Math 341 Exam 1 Preparation Sheet
Supplement

This is a supplement to the "You should be able to" section of the Exam 1 preparation sheet. For each true/false exercise, if a statement is true, justify why it is true or provide a proof, and if a statement is false, justify why it is false or provide a counterexample.

1. Find the least upper bound (supremum) and the greatest lower bound (infimum) of a bounded set.

   (a) $\sup \left\{ \dfrac{n+1}{2n+2} : n \in \mathbb{N} \right\} = 1.$

   (b) $\inf \left\{ \dfrac{1-n}{n} : n \in \mathbb{N} \right\} = -1.$

2. Compute, compare, and prove results on cardinality.

   (a) The set of all functions from $\mathbb{N}$ to $\mathbb{N}$ is countable.

   (b) The power set $\mathcal{P}(A)$ of a finite set A satisfies $|\mathcal{P}(A)| > |A|$.

3. Prove that a sequence converges using the ϵ-N definition.

   (a) $\displaystyle\lim_{n\to\infty} \frac{1}{n^2+1} = 0.$

   (b) $\displaystyle\lim_{n\to\infty} \frac{(-1)^n n}{n+3} = 1.$

   (c) $\displaystyle\lim_{n\to\infty} \frac{2n+5}{n+2} = 2.$

   (d) $\displaystyle\lim_{n\to\infty} \frac{(-1)^n}{2^n} = 0.$

4. Prove limit theorems for sequences.

   (a) If (a_n) converges to a and $c \in \mathbb{R}$, then (ca_n) converges to ca.

   (b) Let $c \in \mathbb{R}$. If (a_n) converges to a and satisfies $c \leq a_n$ for all n, then $c \leq a$.

5. Prove a sequence is monotone and bounded, and find its limit.

   (c) Let $x_1 = 1$ and define x_n recursively by $x_{n+1} = (1/4)x_n + 1$. Then (x_n) is monotone and bounded, with limit $4/3$.

   (b) Let $a_1 = 1$ and define a_n recursively by $a_{n+1}^2 = a_n + 1$. Then (a_n) is monotone and bounded, with limit $(1+\sqrt{5})/2$.

   (c) Let $w_1 = 1$ and define w_n recursively by $w_{n+1} = 2/w_n^2$. Then (w_n) is monotone and bounded with limit $2^{1/3}$.

6. Construct convergent subsequences and prove properties of subsequences.

(a) Let $x_1 = 1/2$ and define x_n recursively by $x_{n+1} = (1 - x_n)/2$. The sequence (x_n) has a monotone decreasing subsequence whose limit is $1/3$. [Hint: consider the subsequence $x_1, x_3, x_5, ...$]

(b) If a sequence (x_n) has two convergent subsequences, then (x_n) converges.

7. Prove a sequence is Cauchy and know the consequences.

(a) If (x_n) is Cauchy, then for any constant c, the sequence (cx_n) is Cauchy.

(b) For a sequence (x_n), if $|x_{n+1} - x_n| < 2^{-n}$ for all $n \in \mathbb{N}$, then (x_n) is Cauchy.

(c) If a sequence is Cauchy, then it converges.

8. Prove convergence for a series using the Tests in Section 2.7.

(a) The series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+1}$ converges.

(b) The series $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{2n-1}$ converges

(c) The series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ converges conditionally.

(d) The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(n^2-1)}{n^3}$ converges absolutely.

9. Prove basic properties for convergent series.

(a) A series $\sum a_n$ converges if and only if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon$$

for all $n > m \geq N$.

(b) If $\sum a_n$ converges, then for a constant c, the series $\sum ca_n$ converges as well.