Math 341 Exam 3 Preparation Sheet
Supplement

This is a supplement to the "You should be able to" section of the Exam 3 preparation sheet. This details some of the basic techniques used in many of the proofs you have seen. Some of the basic techniques are illustrated through examples. For the true/false exercises, if a statement is true, justify why it is true or provide a proof, and if a statement is false, justify why it is false or provide a counterexample.

1. Determine if a function if uniformly continuous and know the consequences.

• One shows that $f : A \to \mathbb{R}$ is uniformly continuous on A by showing for each $\epsilon > 0$ the existence of $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$ with $x, y \in A$.

Example. The function $f(x) = 2x$ is uniformly continuous on $\mathbb{R}$ because the Tchoice of $\delta = \epsilon/2$ is independent of the point c at which continuity was being established.

• One shows that $f : A \to \mathbb{R}$ is not uniformly continuous on A by the Sequential Criterion for Nonuniform Continuity: there exists $\epsilon_0 > 0$ and two sequences (x_n) and (y_n) in A with $|x_n - y_n| \to 0$ and $|f(x_n) - f(y_n)| \geq \epsilon_0$.

Example. The function $g : (0, \infty) \to \mathbb{R}$ defined by $g(x) = \sin(1/x^2)$ is continuous on $(0, \infty)$ but not uniformly continuous on $(0, \infty)$ because for $\epsilon_0 = 1$ and the sequences

$$x_n = \frac{1}{\sqrt{2n\pi}},\ y_n = \frac{1}{\sqrt{2n\pi + \pi/2}}$$

we have $|x_n - y_n| \to 0$ and $|f(x_n) - f(y_n)| = 1 \geq \epsilon_0$.

• Exercises. Decide which of the following statements are true or false.

(a) When A is compact for $f : A \to \mathbb{R}$, the function f is uniformly continuous on A.

(b) The function $f(x) = x^{1/3}$ is uniformly continuous on $[0, \infty)$.

(c) The function x^2 is uniformly continuous on $[0, \infty)$.

2. Know and apply the Intermediate Value Theorem.

• One uses the Intermediate Value Theorem on a continuous function $f : [a, b] \to \mathbb{R}$ to find for any L between $f(a)$ and $f(b)$ a point $c \in (a, b)$ such that $f(c) = L$.

Example. The function $f(x) = \sin(\pi x/2)$ is continuous on $\mathbb{R}$. Since $f(0) = 0$ and $f(1) = 1$, there is $c \in (0, 1)$ such that $f(c) = 1/\pi$.

• Exercises. Decide which of the following statements are true or false.

(a) For $f(x) = x^2 + x - 1$ there exists $c \in [0, 2]$ such that $f(c) = 0$.

(b) For $g(x) = e^{x-1} - x$ there exists $c \in [0, 3]$ such that $g(c) = 1$.

3. Determine if a function is increasing or decreasing.

• One shows that a function $f : A \to \mathbb{R}$ is increasing (decreasing) by showing that for all $x, y \in A$ with $x < y$ we have that $f(x) \leq f(y)$ ($f(x) \geq f(y)$).

Example. The function $f(x) = x^2 - 2x + 1$ is increasing on $A = [1, \infty)$ because for $x < y$ with $x, y \in A$ we have

$$f(x) = x^2 - 2x + 1 = (x - 1)^2 \leq (y - 1)^2 = y^2 - 2y + 1 = f(y).$$

• Exercises. Decide which of the following statements are true or false.

(a) The function $f(x) = x^3 + x$ is increasing on $[0, \infty)$.

(b) The function $f(x) = x^3 - x$ is decreasing on $[-1, 1]$.

4. Prove that a derivative exists for a function using the definition. Let $f : A \to \mathbb{R}$ for A an interval. The definition of the derivative of f a point $c \in A$ is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

provided the limit exists.

• Example. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2/(x+1) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

For $x > 0$ we have

$$\frac{x^2/(x+1) - 0}{x - 0} = \frac{x}{x+1}.$$

The function $x/(x + 1)$ is continuous and so we have

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x}{x+1} = 0.$$

For $x < 0$ we have

$$\frac{f(x) - f(0)}{x - 0} = \frac{0 - 0}{x - 0} = 0$$

and so

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = 0.$$

Since the two one-sided limits exist and are equal, we have that

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0$$

and so f is differentiable at $c = 0$.

• Exercises. Decide which of the following statements are true or false.

(a) The function $f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is differentiable at $c = 0$.

(b) The function $g(x) = \begin{cases} (x^2 + x)/(x^2 - x) & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases}$ is differentiable at $c = 0$.

(c) The function $h(x) = x^{1/3}$ is differentiable at $c = 0$. [Do not use a differentiation formula, but the definition of derivative to answer this.]

5. Know and apply the Mean Value Theorem. If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

• Example. Suppose that $f : A \to \mathbb{R}$ is differentiable on an interval A. Prove that if f' is bounded on A, then f is uniformly continuous.

Proof. We have that there is $M > 0$ such that $|f'(x)| \leq M$ for all $x \in A$.

For $\epsilon > 0$ we choose $\delta = \epsilon/M$. [We will see why this choice for δ in a minute.]

Take $x, y \in A$, and WLOG suppose $x < y$.

By the Mean Value Theorem there is a point $c \in (x, y)$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(c).$$

Taking the absolute value of both sides, multiplying by the denominator, and using the bound on f' gives us

$$|f(x) - f(y)| = |f'(c)|\ |x - y| \leq M|x - y|.$$

Then for $|x - y| < \delta$ we have that

$$|f(x) - f(y)| < M\delta < \epsilon,$$

thus giving us uniform continuity of f on the interval A. □

• Exercises. Decide which of the following statements are true or false.

(a) If $f : (a, b) \to \mathbb{R}$ is differentiable and f is increasing on (a, b), then $f'(x) \geq 0$ for all $x \in (a, b)$.

(b) For $f : [0, \infty) \to \mathbb{R}$ suppose f is continuous, differentiable on $(0, \infty)$, $f(0) = 0$, and $f'(x)$ is increasing on $(0, \infty)$. Then the function $g(x) = f(x)/x$ on $(0, \infty)$ is increasing.

(c) Suppose $f : [0, 10] \to \mathbb{R}$ is continuous with f differentiable on $(0, 10)$. If $f(1) = 5$, $f(5) = 1$, and $f(9) = 4$, then there exists $c \in (0, 10)$ such that $f'(c) = 0$. [In addition to the Mean Value Theorem, you will need Darboux's Theorem to answer this.]

6. Prove pointwise convergence for a sequence of functions. A sequence of functions $f_n : A \to \mathbb{R}$ converges pointwise to a function $f : A \to \mathbb{R}$ if for each $x \in A$ we have convergence of the real numbers $f_n(x)$ converging to $f(x)$.

• Example. The poinwise limit of $f_n(x) = x^n$ on $[0, 1]$ is the piecewise defined function

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ 1 & \text{if } x = 1. \end{cases}$$

• Exercises. Decide which of the following statements are true or false.

(a) Uniform convergence implies pointwise convergence.

(b) The pointwise limit function of

$$f_n(x) = \frac{nx + \sin(nx)}{2n}$$

is $f(x) = x/2$.

7. Prove that a sequence of functions is uniformly convergent and know the consequences. A sequence of functions $f_n : A \to \mathbb{R}$ converges uniformly to $f : A \to \mathbb{R}$ if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$ and all $x \in A$.

The Cauchy Criterion for Uniform Convergence is that f_n converges uniformly to f on A if and only if for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \epsilon$ for all $n, m \geq N$ and all $x \in A$.

Uniform convergence is needed to ensure that the limit function of continuous functions is continuous.

• Example. For $n \in \mathbb{N}$, let $f_n : (0, \infty) \to \mathbb{R}$ be defined by

$$f_n(x) = \frac{n^2x^2}{5 + n^2x^3}.$$

The pointwise limit of f_n is the function $f : (0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{x}.$$

To investigate the convergence of f_n to f, we consider

$$|f_n(x) - f(x)| = \left| \frac{n^2x^2}{5 + n^2x^3} - \frac{1}{x} \right| = \left| \frac{n^2x^3 - (5 + n^2x^3)}{5 + n^2x^3} \right| = \frac{5}{5 + n^2x^3}.$$

Because $5 + n^2x^3 > n^2x^3$, we have

$$|f_n(x) - f(x)| < \frac{5}{n^2x^3}.$$

To get uniform convergence on $(0,\infty)$ would require that for $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x > 0$ we have

$$\frac{5}{n^2 x^3} \leq \frac{5}{N^2 x^3} < \epsilon.$$

This requires that N satisfy

$$N > \left(\frac{5}{\epsilon x^3}\right)^{1/2}.$$

As $x \to 0$ we have that $N \to \infty$, thereby preventing uniform convergence on $(0,\infty)$.

However, if we restrict x to $x \geq a$ where $a > 0$, then as $x^3 \geq a^3$, we get that

$$\frac{5}{N^2 x^3} \leq \frac{5}{N^2 a^3} < \epsilon,$$

and so we can choose

$$N > \left(\frac{5}{\epsilon a^3}\right)^{1/2}.$$

Thus we have uniform convergence of f_n to f on $[a,\infty)$ for any $a > 0$.

• Exercises. Decide which of the following statements are true or false.

(a) The sequence $f_n(x) = 1/(1+n^2x^2)$ converges uniformly on $[0,1]$.

(b) The sequence $g_n(x) = nx(1-x)^n$ converges uniformly on $[0,1]$. [Hint: for each n find the maximum value of $g_n(x)$.]

(c) The sequence $h_n(x) = x^2/(n^2+x)$ converges uniformly on $[0,1]$.

8. Know when the derivative of a limit functions exists. For a sequence of differentiable functions $f_n : [a,b] \to \mathbb{R}$, if we have uniform convergence of f_n' to g and if there exists $x_0 \in [a,b]$ such that $f_n(x_0)$ converges, then f_n converges uniformly, and the limit function f is differentiable with $f' = g$.

• Example. The sequence of functions

$$f_n(x) = \frac{\sin(nx)}{n^3}$$

converges pointwise on $\mathbb{R}$ to $f(x) = 0$.

The sequence of derivatives

$$f_n'(x) = \frac{\cos(nx)}{n^2}$$

converges uniformly to $g(x) = 0$ on $\mathbb{R}$ because

$$|f_n'(x) - 0| = \left|\frac{\cos(nx)}{n^2}\right| \leq \frac{1}{n^2}.$$

Thus we have that $f' = g$

• Exercises. Decide which of the following statements are true or false.

(a) The sequence of derivatives of $f_n(x) = (1/n)\sin(nx)$ converges uniformly on $\mathbb{R}$.

(b) The sequence of differentiable functions

$$f_n(x) = \frac{n^2x^2}{n^2+n}, \ \ x \in \mathbb{R},$$

converges to a differentiable function $f(x)$ for which $f'(x) = 2x$.