

THE DISTRIBUTION OF k -FREE NUMBERS

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Abstract. Let $k \in \{3, 4, 5\}$. Let

$$R_k(x) = \sum_{\substack{n \leq x \\ n \text{ is } k\text{-free}}} 1 - \frac{x}{\zeta(k)}.$$

We give new upper bounds for $R_k(x)$ conditional on the Riemann hypothesis, improving work of S.W. Graham and J. Pintz. The method stays close to that devised by H.L. Montgomery and R.C. Vaughan, with the improvement depending on exponential sum results.

Let $k \geq 2$. A positive integer n is said to be k -free if n is not divisible by the k -th power of a prime. Let

$$R_k(x) = \sum_{\substack{n \leq x \\ n \text{ is } k\text{-free}}} 1 - \frac{x}{\zeta(k)}.$$

An elementary argument yields

$$R_k(x) \ll x^{1/k}.$$

In the opposite direction,

$$R_k(x) = \Omega(x^{1/2k}).$$

(See [5] or [15].)

Assuming the Riemann hypothesis (RH), Montgomery and Vaughan [14] obtained

$$(1.1) \quad R_k(x) \ll x^{1/(k+1)+\epsilon}.$$

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Here ϵ is an arbitrarily small positive number. The exponent in (1.1) has been improved for every value of k . For $k \geq 3$, see Graham and Pintz [9]. For $k = 2$, there have been papers by Graham [7], Baker and Pintz [4] and Jia [13]; the exponent in [13] is $17/54 + \epsilon$.

No-one has yet been able to improve the approximation in Theorem 1 of [14] under RH: for $N \geq 1$,

$$(1.2) \quad R_k(x) = - \sum_{n \leq N} \mu(n) \psi(xn^{-k}) + O(x^{1/2+\epsilon} N^{(1-k)/2} + N^{1/2+\epsilon}).$$

Here $\psi(w) = w - [w] - \frac{1}{2}$. Once (1.2) is applied, exponential sums dominate the discussion.

Jia's paper, which appeared after [9], contains an estimate for exponential sums which has found several applications. Abstractions of the estimate are given by Wu [18]. The exponential sum estimate devised by Heath-Brown [12] is crucial in [4, 9, 13]; it is abstracted in Baker [1]. In the present paper we combine these results with theorems from Robert and Sargos [16] and Baker [2]. The former paper is a natural culmination of the work of Fouvry and Iwaniec [6]; the starting point is the double large sieve. The latter paper is related to earlier work of Baker and Kolesnik [3]. Actually, we shall adapt the results from [2, 18] a little.

THEOREM. *Assume RH. We have*

$$R_k(x) \ll x^{\theta_k + \epsilon} \quad (k = 3, 4, 5)$$

where

$$\theta_3 = \frac{17}{74} = 0.2297\dots, \theta_4 = \frac{17}{94} = 0.1808\dots, \theta_5 = 0.15.$$

For comparison, θ_k replaces $7/(6k+8)$ in [9], and we have

$$\frac{7}{30} = 0.2333\dots, \frac{7}{38} = 0.1842\dots, \frac{7}{46} = 0.1521\dots$$

In contrast with [14, 7, 4, 9, 13], we apply Heath-Brown's "generalized Vaughan identity" [12], or more precisely, the variant provided in [2]. We add a new result on the application of this decomposition (the case $\frac{1}{6} < a \leq \frac{1}{5}$ of Lemma 3(iii) below) which may be of independent interest.

We close this section with a few remarks on notation. Implied constants depend at most on k and ϵ , except in section 3 where they may depend on $u, v, \alpha, \beta, \gamma, \kappa$ and λ . We write " $A \asymp B$ " if $A \ll B \ll A$. We write " $n \sim N$ " for $N < n \leq 2N$. As usual, $e(z) = e^{2\pi iz}$. The cardinality of a finite set E is written $|E|$.

1. Decomposition of sums involving the Möbius function

Let $2 \leq D < D' \leq 2D$ and let f be a complex function on $[D, D']$. The sum

$$\sum_{D < n \leq D'} \mu(n) f(n)$$

can be decomposed into $O((\log D)^{2l-1})$ sums of the form

$$(2.1) \quad \sum_{\substack{n_i \sim N_i \\ D < n_1 \dots n_{2l-1} \leq D'}} \mu(n_1) \dots \mu(n_{2l-1}) f(n_1 \dots n_{2l-1}).$$

Here $N_i > \frac{1}{2}$, $\prod_{i=1}^{2l-1} N_i \asymp D$ and

$$(2.2) \quad 2N_i \leq (2D)^{1/l} \text{ if } i \geq l.$$

See [2, Section 2].

To apply the decomposition, we need results of the following kind. The numbers α_i ($1 \leq i \leq r$) in Lemma 1 arise as exponents for which $N_i \asymp D^{\alpha_i}$. (To be precise, let $r = 2l - 1$, $N_0 = 2^{2l-1}N_1 \dots N_{2l-1}$ and define α_i by

$$2N_i = N_0^{\alpha_i} \text{ for } 1 \leq i \leq 2l - 1.)$$

LEMMA 1. *Let $0 \leq \alpha_1 \leq \dots \leq \alpha_r$, $\alpha_1 + \dots + \alpha_r = 1$. For $S \subseteq \{1, \dots, r\}$, we write $\sigma_s = \sum_{i \in S} \alpha_i$.*

(i) *Let h be an integer, $h \geq 3$. Suppose that $\alpha_r \leq \frac{2}{h+1}$. Then some $\sigma_s \in [\frac{1}{h}, \frac{2}{h+1}]$.*

(ii) *Let $\lambda \geq \frac{2}{3}$. Suppose that $\alpha_r \leq \lambda$. Then some $\sigma_s \in [1 - \lambda, \lambda]$.*

PROOF. See [2, Lemma 1]. \square

The following lemma and Lemma 3(iii) are based on a result in Heath-Brown [2], which it strengthens in the case $\frac{1}{6} < a \leq \frac{1}{5}$.

LEMMA 2. *Make the hypotheses of Lemma 1.*

(i) *Let $\frac{1}{5} < a \leq \frac{1}{3}$. Suppose that $\alpha_r \leq \frac{1-a}{2}$. Then some $\sigma_s \in [a, 2a]$.*

(ii) *Let $0 < a \leq \frac{1}{5}$. Suppose that $\alpha_r \leq \frac{1-a}{2}$. Then some $\sigma_s \in [a, \frac{1}{3}]$.*

PROOF. (i) We have $\frac{1}{h+1} < a \leq \frac{1}{h}$ for some natural number h , which must be 3 or 4. Now

$$\alpha_r \leq \frac{1-a}{2} < \frac{2}{5} \leq \frac{2}{h+1}.$$

By Lemma 1(i), some $\sigma_s \in [\frac{1}{h}, \frac{2}{h+1}] \subseteq [a, 2a]$.

(ii) Suppose that no $\sigma_s \in [a, \frac{1}{3}]$. Let

$$T = \{j : \alpha_j \leq \frac{1}{3} - a\} \quad , \quad U = \{j : \frac{1}{3} - a < \alpha_j < a\}$$

(U is empty if $a \leq \frac{1}{6}$), and

$$V = \{j : \alpha_j > \frac{1}{3}\}.$$

Thus

$$(2.3) \quad \sigma_T + \sigma_U + \sigma_V = 1.$$

Clearly

$$(2.4) \quad |V| \leq 2 \quad , \quad \sigma_V \leq |V| \left(\frac{1-a}{2} \right) \leq 1-a.$$

Suppose for a moment that U is nonempty. Then for any $j \in U$,

$$(2.5) \quad \sigma_T + \alpha_j < a.$$

To see this suppose the contrary, and take the smallest $\sigma_W \geq a - \alpha_j$ with $W \subseteq T$. Then

$$\sigma_W - \alpha_k < a - \alpha_j$$

for any $k \in W$. Hence $a \leq \sigma_W + \alpha_j < a + \alpha_k \leq \frac{1}{3}$, contrary to our hypothesis. A similar argument gives $\sigma_T < a$ if U is empty, and it follows that

$$(2.6) \quad \sigma_T + \sigma_U < a + \max(0, |U| - 1)a = \max(1, |U|)a.$$

Suppose for a moment that $|U| \geq 2$. Take α_i, α_j in U , $i \neq j$. Then

$$\alpha_i + \alpha_j > \frac{2}{3} - 2a > a.$$

Consequently, $\alpha_i + \alpha_j > \frac{1}{3}$. This yields

$$(2.7) \quad \text{If } |U| \geq 2, \text{ then } \sigma_U > \frac{|U|}{2} \frac{1}{3} = \frac{|U|}{6}.$$

In particular, $|U| \leq 5$.

We now consider all possibilities for $|U|$.

Suppose $|U| = 0$ or 1 . From (2.6),

$$\sigma_T + \sigma_U < a,$$

and from (2.4),

$$\sigma_T + \sigma_U + \sigma_V < 1,$$

which is absurd.

Suppose $|U| = 2$ or 3 . Then $\sigma_U > \frac{1}{3}$ from (2.7), and so $\sigma_V < \frac{2}{3}$ and $|V| \leq 1$,

$$\sigma_V \leq \frac{1-a}{2}.$$

In conjunction with (2.6), this yields

$$\sigma_T + \sigma_U + \sigma_V < 3a + \frac{1-a}{2} \leq 1,$$

which is absurd.

Suppose finally that $|U| = 4$ or 5 . From (2.6) and (2.7),

$$\frac{2}{3} < \sigma_T + \sigma_U < 5a \leq 1,$$

so that $\sigma_V \in (0, \frac{1}{3})$. This is absurd and the lemma is proved. \square

LEMMA 3. *Let f be a complex function on $(D, D']$ where $2 \leq D < D' \leq 2D$.*

(i) *Suppose that*

$$(2.8) \quad \sum_{\substack{m \sim M \\ D < mn \leq D'}} \sum_{n \sim N} a_m b_n f(mn) \ll B$$

whenever $|a_m| \leq 1$, $|b_n| \leq 1$ and

$$D^{1/h} \ll N \ll D^{2/(h+1)}$$

where h is a natural number, $h \geq 3$. Suppose further that

$$(2.9) \quad \sum_{\substack{m \sim M \\ D < mn \leq D'}} a_m \sum_{n \sim N} f(mn) \ll B$$

whenever $|a_m| \leq 1$ and

$$N \gg D^{2/(h+1)}.$$

Then

$$(2.10) \quad \sum_{D < d \leq D'} \mu(d) f(d) \ll BD^\epsilon.$$

(ii) *Let $\lambda \geq \frac{2}{3}$. Suppose that (2.8) holds whenever $|a_m| \leq 1$, $|b_n| \leq 1$ and*

$$D^{1-\lambda} \ll N \ll D^{1/2},$$

while (2.9) holds whenever $|a_m| \leq 1$ and

$$N \gg D^\lambda.$$

Then (2.10) holds.

(iii) Let $0 < a \leq \frac{1}{3}$. Let $b = 2a$ if $a > \frac{1}{5}$ and $b = \frac{1}{3}$ if $a \leq \frac{1}{5}$. Suppose that (2.8) holds whenever $|a_m| \leq 1$, $|b_n| \leq 1$ and

$$D^a \ll N \ll D^b,$$

while (2.9) holds whenever $|a_m| \leq 1$ and

$$N \gg D^{(1-a)/2}.$$

Then (2.10) holds.

PROOF. We prove (iii); the other proofs are similar. Take $l = 4$ in the decomposition into sums (2.1). We may suppose that D is large. Given one of the sums (2.1), let N_0 and $\alpha_1, \dots, \alpha_{2l-1}$ have the values assigned before Lemma 1. We claim that the sum (2.1) is $O(BN^{\epsilon/2})$. If we can group the variables as

$$n = \prod_{i \in S} n_i \quad , \quad m = \prod_{i \notin S} n_i$$

with $\sigma_S \in [a, b]$, then from (2.8) the sum in (2.1) is $O(BN^{\epsilon/2})$, since the corresponding coefficients a_m, b_n are $O(N^{\epsilon/2})$. If this grouping is not possible, then by Lemma 2 there is an $\alpha_j > \frac{1-a}{2}$. Now we group the variables as

$$m = \prod_{i \neq j} n_i \quad , \quad n = n_j.$$

Note that $j < l$ from (2.2). It follows from (2.9) that the sum in (2.1) is $O(BN^{\epsilon/2})$. Now (iii) follows at once. \square

2. Lemmas on Exponential Sums

We quote a preliminary lemmas from Graham and Kolesnik [8].

LEMMA 4. *Let*

$$E(H) = \sum_{i=1}^u A_i H^{a_i} + \sum_{j=1}^v B_j H^{-b_j}$$

where A_i, B_j, a_i and b_j are positive. Let $0 \leq H_1 < H_2$. Then there is some $H \in (H_1, H_2]$ with

$$E(H) \ll \sum_{i=1}^u \sum_{j=1}^v (A_i^{b_j} B_j^{a_i})^{1/(a_i+b_j)} + \sum_{i=1}^u A_i H_1^{a_i} + \sum_{j=1}^v B_j H_2^{-b_j}.$$

PROOF. This follows at once from [8], Lemma 2.4. \square

In the remainder of this section, we write $L = \log(D + 2)$. Our first lemma is a variant of [2, Theorem 5].

LEMMA 5. *Let (κ, λ) be an exponent pair. Let α, β be real constants, $\beta \neq 0$, $\beta < 1$, $\alpha < 0$. Suppose that*

$$(3.1) \quad X > 0, \frac{1}{2} \leq N \ll M, MN \asymp D, D < D' \leq 2D.$$

Let

$$(3.2) \quad S_1 = \sum_{\substack{m \sim M \\ D \leq mn < D'}} a_m \sum_{n \sim N} b_n e\left(\frac{Xm^\alpha n^\beta}{M^\alpha N^\beta}\right)$$

where $|a_m| \leq 1$, $|b_n| \leq 1$. Then

$$(3.3) \quad S_1 \ll L^{7/4} \left(DN^{-1/2} + DM^{-1/4} + D^{5/4} X^{-1/4} N^{-3/4} + D^{5/4} X^{-1/4} M^{-3/8} \right. \\ \left. + (D^{11+10\kappa} X^{1+2\kappa} N^{2(\lambda-\kappa)})^{1/(14+12\kappa)} + (D^{5+4\kappa} N^{\lambda-\kappa})^{1/(6+4\kappa)} \right).$$

PROOF. In [2], proof of Theorem 5, it is shown that

$$(3.4) \quad L^{-7} S_1^4 \ll \frac{D^4}{Q^2} + D^3 \left(\left(\frac{XQ^3}{D} \right)^{1/2+\kappa} N^{\lambda-\kappa} + N + D^2 X^{-1} Q^{-3} \right)$$

for any natural number $Q \leq \min(c^2 N, cM^{-1/2})$. Here and in the remainder of the paper, c is a small positive constant. With a little thought we see that (3.4) is true for any real Q with $0 < Q \leq \min(c^2 N, cM^{-1/2})$. Now Lemma 4 yields

$$L^{-7} S_1^4 \ll D^4 N^{-2} + D^4 M^{-1} + (D^4)^{(3+6\kappa)/(7+6\kappa)} (D^{5/2-\kappa} X^{1/2+\kappa} N^{\lambda-\kappa})^{4/(7+6\kappa)} \\ + (D^5 X^{-1})^{(3+6\kappa)/(9+6\kappa)} (D^{5/2-\kappa} X^{1/2+\kappa} N^{\lambda-\kappa})^{6/(9+6\kappa)} \\ + D^5 X^{-1} M^{-3/2} + D^5 X^{-1} N^{-3}.$$

After some simplification, we obtain (3.3). \square

Our next lemma improves (2.1) of [18, Theorem 2] in the case $\kappa = \lambda = \frac{1}{2}$; three terms have been dropped from the bound for S_1 , as explained below.

LEMMA 6. *Suppose that (3.1) holds and $|a_m| \leq 1$, $|b_n| \leq 1$. The sum S_1 in (3.2) satisfies*

$$(3.5) \quad S_1 \ll L^3 \left((X^4 M^{13} N^{15})^{1/20} + X^{1/6} M^{2/3} N^{7/9} + (X M^7 N^{10})^{1/11} + M N^{1/2} + X^{-1/2} M N \right).$$

PROOF. Define S_0 in the same way as S_1 but without the condition $D < mn \leq D'$. It suffices to prove (3.5) with S_1 , L^3 replaced by S_0 , L^2 , as explained in Harman [10, pp. 49-50].

If $X \leq M$, the double large sieve [6, Theorem 2] yields

$$S_0 \ll L \left((X M N)^{1/2} + M N^{1/2} + X^{-1/2} M N \right)$$

which is satisfactory since $(X M N)^{1/2} \ll M N^{1/2}$.

Suppose now that $X > M$. From Theorem 2 of Wu [18], with $\kappa = \lambda = \frac{1}{2}$,

$$(3.6) \quad S_0 \ll L^2 \left((X^4 M^{13} N^{15})^{1/20} + X^{1/6} M^{2/3} N^{7/9} + (X M^3 N^4)^{1/5} \right. \\ \left. + (X M^7 N^{10})^{1/11} + M^{2/3} N^{17/18} + M N^{1/2} + (X^{-1} M^{14} N^{23})^{1/22} + X^{-1/2} M N \right)$$

Moreover,

$$(XM^7N^{10})^{1/11} \gg (X^{-1}M^{14}N^{23})^{1/22}$$

since $X > M \gg N$; and

$$(X^4M^{13}N^{15})^{1/20} \gg (XM^3N^4)^{1/5}$$

since $M \gg N$.

Finally,

$$X^{1/6}M^{2/3}N^{7/9} \gg M^{2/3}N^{17/18}$$

since $X > M \gg N$. Thus three terms on the right-hand side of (3.6) can be discarded, and the lemma follows. \square

LEMMA 7. Let α, β, γ be real constants with $\alpha < 1, \alpha\beta\gamma \neq 0$. Let (κ, λ) be an exponent pair. Let M, M_1, M_2 be real numbers $\geq \frac{1}{2}$ and $X \geq M_1M_2$. Let

$$S_2 = \sum_{m \sim M} \sum_{\substack{m_1 \sim M_1 \\ D \leq mm_1 < D'}} \sum_{m_2 \sim M_2} a_m b_{m_1, m_2} e\left(\frac{Xm^\alpha m_1^\beta m_2^\gamma}{M^\alpha M_1^\beta M_2^\gamma}\right)$$

where $|a_n| \leq 1, |b_{m_1, m_2}| \leq 1$. Then

$$S_2 \ll MM_1M_2(\log 12MM_1M_2)^2\{(M_1M_2)^{-1/2} + \left(\frac{X}{M_1M_2}\right)^{\kappa/(2+2\kappa)} M^{-(1+\kappa-\lambda)/(2+2\kappa)}\}.$$

PROOF. By a similar remark to that at the beginning of the last proof, this follows from [1, Theorem 2]. \square

LEMMA 8. Let α, β, γ be real constants with $\alpha(\alpha-1)\beta\gamma \neq 0$. Let H, M, N be real numbers $\geq \frac{1}{2}$, let $X > 0$ and

$$S = \sum_{h \sim H} \sum_{m \sim M} \max \left| \sum_{n=N_1}^{N_2} e\left(\frac{Xm^\alpha h^\beta n^\gamma}{M^\alpha H^\beta N^\gamma}\right) \right|$$

where the maximum is taken over $1 \leq N_1 \leq N_2 \leq N$. Then

$$S \ll (HNM)^{1+\epsilon} \left(\left(\frac{X}{HMN^2}\right)^{1/4} + \frac{1}{N^{1/2}} + \frac{1}{X} \right).$$

PROOF. See [16, Theorem 3]. \square

3. Proof of Theorem

Throughout this section, $k \in \{3, 4, 5\}$. We write

$$y_k = x^{(1-2\theta_k)/(k-1)},$$

so that (1.2) yields

$$R_k(x) = - \sum_{n \leq y_k} \mu(n) \psi(xn^{-k}) + O(x^{\theta_k + \epsilon}).$$

Accordingly it suffices to prove that

$$(4.1) \quad \sum_{n \sim D} \mu(n) \psi(xn^{-k}) \ll x^{\theta_k + 3\epsilon/4}$$

for

$$(4.2) \quad x^{\theta_k} < D < x^{(1-2\theta_k)/(k-1)}.$$

((4.1) is trivial for smaller D .)

Let $H \geq 1$. Vaaler [17] (see also [8, Appendix]) gave the approximation

$$\left| \psi(w) - \sum_{0 < |h| \leq H} a_h e(hw) \right| \leq B(w),$$

where $B(w) = \sum_{|h| \leq H} b_h e(hw)$ is non-negative; the a_h, b_h are given explicitly and satisfy

$$(4.3) \quad a_h \ll \frac{1}{h}, \quad b_h \ll \frac{1}{H}.$$

It follows that

$$\left| \sum_{n \sim D} \mu(n) \psi(xn^{-k}) - \sum_{0 < |h| \leq H} a_h \sum_{n \sim D} \mu(n) e(hxn^{-k}) \right| \leq \sum_{|h| \leq H} b_h \sum_{n \sim D} e(hxn^{-k}).$$

We select $H = [Dx^{-\theta_k}]$. The contribution to the right-hand side from $h = 0$ is

$$\ll DH^{-1} \ll x^{\theta_k}$$

from (4.3). Accordingly, after a splitting-up argument, we need to show that

$$(4.4) \quad \sum_{h \sim K} c_h \sum_{n \sim D} \mu(n) e(hxn^{-k}) \ll Kx^{\theta_k + 2\epsilon/3}$$

whenever (4.2) holds, $|c_h| \leq 1$ and

$$(4.5) \quad \frac{1}{2} \leq K \leq Dx^{-\theta_k};$$

for our proof will show that (4.4) remains valid with 1 in place of $\mu(n)$.

In the rest of this section, we write

$$S_I(D, K, N) = \sum_{h \sim K} c_h \sum_{\substack{m \sim M \\ mn \sim D}} a_m \sum_{n \sim N} e\left(\frac{hx}{(mn^k)}\right),$$

$$S_{II}(D, K, N) = \sum_{h \sim K} \sum_{\substack{m \sim M \\ mn \sim D}} a_m \sum_{n \sim N} b_n e\left(\frac{hx}{(mn^k)}\right),$$

where c_h, a_m, b_n are unspecified numbers of absolute value ≤ 1 , and D, K are assumed to satisfy (4.2), (4.5).

LEMMA 9. (i) Suppose that

$$(4.6) \quad S_{II}(D, K, N) \ll Kx^{\theta_k + \epsilon/3}$$

whenever

$$(4.7) \quad D^2 x^{-2\theta_k} \ll N \ll D^{1/2}.$$

Then (4.4) holds.

(ii) Suppose that

$$(4.8) \quad K < x^{5\theta_k - 1} D^{k-7/2}.$$

Then (4.4) holds.

(iii) Suppose that

$$(4.9) \quad KD^{8-k} < x^{10\theta_k - 1}.$$

Then (4.4) holds.

PROOF. (i) Since $\theta_k > 5/(6k+4)$, we deduce from (4.2) that

$$(4.10) \quad D^2 x^{-2\theta_k} \ll D^{1/3-c}.$$

We can thus apply Lemma 3(ii) with the choice

$$D^\lambda = x^{2\theta_k} D^{-1}, \quad D^{1-\lambda} = D^2 x^{-2\theta_k}.$$

Our hypothesis gives (4.6) for the range

$$D^{1-\lambda} \ll N \ll D^{1/2}.$$

We claim that

$$(4.11) \quad S_I(D, K, N) \ll K x^{\theta_k + \epsilon/3}$$

for

$$(4.12) \quad N \gg x^{1-4\theta_k} D^{3-k}.$$

This is a consequence of Lemma 8; we must show that

$$KD \left(\frac{x}{ND^{k+1}} \right)^{1/4} + KDN^{-1/2} + KD(KxD^{-k})^{-1} \ll Kx^{\theta_k}.$$

The first term on the left gives rise to the condition (4.12). The second term produces the condition

$$(4.13) \quad N \gg D^2 x^{-2\theta_k}.$$

Moreover, (4.2) gives

$$D^2 x^{-2\theta_k} < x^{1-4\theta_k} D^{3-k},$$

so that (4.13) follows from (4.12). Finally, it is easy to see that

$$x^{-1} D^{1+k} \ll x^{\theta_k},$$

so that the third term gives no difficulty, proving our claim.

Now

$$D^\lambda > x^{1-4\theta_k} D^{3-k},$$

or equivalently,

$$D^{4-k} < x^{6\theta_k-1}.$$

For $k=3$, this follows from $\theta_3 > 3/14$. For $k=4$, we use $\theta_4 > 1/6$, and for $k=5$, we require $D > x^{1/10}$, which is a consequence of (4.2). We conclude that (4.11) holds for $N \gg D^\lambda$, and (i) follows at once.

(ii) We apply Lemma 6 to show that (4.6) holds in the range (4.7) under the hypothesis (4.8). Treating the variable h trivially, this reduces to verifying that

$$\begin{aligned} & (K^4 x^4 D^{-4k} M^{13} N^{15})^{1/20} + (KxD^{-k})^{1/6} M^{2/3} N^{7/9} + (KxD^{-k} M^7 N^{10})^{1/11} \\ & + MN^{1/2} + (KxD^{-k})^{-1/2} D \ll x^{\theta_k}. \end{aligned}$$

The term $MN^{1/2}$ gives rise to the lower bound for N in (4.7). Next,

$$(KxD^{-k})^{-1/2} D \ll D^{k/2+1} x^{-1/2} \ll x^{\theta_k}$$

follows from (4.2), because

$$\theta_k > \frac{3}{4k+2}.$$

The condition

$$(K^4 x^4 D^{-4k} M^{13} N^{15})^{1/20} \ll x^{\theta_k}$$

may be rewritten

$$K^4 N^2 \ll x^{20\theta_k - 4} D^{4k - 13}.$$

In this form, it follows from (4.8) for $N \ll D^{1/2}$.

The condition

$$(Kx D^{-k} M^7 N^{10})^{1/11} \ll x^{\theta_k}$$

may be rewritten

$$KN^3 \ll x^{11\theta_k - 1} D^{k - 7}.$$

To show that this follows from (4.8), it suffices to verify that

$$x^{5\theta_k - 1} D^{k - 7/2} \ll x^{11\theta_k - 1} D^{k - 17/2}$$

or

$$(4.14) \quad D^5 \ll x^{6\theta_k}.$$

This in turn follows from (4.2), since

$$\theta_k > \frac{5}{6k + 4}.$$

Finally, the condition

$$(xKD^{-k})^{1/6} M^{2/3} N^{7/9} \ll x^{\theta_k}$$

reduces to

$$K \ll D^{k-4} x^{6\theta_k - 1} N^{-2/3},$$

and it suffices if

$$K \ll D^{k-13/3} x^{6\theta_k - 1}.$$

We need only verify that

$$x^{5\theta_k - 1} D^{k - 7/2} \ll D^{k - 13/3} x^{6\theta_k - 1}.$$

This reduces to (4.14), and the proof of (ii) is complete.

(iii) Similarly, on applying Lemma 5 with $\kappa = \lambda = 1/2$, we must show that (4.7) and (4.9) together imply

$$\begin{aligned} & DN^{-1/2} + DM^{-1/4} + D^{5/4} (Kx D^{-k})^{-1/4} N^{-3/4} \\ & + D^{5/4} (Kx D^{-k})^{-1/4} M^{-3/8} + D^{4/5} (Kx D^{-k})^{1/10} + D^{7/8} \ll x^{\theta_k}. \end{aligned}$$

To begin with, we observe that (4.9) implies

$$(4.15) \quad D \ll x^{(10\theta_k - 1)/(8 - k)}.$$

Thus

$$D^{7/8} \ll x^{(70\theta_k - 7)/(64 - 8k)} < x^{\theta_k}$$

because

$$(4.16) \quad \theta_k < \frac{7}{6 + 8k}.$$

Since $M \gg D^{1/2}$, it follows also that

$$DM^{-1/4} \ll x^{\theta_k}.$$

As before, the term $DN^{-1/2}$ gives rise to the lower bound on N in (4.7). Next,

$$D^{4/5} (Kx D^{-k})^{1/10} = (KD^{8-k})^{1/10} x^{1/10} < x^{\theta_k}$$

from (4.9). Next,

$$D^{5/4} (Kx D^{-k})^{-1/4} N^{-3/4} \ll (D^{5+k} x^{-1})^{1/4} (D^2 x^{-2\theta_k})^{-3/4}$$

$$= (D^{k-1}x^{-1+6\theta_k})^{1/4} < x^{\theta_k}$$

from (4.2).

Finally, since $M \gg D^{1/2}$,

$$D^{5/4}(KxD^{-k})^{-1/4}M^{-3/8} \ll (D^{5+k}x^{-1})^{1/4}D^{-3/16} = (D^{17+4k})^{1/16}x^{-1/4} \ll x^{\theta_k}.$$

To see this, we use (4.15):

$$D^{17+4k} \ll x^q \quad , \quad \text{where}$$

$$q = \frac{(17+4k)(10\theta_k-1)}{8-k} < 16\theta_k + 4,$$

as a consequence of (4.16). This completes the proof of (iii). \square

We have yet to use Lemma 7, which is capable of extending the range (4.7) on the left.

LEMMA 10. *Suppose that*

$$(4.17) \quad K^{-1}D^2x^{-2\theta_k} \ll N \ll \min(D^{k-4}x^{6\theta_k-1}, D^{1/2}).$$

Then (4.6) holds.

PROOF. We apply Lemma 7 with N, K in place of M_1, M_2 , and $\kappa = \lambda = 1/2$. The condition $X \gg M_1M_2$ reduces to

$$xD^{-k} \gg N.$$

Since $N \ll D^{1/2}$, this reduces in turn to $\theta_k \geq 3/(4k+2)$, which was used earlier.

Now we need to show that (4.17) implies

$$KD(NK)^{-1/2} + KD(xD^{-k}N^{-1})^{1/6}M^{-1/3} \ll Kx^{\theta_k}.$$

The term $KD(NK)^{-1/2}$ gives rise to the lower bound for N in (4.17). The condition

$$KD(xD^{-k}N^{-1})^{1/6}M^{-1/3} \ll Kx^{\theta_k}$$

is equivalent to

$$D^6 \left(\frac{x}{D^{k+2}} \right) N \ll x^{6\theta_k},$$

which reduces to the upper bound for N in (4.17). \square

PROOF OF THE THEOREM. Let D, K satisfy (4.2), (4.5). We must show that (4.4) holds.

Suppose first that $k = 3$. If

$$K < x^{11/74}D^{-1/2}$$

then we are done, by Lemma 9(ii). Suppose now that

$$(4.18) \quad K \geq x^{11/74}D^{-1/2}.$$

We apply Lemma 3(i). Thus it suffices to prove (4.6) for

$$(4.19) \quad D^{1/4} \ll N \ll D^{2/5}$$

and (4.11) for

$$(4.20) \quad N \gg D^{2/5}.$$

Lemma 10 gives (4.6) in the range

$$K^{-1}D^2x^{-17/37} \ll N \ll \min(D^{-1}x^{14/37}, D^{1/2}).$$

From (4.18), (4.2),

$$\begin{aligned} K^{-1}D^2x^{-17/37} &\leq D^{5/2}x^{-45/74} < D^{1/4}, \\ D^{-1}x^{14/37} &> D^{2/5}. \end{aligned}$$

Thus (4.6) holds in the required range.

As shown in the proof of Lemma 9, (4.11) holds in the range (4.12), which becomes

$$N \gg x^{3/37}$$

for $k = 3$. Since $x^{3/37} < D^{2/5}$ from (4.2), the discussion of the case $k = 3$ is complete.

Now let $k = 4$. From Lemma 9(ii), we may suppose that

$$(4.21) \quad K \geq x^{-9/94}D^{1/2}.$$

As above, it suffices to prove (4.6) in the range (4.19), and (4.11) in the range (4.20).

Lemma 10 gives (4.6) in the range

$$(4.22) \quad K^{-1}D^2x^{-17/47} \ll N \ll x^{4/47},$$

since $D^{1/2} > x^{17/188} > x^{4/47}$. Now

$$K^{-1}D^2x^{-17/47} < x^{-25/94}D^{3/2} < D^{1/4},$$

since $D < x^{10/47}$ from (4.2). Moreover,

$$x^{4/47} > D^{2/5},$$

also from (4.2). This gives (4.6) in the desired range.

As for (4.11), the range (4.12) becomes

$$(4.23) \quad N \gg x^{13/47}D^{-1}.$$

If $D \geq x^{9/47}$, then we obtain (4.11) in the desired range by combining the ranges (4.22) and (4.23).

It remains to prove (4.4) in the case $D < x^{9/47}$. In this case, (4.5) yields

$$KD^4 \leq D^5x^{-17/94} < x^{73/94},$$

and (4.4) follows from Lemma 9(iii).

Finally, let $k = 5$. In view of Lemma 9(ii), (iii), we may suppose that

$$(4.24) \quad K \geq x^{-1/4}D^{3/2}$$

and

$$(4.25) \quad KD^3 \geq x^{1/2}.$$

We may apply Lemma 3(iii) with

$$D^a = K^{-1}D^2x^{-3/10},$$

which was shown in (4.10) to be smaller than $D^{1/3}$. We must show that (4.6) holds for

$$K^{-1}D^2x^{-3/10} \ll N \ll \max(D^{1/3}, K^{-2}D^4x^{-3/5}),$$

and that (4.11) holds for

$$N \gg K^{1/2}D^{-1/2}x^{3/20}.$$

We apply Lemma 10. We have to show that

$$\max(D^{1/3}, K^{-2}D^4x^{-3/5}) \ll Dx^{-1/10}$$

(since $Dx^{-1/10} < D^{1/2}$ from (4.2)). The bound

$$D^{1/3} < Dx^{-1/10}$$

follows from (4.2). Also,

$$K^{-2}D^4x^{-3/5}(Dx^{-1/10})^{-1} = K^{-2}D^3x^{-1/2} \leq 1$$

from (4.24). This gives (4.6) in the required range.

As for (4.11), the range (4.12) satisfies our requirements, because we obtain

$$x^{2/5}D^{-2} \leq K^{1/2}D^{-1/2}x^{3/20}$$

on rearranging (4.25). This finishes the discussion for $k = 5$, and completes the proof of the Theorem. \square

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