Math 341 Exam 2 Preparation Sheet
Supplement

This is a supplement to the "You should be able to" section of the Exam 2 preparation sheet. This details some of the basic techniques used in many of the proofs you have seen. Some of the basic techniques are illustrated through examples. For the true/false exercises, if a statement is true, justify why it is true or provide a proof, and if a statement is false, justify why it is false or provide a counterexample.

1. Prove or disprove statements about convergence of infinite series.

• One way to show that a series $\sum_{n=1}^{\infty} a_n$ converges is by applying the Cauchy Criterion.

Example. Prove that if $\sum_{n=1}^{\infty} a_n$ converges absolutely and (b_n) is a bounded sequence, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

Let $M > 0$ be a bound on the bounded sequence (b_n), i.e., $|b_n| \leq M$ for all $n \in \mathbb{N}$.

By the Cauchy Criterion, we have for $\epsilon > 0$ the existence of $N \in \mathbb{N}$ such that for all $n > m \geq N$ we have

$$|a_{m+1}| + |a_{m+2}| + \cdots + |a_n| < \frac{\epsilon}{M}.$$

Then

$$\begin{aligned} |a_{m+1}b_{m+1}| + |a_{m+2}b_{m+2}| + \cdots + |a_n b_n| &\leq |a_{m+1}|M + |a_{m+2}|M + \cdots + |a_n|M \\ &= M(|a_{m+1}| + |a_{m+2}| + \cdots + |a_n|) \\ &\leq M\left(\frac{\epsilon}{M}\right) = \epsilon. \end{aligned}$$

By the Cauchy Criterion, the series $\sum_{n=1}^{\infty} |a_n b_n|$ converges.

Hence the series $\sum_{n=1}^{\infty} a_n b_n$ converges absolutely.

By the Absolute Convergence Test, the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

Exercises. Decide which of the following statements are true or false.

(a) There exists two divergent series $\sum_{n=1}^{\infty} x_n$, $\sum_{n=1}^{\infty} y_n$ such that $\sum_{n=1}^{\infty} x_n y_n$ converges.

(b) If $\sum_{n=1}^{\infty} a_n$ converges conditionally, then $\sum_{n=1}^{\infty} n^2 a_n$ diverges.

2. Prove or disprove sets are open and closed and know the consequences of both.

• One shows that a subset O of $\mathbb{R}$ is open by (a) exhibiting for each $x \in O$ a $V_\epsilon(x) \subseteq O$, or (b) showing that O^c is closed.

Example. Show that $O = \bigcup_{n=3}^{\infty} \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$ is open.

For $x \in O$ there exists $m \geq 3$ such that

$$x \in \left[\frac{1}{m}, 1 - \frac{1}{m}\right].$$

Because $m+1>m$, we have

$$x \in \left[\frac{1}{m}, 1-\frac{1}{m}\right] \subseteq \left(\frac{1}{m+1}, 1-\frac{1}{m+1}\right) \subseteq \left[\frac{1}{m+1}, 1-\frac{1}{m+1}\right] \subseteq O.$$

Choose

$$\epsilon = \min\left\{x-\frac{1}{m+1}, 1-\frac{1}{m+1}-x\right\}.$$

Then

$$V_\epsilon(x) \subseteq \left(\frac{1}{m+1}, 1-\frac{1}{m+1}\right) \subseteq O$$

and so O is open. [Here of course $O=(0,1)$.]

• One shows that a subset C of $\mathbb{R}$ is closed by (a) showing that C^c is open, or (b) that C contains all of its limit points.

For option (b) here, one starts with a limit point x of C and a convergent sequence (x_n) in C with $x=\lim x_n$ and $x \neq x_n$ for all $n \in \mathbb{N}$, and shows that $x \in C$.

Example. Show that $C = \bigcap_{m=1}^{\infty}\left[-\frac{1}{m}, 1+\frac{1}{m}\right]$ is closed.

Let x be a limit point C.

Then there is a sequence (x_n) in C with $x_n \neq x$ for all $n \in \mathbb{N}$ and $x_n \to x$.

Because $x_n \in C$ we have

$$-\frac{1}{m} \leq x_n \leq 1+\frac{1}{m} \text{ for all } m \in \mathbb{N}.$$

By the Order Limit Theorem we have

$$-\frac{1}{m} \leq x \leq 1+\frac{1}{m} \text{ for all } m \in \mathbb{N}.$$

Thus $x \in C$ as well, so that C is closed. [Here of course $C=[0,1]$.]

• Exercises. Decide which of the following statements are true or false.

(a) If a subset A of $\mathbb{R}$ has an isolated point, it cannot be open.

(b) Every finite set is closed.

(c) If O is an open subset of $\mathbb{R}$ that contains $\mathbb{Q}$, then $O=\mathbb{R}$.

3. Determine if a point is a limit point or an isolated point of a given subset A of $\mathbb{R}$.

• One shows that a point x is a limit point of A by constructing a sequence (x_n) in A with $x_n \neq x$ for all $n \in \mathbb{N}$ and $x_n \to x$.

Example. The middle thirds Cantor set C is the countable intersection of closed subsets C_n of $[0,1]$.

We show that each $x \in C$ is a limit point of C as follows.

For each $n \in \mathbb{N}$ the point x belongs to one of the distinct 2^n closed subintervals of C_n, each of length $1/3^n$.

We choose x_n to be an endpoint of a closed subinterval of C_n that is adjacent to the closed subinterval containing x.

In this way the sequence (x_n) is in C with $x_n \neq x$ for all $n \in \mathbb{N}$.

We get convergence of (x_n) to x because the distance between adjacent closed subintervals of C_n goes to 0 as $n \to \infty$.

• One shows that a point $a \in A$ is an isolated point of A by finding a $V_\epsilon(a)$ such that $A \cap V_\epsilon(a) = \{a\}$.

Example. Usually isolated points are easy to spot, as in the set $A = [0,1] \cup \{2\}$.

• Exercises. Decide which of the following statements are true or false.

(a) If A is a bounded subset of $\mathbb{R}$, then $s = \sup A$ is a limit point of A.

(b) Every point in the set
$$A = \left\{ \frac{(-1)^n n}{n+1} : n \in \mathbb{N} \right\}$$
is isolated.

4. <u>Find the closure of a given subset A of $\mathbb{R}$.</u>

• This requires identifying the set of limit points L of A since the closure of A is $\overline{A} = A \cup L$.

Example. The closure of $A = \{1/n : n \in \mathbb{N}\}$ is $A \cup \{0\}$ because 0 is the only limit point of A.

• Exercises. Decide which of the following statements are true or false.

(a) For any subset A of $\mathbb{R}$, the set $\overline{A}^c$ is open.

(b) A subset A of $\mathbb{R}$ is closed if and only if $\overline{A} = A$.

5. <u>Know the properties of compact sets and how to determine if $A \subseteq \mathbb{R}$ is compact.</u>

• One shows that A is compact by showing that A is closed and bounded (Heine-Borel Theorem).

Example. The middle thirds Cantor set C is bounded because it is a subset of the bounded set $[0,1]$.

It is closed because its complement is the union of open intervals, and hence open:
$$C^c = (-\infty, 0) \cup (1, \infty) \cup \left(\bigcup_{n=1}^{\infty} C_n^c \right).$$

• Exercises. Decide which of the following statements are true or false.

(a) If $A_1, A_2, A_3, \ldots$ are compact subsets of $\mathbb{R}$, then

$$\bigcap_{n=1}^{\infty} A_n$$

is compact as well.

(a) If closed subsets A_n, $n \in \mathbb{N}$ satisfy $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$, then

$$\bigcap_{n=1}^{\infty} A_n$$

is compact as well.

6. Determine if a subset A of $\mathbb{R}$ is perfect.

• One shows that A is perfect by showing it is nonempty, closed, and does not contain isolated point.

Example. The Cantor set C is perfect because it is nonempty (it contains the rational endpoints of every C_n), it is closed, and as reviewed earlier, it has no isolated points.

• Exercises. Decide which of the following statements are true or false.

(a) A perfect set is uncountable.

(b) If C is the middle thirds Cantor set, then $C \cap [0, 1/2]$ is perfect.

(c) If C is the middle thirds Cantor set, then $C \cap \mathbb{Q}$ is perfect.

7. Determine if a set is disconnected or connected and know the consequences.

• One shows that a subset E of $\mathbb{R}$ is disconnected if there are sets A and B with $E = A \cup B$ such that $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$.

• One shows that a subset E of $\mathbb{R}$ is connected if for every pair of sets A and B with $E = A \cup B$, we have that $\overline{A} \cap B \neq \emptyset$ or $A \cap \overline{B} \neq \emptyset$.

Example. The set $\mathbb{Q}$ is disconnected because $A = \mathbb{Q} \cap (-\infty, \sqrt{2})$ and $B = \mathbb{Q} \cap (\sqrt{2}, \infty)$ satisfy $\mathbb{Q} = A \cup B$ with $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$.

• Exercises. Decide which of the following statements are true or false.

(a) The middle thirds Cantor set C is disconnected.

(b) The only connected subsets of $\mathbb{R}$ are the intervals.

8. Find the limit of a function if it exists or be able to prove one does not exist.

• One show for $f : A \to \mathbb{R}$ and c a limit point of A, that $f(x) \to L$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$ with $x \in A$.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x$, and let $c = 1$.

We guess the limit of $f(x)$ as $x \to 1$ to be $L = 2$.

To confirm this guess, we want to control $|f(x) - L| = |2x - 2| = 2|x - 1|$.

Choosing $\delta = \epsilon/2$ we have $|f(x) - L| < 2(\epsilon/2) = \epsilon$ whenever $0 < |x - 1| < \delta$.

• One shows that a limit of a function $f(x)$ does not exist as $x \to c$ by finding two sequences (x_n) and (y_n) in A such that $x_n \to c$, $y_n \to c$, but that $\lim f(x_n) \neq \lim f(y_n)$.

Example. The function

$$g(x) = \begin{cases} \sin(1/x^2) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is not continuous at $c = 0$ because the sequences

$$x_n = \frac{1}{\sqrt{2n\pi}}, \quad y_n = \frac{1}{\sqrt{2n\pi + \pi/2}}$$

both converge to 0 but $f(x_n) = 0$ and $f(y_n) = 1$ for all $n \in \mathbb{N}$.

• Exercises. Decide which of the following statements are true or false.

(a) $\lim_{x \to 1}(x^2 + x) = 2$.

(b) $\lim_{x \to 0} \sin(\ln x) = 0$.

9. <u>Determine if a function if continuous at a point and what this implies</u>.

• One shows that $f : A \to \mathbb{R}$ is continuous at a limit point $c \in A$ by one of the four characterizations of continuity such as if $(x_n) \to c$ with $x_n \in A$ for all $n \in \mathbb{N}$, then $f(x_n) \to f(c)$.

Example. The function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} x\sin(1/x^2) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is continuous at $c = 0$ because for any sequence (x_n) with $x_n \neq 0$ for all $\mathbb{N}$ and $x_n \to 0$ we have $|\sin(1/x_n^2)|$ is bounded by 1 and $|x_n| \to 0$, so that $|x_n \sin(1/x_n^2)| \to 0$.

• Exercises. Decide which of the following statements are true or false.

(a) Let C be the middle third Cantor set, and define $g : [0, 1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & \text{if } x \in C, \\ 0 & \text{if } x \notin C. \end{cases}$$

The function g is continuous at every $c \in C$.

(b) If $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are both continuous at $c \in A$, then $f + g$ is continuous at c.