx = -\frac{2}{3}$$

12. Suppose the radius of a spherical object is expanding at a rate of 0.5 m/hr. How fast is the volume increasing when the radius is 5m?

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Given $dr/dt = 0.5$ m/hr, so when $r = 5$ m,

$$\frac{dV}{dt} = 4\pi(5)^2 0.5 = 50\pi \quad (\text{m}^3/\text{hr})$$

13. Find the limit. If the limit does not exist, write "Limit = ∞ ," or "Limit = $-\infty$," or *if neither of these is true*, write "Limit doesn't exist." You must show your work.

(a) $\lim_{x \rightarrow \infty} (x - \sqrt{x^2 - 5x})$

$$\begin{aligned} \lim_{x \rightarrow \infty} (x - \sqrt{x^2 - 5x}) &= \lim_{x \rightarrow \infty} \frac{x^2 - (x^2 - 5x)}{x + \sqrt{x^2 - 5x}} = \lim_{x \rightarrow \infty} \frac{5x}{x + \sqrt{x^2 - 5x}} \\ &= \lim_{x \rightarrow \infty} \frac{5}{1 + \sqrt{1 - \frac{5}{x}}} = \frac{5}{2} \end{aligned}$$

(b) $\lim_{x \rightarrow 0^+} \left(1 + \frac{3}{x}\right)^x$

if $y = \left(1 + \frac{3}{x}\right)^x$, then

$$\ln y = x \ln \left(1 + \frac{3}{x}\right) = \frac{\ln \left(1 + \frac{3}{x}\right)}{1/x}$$

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} \frac{\ln \left(1 + \frac{3}{x}\right)}{1/x}$$

Quotient is of the form ∞/∞ , so applying L'Hopital's rule gives

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} \frac{\frac{-3/x^2}{1 + \frac{3}{x}}}{-1/x^2} = \lim_{x \rightarrow 0^+} \frac{3x}{x + 3} = 0$$

Hence

$$\lim_{x \rightarrow 0^+} \left(1 + \frac{3}{x}\right)^x = \lim_{x \rightarrow 0^+} y = e^0 = 1$$

14. A hot ingot is removed from the furnace and placed for cooling outdoors, where the temperature is 70°F . After one hour, the temperature of the ingot is 250° , and after 2 hours, the temperature is 100° . How hot was the ingot when it was removed from the furnace?

(Newton's cooling model is $\frac{dT}{dt} = -k(T - A)$.) From given information, $A = 70$ ($^\circ\text{F}$)

Separating variables, integrating and writing the integration constant as $-\ln C$,

$$\int \frac{dT}{T - A} = \int (-k) dt \Rightarrow \ln(T - A) - \ln C = -kt$$

$$T = T(t) = A + Ce^{-kt}$$

$$T(1) = 250 = 70 + Ce^{-k(1)} \Rightarrow Ce^{-k} = 180$$

$$T(2) = 100 = 70 + Ce^{-k(2)} \Rightarrow Ce^{-2k} = 30$$

Take the ratio,

$$e^k = 6 \Rightarrow C = 1080 \Rightarrow T = 70 + \frac{1080}{6^t}$$

$$T(0) = 70 + 1080 = 1150$$
 ($^\circ\text{F}$)

15. Find $\frac{df}{dx}(0)$ for

(a) $f(x) = \sec(3x) \tan(4x)$.

$$\begin{aligned} \frac{df}{dx} &= \frac{d \sec(3x)}{dx} \tan(4x) + \sec(3x) \frac{d \tan(4x)}{dx} \\ &= (3 \sec(3x) \tan(3x)) \tan(4x) + \sec(3x) \sec^2(4x)(4) \end{aligned}$$

so

$$\frac{df}{dx}(0) = 0 + 4 = 4$$

(b) $f(x) = u^3 + 1$, $u = 3v - 2$, $v = x^3 + 1$.

$$\begin{aligned} \frac{df}{dx} &= \frac{df}{du} \frac{du}{dv} \frac{dv}{dx} \\ \frac{df}{du} &= 3u^2, \quad \frac{du}{dv} = 3, \quad \frac{dv}{dx} = 3x^2 \end{aligned}$$

At $x = 0$,

$$v = 1, u = 1,$$

so

$$\frac{df}{dx} = (3)(3)(0) = 0$$

16. Compute the following integrals:

$$(a) \int \frac{6}{\sqrt{x}(1 + \sqrt{x})} dx$$

Let $s = (1 + \sqrt{x})$, then

$$ds = \frac{1}{2\sqrt{x}} dx \Rightarrow \frac{1}{\sqrt{x}} dx = 2 ds$$

$$\int \frac{6}{\sqrt{x}(1 + \sqrt{x})} dx = \int \frac{12}{s} ds = 12 \ln |s| + C = 12 \ln(1 + \sqrt{x}) + C$$

$$(b) \int \frac{\sin(\ln x)}{x} dx$$

Let $s = \ln x$,

$$ds = \frac{1}{x} dx$$

$$\int \frac{\sin(\ln x)}{x} dx = \int \sin(s) ds = -\cos(s) + C = -\cos(\ln x) + C$$

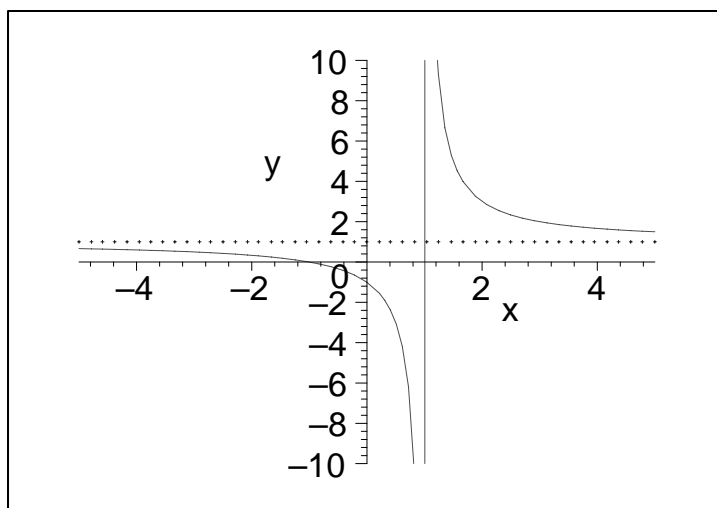
17. (a) Find the horizontal asymptote of $f(x) = \frac{x+1}{x-1}$, and sketch the graph of f .

$$f(x) = \frac{x+1}{x-1} = 1 + \frac{2}{x-1}$$

so as $x \rightarrow \pm\infty$, $f(x) \rightarrow 1$, i.e. horizontal asymptote is

$$y = 1$$

Function is a hyperbola with vertical asymptote $x = 1$.



(b) Show that the polynomial $p(x) = x^3 - x^2 - 2x + 2$ has a zero in the interval $[-2, 0]$.

$$p(-2) = -8 - 4 + 4 + 2 = -6 < 0, \quad p(0) = 2 > 0$$

so by intermediate value theorem, there exists $x_0 \in [-2, 0]$, such that $p(x_0) = 0$.

18. A piece of commercial property is to be enclosed by fencing on the front and two sides. Fencing for the sides costs \$2.50 per foot, and fencing for the front costs \$4.00 per foot. What are the dimensions of the largest rectangular lot that can be so enclosed for \$800?

Let y be the length of the front fence and x be the length of a side fence.

$$2x(2.50) + y(4.00) = 800 \Rightarrow 5x + 4y = 800 \Rightarrow y = \frac{800 - 5x}{4}$$

Area $A = xy$, so

$$A = xy = x \frac{800 - 5x}{4} = 200x - \frac{5}{4}x^2$$

$$\frac{dA}{dx} = 200 - \frac{5}{2}x$$

Area is maximized when $dA/dx = 0$,

$$200 - \frac{5}{2}x = 0 \Rightarrow x = 80$$

and thus

$$y = \frac{800 - 5(80)}{4} = 100.$$

So the maximizing dimensions are

$$x = 80, y = 100.$$

Alternatively, differentiating $5x + 4y = 800$,

$$5 + 4 \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{5}{4}$$

Area $A = xy$, so

$$\frac{dA}{dx} = \frac{d(xy)}{dx} = y + x \frac{dy}{dx} = y - \frac{5}{4}x$$

Area is maximized when $dA/dx = 0$, so $y = 5x/4$.

$$5x + 4y = 800 \Rightarrow 5x + 4 \frac{5}{4}x = 800 \Rightarrow x = 80, y = 100$$

19. a) State the definition of *derivative* of $f(x)$ at the point a .

Any of the following will do

$$\begin{aligned} & \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \\ & \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} \\ & \lim_{h \rightarrow 0} \frac{f(a + h) - f(a - h)}{2h} \\ & \lim_{x \rightarrow a} \frac{f(a) - f(x)}{a - x} \\ & \lim_{h \rightarrow 0} \frac{f(a) - f(a - h)}{h} \end{aligned}$$

etc

There should be no $a \rightarrow x$ expression in the limit as a is a fixed value.

b) Use definition in a) to find the derivative of $f(x) = \begin{cases} x^2 \cos \frac{1}{x}, & x \neq 0; \\ 0, & x = 0 \end{cases}$ at $x = 0$.

$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x^2 \cos \frac{1}{x} - 0}{x} = \lim_{x \rightarrow 0} x \cos \frac{1}{x}$$

Since $-1 \leq \cos \frac{1}{x} \leq 1$, so

$$-x \leq x \cos \frac{1}{x} \leq x$$

As $\lim_{x \rightarrow 0} \pm x = 0$, thus by the pinching/ sandwich theorem,

$$f'(0) = 0$$

Alternatively,

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h^2 \cos \frac{1}{h} - 0}{h} = \lim_{h \rightarrow 0} h \cos \frac{1}{h}$$

Arguing as above gives $f'(0) = 0$.